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A UNIQUE COMMON FIXED POINT THEOREM FOR OCCASIONALLY WEAKLY COMPATIBLE MAPS

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Abstract. The aim of this paper is to establish a unique common fixed point theorem for two pairs of occasionally weakly compatible single and multi-valued maps in a metric space. This result improves the result of Türkoğlu et al. [6] and references therein.

1 Introduction and preliminaries

Throughout this paper, $(\mathcal{X}, d)$ denotes a metric space and $CB(\mathcal{X})$ the family of all nonempty closed and bounded subsets of $\mathcal{X}$. Let H be the Hausdorff metric on $CB(\mathcal{X})$ induced by the metric d; i.e.,

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(A,y)\}$$

for A, B in $CB(\mathcal{X})$, where

$$d(x,A) = \inf\{d(x,y) : y \in A\}.$$

Let f, g be two self-maps of a metric space (X, d). In his paper [5], Sessa defined f and g to be weakly commuting if for all $x \in \mathcal{X}$

$$d(fgx, gfx) \leq d(gx, fx).$$

It can be seen that two commuting maps ($fgx = gfx$ $\forall x \in \mathcal{X}$) are weakly commuting, but the converse is false in general (see [5]).

Afterwards, Jungck [2] extended the concepts of commutativity and weak commutativity by giving the notion of compatibility. Maps f and g above are compatible if

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

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whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in \mathcal{X}$. Obviously, weakly commuting maps are compatible, but the converse is not true in general (see [2]).

Further, Kaneko and Sessa [4] extended the concept of compatibility for single valued maps to the setting of single and multi-valued maps as follows: $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are said to be compatible if $fFx \in CB(\mathcal{X})$ for all $x \in \mathcal{X}$ and

$$\lim_{n\to\infty} H(Ffx_n, fFx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $Fx_n \to A \in CB(\mathcal{X})$ and $fx_n \to t \in A$.

In 2002, Türkoğlu et al. [6] gave another generalization of commutativity and weak commutativity for single valued maps by introducing the next definition: $f : \mathcal{X} \to \mathcal{X}$ and $F : \mathcal{X} \to CB(\mathcal{X})$ are called compatible if

$$\lim_{n\to\infty} d(fy_n, Ffx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathcal{X}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = t$ for some $t \in \mathcal{X}$, where $y_n \in Fx_n$ for $n = 1, 2, \dots$.

In [3], Jungck and Rhoades weakened the notion of compatibility for single and multi-valued maps by giving the concept of weak compatibility. They define maps f and F above to be weakly compatible if they commute at their coincidence points; i.e., if $fFx = Ffx$ whenever $fx \in Fx$.

Recently, Abbas and Rhoades [1] generalized the concept of weak compatibility in the setting of single and multi-valued maps by introducing the notion of occasionally weak compatibility (owc). Maps f and F are said to be owc if and only if there exists some point x in $\mathcal{X}$ such that

$$fx \in Fx \text{ and } fFx \subseteq Ffx.$$

For our main results we need the following lemma which whose proof is obvious.

Lemma 1. *let A, B in $CB(\mathcal{X})$, then for any $a \in A$ we have*

$$d(a, B) \leq H(A, B).$$

In their paper [6], Türkoğlu et al. proved the next result.

Theorem 2. *Let $(\mathcal{X}, d)$ be a complete metric space. Let $f, g : \mathcal{X} \to \mathcal{X}$ be continuous maps and $S, T : \mathcal{X} \to CB(\mathcal{X})$ be H-continuous maps such that $T(\mathcal{X}) \subseteq f(\mathcal{X})$ and $S(\mathcal{X}) \subseteq g(\mathcal{X})$, the pair S and g are compatible maps and*

$$\begin{aligned} H^p(Sx, Ty) \leq\ & \max\{ad(fx, gy)d^{p-1}(fx, Sx), ad(fx, gy)d^{p-1}(gy, Ty), \\ & ad(fx, Sx)d^{p-1}(gy, Ty), cd^{p-1}(fx, Ty)d(gy, Sx)\} \end{aligned}$$

for all $x, y \in \mathcal{X}$, where $p \geq 2$ is an integer, $0 < a < 1$ and $c \geq 0$. Then there exists a point $z \in \mathcal{X}$ such that $fz \in Sz$ and $gz \in Tz$, i.e., z is a coincidence point of f, S and of g, T. Further, z is unique when $0 < c < 1$.

