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Combinatorial set theory: Partition relations for cardinals. (In English)

Studies in Logic and the Foundations of Mathematics, 106. Amsterdam-New York-Oxford: North-Holland Publishing Company; Budapest: Akadémiai Kiadó. 347 p. \$ 53.75; Dfl. 140.00 (1984).

Constitutive parts of the present book are: Preface (pp. 5-6), Contents (7-8), Chapter I. Introduction (9-33, sections 1-4), II. Preliminaries (34-51, 5-7), III. Fundamentals about partition relations (52-79, 8-12), IV. Trees and positive ordinary partition relations (80-104, 13-18), V. Negative o.p.r.'s and the discussion of the finite case (105-157, 19-26), VI. The canonization lemmas (158-167, 27-28), VII. Large cardinals (168-214, 29-34), VIII. Discussion of the o.p.r. with superscript 2 (215-232, 35-37), IX. Discussion of the o.p.r. with superscript ≥ 3 (233-262, 38-42), X. Some applications of combinatorial methods (263-312, 43-48), XI. A brief survey of the square bracket relation (313-333, 49-55), Bibliography (335-340; 109 items), Author index (341-342), Subject index (343-347).

The Preface beginns like this: "Ramsey's classical theorem in its simplest form, published in 1930, says that if we put the edges of an infinite graph into two classes, then there will be an infinite subgraph all edges of which belong to the same class. The partition calculus has been developed as a collection of generalizations of this theorem". The o.p.r. (ordinary partition relation) (1) $a \to (b_h)^r_{h\in H}$ means: for any partition $I =< I_h : h \in H >$ of $[A]^r$, where A is any set of cardinality a, there are $h \in H$ and B_h such that $[B_h]^r \subseteq I_h$. Similarly, (1) is defined if a, b_h are order types. E.g. $\omega \to (\omega)^m_n$ for any integers m, n (Ramsey); this implies the finite Ramsey's Theorem: For any positive integers k, r, m there is an integer n such that $n \to (m)^r_k$. Any infinite cardinal κ satisfies $\kappa \to (\kappa, \omega)^2$ (Erdős-Dushnik- Miller, 1941), and $2^\kappa \not\to (\kappa^+, \kappa^+)^2$ [cf. *W. Sierpiński*, Ann. Sc. Norm. Super. Pisa, II. Ser. 2, 285-287 (1933; Zbl 007.09702)] by solving affirmatively a Knaster's problem: as B. Knaster (in reviewing Sierpiński's paper [loc. cit.]) wrote: "Für höhere Mächtigkeiten bleibt das Problem offen". The present reviewer found independently the requested partition in 1937, and published it in Revista Ci. 42, 827-846 (1940; Zbl 063.03392), 43, 483-500 (1941; Zbl 063.03392); his method is transferable to higher cardinals substituting $\bar{H}_0$ by $\bar{H}_\beta$, for any ordinal β ; H_β is the Hausdorff set $(l + m + n)^{\omega^*_\beta}$ (cf. reviewer's Thèse). Many inaccessible cardinals satisfy $x \not\to (x)^2_2$ (Hanf- Tarski). Problem: Does GCH imply $\aleph_{\omega+1} \to [\aleph_2]^{\aleph_0}_{\aleph_1}$?

Some other p.r.'s are considered: polarized, simultaneous, for reals etc. The book contains many propositions on p.r.'s in dependence of variables in corresponding symbols. It is instructive to compare "positive" or $\to$ results and "negative" ($\not\to$) ones, e.g. the stepping up 16.1, the negative stepping up 24.1, pseudo-stepping up 25.6. A great majority of results were published previously but e.g. Baumgartner's Theorem 31.4 was not published elsewhere. As to applications, they concern: topology, set mapping theorems (Fodor, Hajnal), powers of cardinals with numerous inequalities gathered around Silver's

Theorem that GCH cannot first fail at a singular cardinal of cofinality $> \omega_0$. The book is almost self-contained. The main method of proofs are tree considerations. Technically, the book is at a high level.
Reviewer's remarks. 1. In connection with the quoted part of the Preface, it is a pity that the book contains no trace of the fact that the reviewer arrived at Ramsey results and partition calculus independently of Ramsey, Erdős, Rado,... via his "relation fondamentale" (cf. p. 1197 of the reviewer's paper in C. R. Acad. Sci., Paris 205, 1196-1198 (1937; Zbl 018.05504), the proof of which was published in C. R. Soc. Sci. Varsovie, Cl. III 32, 62-67 (1939), reprinted in Periodicum Math.-Phys. Astron., II. Ser. 14, 205-210 (1959; Zbl 094.03301), and reviewed by *Erdős* in MR 23,A2341; cf. also the reviewer's paper in Conferenze Sem. Mat. Univ. Bari 89, 18 p. (1963; Zbl 149.40301) and the reviewer's review of a paper by *W.G.Fleissner* in Zbl 411.54007).
2. The proof of Th.52.3 is not completely correct because the orderings $<_\xi$ ($\xi < \aleph_1$) are not independent: completely independent are total orderings $<_n$ ($n \in N$, N denoting the system of all nodes or knots of the tree considered; node partition of any given tree is missing in the present book; this notion is important for the "natural ordering" of $(T, \leq)$ as total order extension of $<$).

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Classification:
03E05 Combinatorial set theory (logic)
05C55 Generalized Ramsey theory
04A20 Combinatorial set theory
03-02 Research monographs (mathematical logic)
04-02 Research monographs (set theory)
05-02 Research monographs (combinatorics)
03E55 Large cardinals

Keywords:
bracket arrow; partition tree; cofinality; partition relations; infinite graph; inaccessible cardinals; GCH