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Intersection properties of systems of finite sets. (In English)

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The authors use a theorem of Erdős-Rado [*P.Erdős* and *R.Rado*, J. London Math. Soc. 35, 85-90 (1960; Zbl 103.27901)] to generalize theorems of Erdős-Ko-Rado [*P.Erdős, Chao Ko* and *R.Rado*, Quart. J. Math., Oxford II. Ser. 12, 313-320 (1961; Zbl 100.01902)] , *M.Deza* [J. Comb. Theory, Ser. B 16, 166-167 (1974; Zbl 263.05007)] , *A.Hajnal* and *R. Rothschild* [J. Comb. Theory, Ser. B 15, 359-362 (1973; Zbl 269.05003)] and *A.J.W.Hilton* and *E.C.Milner* [Theorem 2 in Quart. J. Math. Oxford II. Ser. 18, 369-384 (1967; Zbl 168.26205)]. X is a finite set with $|X| = n$, $L = \{l_1, \dots, l_r\}$, $l_1 < \dots < l_r$ and $K = \{k_1, \dots, k_s\}$, $k_1 < \dots k_s$ are sets of integers: an (n, L, K)- system is a collection $\mathcal{A}$ of subsets of X such that for each $A_1, A_2 \in \mathcal{A}$, $|A_1|$,$|A_2| \in K$ and $|A_1 cap A_2| \in L$. Define $K_i = K \cap \{l_i * 1, \dots, L_{i+1}\}$, $0 \le i \le r$, where $l_0 = -1, L_{r+1} = k_s$, and $k_1^* = min\{k | k \in K_i\}$. Theorem 7. (1) If $|\mathcal{A}| > k_s c(k_s, L) \prod_{i=2}^r (n - l_i)/(k_i^* - l_i)$ then there exists a set D such that $|D| = l_1$ and $D \subseteq A$ for every $A \in \mathcal{A}$. (ii) If $|\mathcal{A}| > k_s^3 2^{r-1} n^{r-1}$ then there exists a $k \in K_r$ such that $l_i - l_{i-1}$ divides $l_{i+1} - l_i$, $2 \le i \le r$, $l_{r+1} = k$. (iii) $|\mathcal{A}| \le \sum_{i=0}^r \varepsilon_1 \prod (n - l_j)/(k_i^* - l_j)$ where $\varepsilon = 0 \text{or} 1$ according as $K_i = \emptyset$ or not, and the product is taken over those j, $1 \le j \le r$ for which $l_j < k_i^*$.

Theorem 8. If $K = \{k\}$ and for a fixed $q \ge 1$ we can find, among any $A_1, \dots, A_{q+1} \in \mathcal{A}$, two of them A_1, A_j such that $|A_i \cap A_j| \in L$, then there is a constant $c = c(k, q)$ such that if $|\mathcal{A}| > (q-1) \prod_{i=1}^r (n - l_i)/(k - l_i) + cn^{r-1}$ then there are sets $D_1, \dots, D_s$, each of cardinality l_1, such that for every $A \in \mathcal{A}$ there is an i for which $D_i \subset A$. Further, if q_i is the maximum number of sets A_j, $1 \le j \le q_i$, such that $D_i \subset A_j$, but for $h \ne i$, $D_n \not\subset A_j$ and $|A_{j_1} \cap A_{j_2}| \notin L$ for $1 \le j_1 < j_2 \le q_1$, then $\sum_{i=1}^s q_i = q$. Also, for $n > n_0(k, q)$, $|\mathcal{A}| \leqq \prod_{i=1}^r (n - l_i)/(k - l_i) + 0(n^{r-1})$.

Theorem 9. If, for any t different members of $\mathcal{A}$, $> |A_1 \cap \dots \cap A_t| \in L$, then there is a constant $c = c(k, t)$ such that if $|\mathcal{A}| > cn^{r-1}$, then there is a set D, $|D| = l_1$, $D \subset A$ for every $A \in \mathcal{A}$, and $l_i - l_{i-1}$ divides $l_{i+1} - l_i$, $2 \le i \le r$. Also, for $n > n_0(k, t)$, $|\mathcal{A}| \le (t-1) \prod_{i=1}^r (n - l_i)/(k - l_i)$. The authors ask if it is true that $L' \subset L$ implies the existence, for large enough n, of (n, L, k)- and (n, L', k)-systems $\mathcal{A}$ and $\mathcal{A}'$, each of maximum cardinaly with $\mathcal{A}' \subseteq \mathcal{A}$. They note that Theorem 7 and 9 may be simultaneously generalized to the families called quasi-block- designs by *Vera T. Sós* [Colloq. int. Teorie comb., Roma 1973, Tomo II, 223-233 (1976; Zbl 261.05022)].

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