Our aim here is to establish and prove a unique common fixed point theorem by dropping the hypothesis of continuity required on the four maps in the above result, and deleting the two conditions $T(\mathcal{X}) \subseteq f(\mathcal{X})$ and $S(\mathcal{X}) \subseteq g(\mathcal{X})$ with $a \geq 0$ in a metric space instead of a complete metric space, by using the concept of occasionally weakly compatible maps given in [6].

2 Main results

Theorem 3. *Let $(\mathcal{X}, d)$ be a metric space. Let $f, g : \mathcal{X} \to \mathcal{X}$ and $F, G : \mathcal{X} \to CB(\mathcal{X})$ be single and multi-valued maps, respectively such that the pairs $\{f, F\}$ and $\{g, G\}$ are owc and satisfy inequality*

$$\textbf{(2.1)} \quad \begin{aligned} H^p(Fx, Gy) \leq \; & \max\{ad(fx, gy)d^{p-1}(fx, Fx), ad(fx, gy)d^{p-1}(gy, Gy), \\ & ad(fx, Fx)d^{p-1}(gy, Gy), cd^{p-1}(fx, Gy)d(gy, Fx)\} \end{aligned}$$

for all x, y in $\mathcal{X}$, where $p \geq 2$ is an integer, $a \geq 0$, $0 < c < 1$. Then f, g, F and G have a unique common fixed point in $\mathcal{X}$.

Proof. Since the pairs $\{f, F\}$ and $\{g, G\}$ are owc, then there exist two elements u and v in $\mathcal{X}$ such that $fu \in Fu$, $fFu \subseteq Ffu$ and $gv \in Gv$, $gGv \subseteq Ggv$.
First we prove that $fu = gv$. By Lemma 1 and the triangle inequality we have $d(fu, gv) \leq H(Fu, Gv)$. Suppose that $H(Fu, Gv) > 0$. Then, by inequality (2.1) we get

$$\begin{aligned} H^p(Fu, Gv) &\leq \max\{ad(fu, gv)d^{p-1}(fu, Fu), ad(fu, gv)d^{p-1}(gv, Gv), \\ &\quad ad(fu, Fu)d^{p-1}(gv, Gv), cd^{p-1}(fu, Gv)d(gv, Fu)\} \\ &= \max\{0, cd^{p-1}(fu, Gv)d(gv, Fu)\}. \end{aligned}$$

Since $d(fu, Gv) \leq H(Fu, Gv)$ and $d(gv, Fu) \leq H(Fu, Gv)$ by Lemma 1, and then

$$H^p(Fu, Gv) \leq cd^{p-1}(fu, Gv)d(gv, Fu) \leq cH^p(Fu, Gv) < H^p(Fu, Gv)$$

which is a contradiction. Hence $H(Fu, Gv) = 0$ which implies that $fu = gv$.
Again by Lemma 1 and the triangle inequality we have

$$d(f^2u, fu) = d(ffu, gv) \leq H(Ffu, Gv).$$

We claim that $f^2u = fu$. Suppose not. Then $H(Ffu, Gv) > 0$ and using inequality (2.1) we obtain

$$\begin{aligned} H^p(Ffu, Gv) &\leq \max\{ad(f^2u, gv)d^{p-1}(f^2u, Ffu), ad(f^2u, gv)d^{p-1}(gv, Gv), \\ &\quad ad(f^2u, Ffu)d^{p-1}(gv, Gv), cd^{p-1}(f^2u, Gv)d(gv, Ffu)\} \\ &= cd^{p-1}(f^2u, Gv)d(gv, Ffu). \end{aligned}$$

But $d(f^2u, Gv) \leq H(Ffu, Gv)$ and $d(gv, Ffu) \leq H(Ffu, Gv)$ by Lemma 1 and so

$$H^p(Ffu, Gv) \leq cH^p(Ffu, Gv) < H^p(Ffu, Gv),$$

a contradiction. This implies that $H(Ffu, Gv) = 0$, thus $f^2u = fu = gv$.
Similarly, we can prove that $g^2v = gv$.
Putting $fu = gv = z$, then, $fz = z = gz$, $z \in Fz$ and $z \in Gz$. Therefore z is a common fixed point of maps f, g, F and G.
Now, suppose that f, g, F and G have another common fixed point $z' \neq z$. Then, by Lemma 1 and the triangle inequality we have

