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The tails of infinitely divisible laws and a problem in number theory. (In English)
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A function defined on the positive integers is said to be additive if it satisfies the relation $f(mn) = f(m)+f(n)$ for all relatively prime m and n. An example of an additive function is $\omega(n)$, the number of distinct prime divisors of the positive integer n. A great deal of work has gone into proving central limit type theorem for additive functions satisfying certain moment conditions. See *P.Erdős* and *M.Kac* [Am. J. Math. 62, 738- 742 (1940; Zbl 024.10203)] for early such work, and *J.Kubilius*'s monograph [Probabilistic methods in the theory of numbers (1962; Zbl 127.27402)]. See also *W.Philipp* [Analytic Number Theory, Proc. Symp. pure Math. 24, St. Louis Univ. Missouri 1972, 233-246 (1973; Zbl 269.10031)] for functional central limit theorems for additive functions. In many of these investigations the theory of sums of independent random variables come into play, and the limit laws are infinitely divisible.
In the present paper the additive function f generated by $f(p^m) = (\log p^m)^\alpha$ with $0 < \alpha < 1$ is considered, and the limit law.

$$(*) \qquad K_\alpha 8x) = (\text{weak}) \lim_{N\to\infty} N^{-1}\sharp\{n \le N | f(n) < x(\log N)^\alpha\}$$

is studied. *P. Erdős* in [Ann. of Math., II Ser. 47, 1-20 (1946; Zbl 061.07902)] had noted that K_α exists for all α. If $\alpha = 1$, then K_α is degenerate (consists of a single jump). For $\alpha > 1$, *B.V.Levin* and *N.M.Timofeev* [Acta arithmetica 26, 333-364 (1975; Zbl 318.10041)] have shown that the support of K_α lies in the unit interval. Non-degenerate laws with bounded support are not infinitely dibisible, and hence K_α is not infinitely divisible in this case. For $0 < \alpha < 1$ it turns out the support of K_α is not concentrated on a bounded interval but the authors show that K_α is nevertheless not infinitely divisible. The proof depends on a probabilistic theorem of independent interest. The authors show that if the tail of an infinitely divisible probality law approaches zero sufficiently rapidly, then the law must be the normal law. For K_α to be infinitely divisible, it must be normal, which it clearly is not. Hence K_α is not infinitely divisible for any positive $\alpha \ne 1$. This implies that any attempt to study K_α by applying the theory of sums of independent random variables to (*) in a straightforward way seems doomed to failure. For a general picture of probabilistic methods in the theory of additive functions see *P.Billingsley* [Ann. of Probab. 2, 749-791 (1974; Zbl 327.10055)].

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