$$d(z, z') = d(fz, gz') \leq H(Fz, Gz').$$

Assume that $H(Fz, Gz') > 0$. Then the use of inequality (2.1) gives

$$\begin{aligned} H^p(Fz, Gz') &\leq \max\{ad(fz, gz')d^{p-1}(fz, Fz), ad(fz, gz')d^{p-1}(gz', Gz'), \\ &\quad ad(fz, Fz)d^{p-1}(gz', Gz'), cd^{p-1}(fz, Gz')d(gz', Fz)\} \\ &= cd^{p-1}(fz, Gz')d(gz', Fz). \end{aligned}$$

Then since $d(fz, Gz') \leq H(Fz, Gz')$ and $d(gz', Fz) \leq H(Fz, Gz')$, we have

$$H^p(Fz, Gz') \leq cH^p(Fz, Gz') < H^p(Fz, Gz'),$$

a contradiction. Then $H(Fz, Gz') = 0$ and hence $z' = z$. □

If we put in Theorem 3 $f = g$ and $F = G$, we obtain the following result.

Corollary 4. *Let $(\mathcal{X}, d)$ be a metric space and let $f : \mathcal{X} \to \mathcal{X}$, $F : \mathcal{X} \to CB(\mathcal{X})$ be a single and a multi-valued map respectively. Suppose that f and F are owc and satisfy the inequality*

$$\begin{aligned} H^p(Fx, Fy) &\leq \max\{ad(fx, fy)d^{p-1}(fx, Fx), ad(fx, fy)d^{p-1}(fy, Fy), \\ &\quad ad(fx, Fx)d^{p-1}(fy, Fy), cd^{p-1}(fx, Fy)d(fy, Fx)\} \end{aligned}$$

for all x, y in $\mathcal{X}$, where $p \geq 2$ is an integer, $a \geq 0$ and $0 < c < 1$. Then, f and F have a unique common fixed point in $\mathcal{X}$.

Now, letting $f = g$ we get the next corollary.

Corollary 5. *Let $(\mathcal{X}, d)$ be a metric space, $f : \mathcal{X} \to \mathcal{X}$ be a single map and $F, G : \mathcal{X} \to CB(\mathcal{X})$ be two multi-valued maps such that*
(i) the pairs $\{f, F\}$ and $\{f, G\}$ are owc,
(ii) the inequality

$$\begin{aligned} H^p(Fx, Gy) &\leq \max\{ad(fx, fy)d^{p-1}(fx, Fx), ad(fx, fy)d^{p-1}(fy, Gy), \\ &\quad ad(fx, Fx)d^{p-1}(fy, Gy), cd^{p-1}(fx, Gy)d(fy, Fx)\} \end{aligned}$$

holds for all x, y in $\mathcal{X}$, where $p \geq 2$ is an integer, $a \geq 0$ and $0 < c < 1$. Then, f, F and G have a unique common fixed point in $\mathcal{X}$.

Now, we give an example which illustrate our main result.

Example 6. *Let* $\mathcal{X} = [0,2]$ *endowed with the Euclidean metric* d*. Define* $f, g : \mathcal{X} \to \mathcal{X}$ *and* $F, G : \mathcal{X} \to CB(\mathcal{X})$ *as follows:*

$$fx = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2, \end{cases} \quad Fx = \begin{cases} \{1\} & \text{if } 0 \le x \le 1 \\ \{0\} & \text{if } 1 < x \le 2, \end{cases}$$

$$gx = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 2 & \text{if } 1 < x \le 2, \end{cases} \quad Gx = \begin{cases} \{1\} & \text{if } 0 \le x \le 1 \\ \{\frac{x}{2}\} & \text{if } 1 < x \le 2. \end{cases}$$

First, we have

$$f(1) = 1 \in F(1) = \{1\} \text{ and } fF(1) = \{1\} = Ff(1)$$

and

$$g(1) = 1 \in G(1) = \{1\} \text{ and } gG(1) = \{1\} = Gg(1);$$

i.e., f *and* F *as well as* g *and* G *are owc.*
Also, for all x *and* y *in* $\mathcal{X}$*, inequality* (2.1) *is satisfied for* ***a large enough*** a*.*
So, all hypotheses of Theorem 3 are satisfied and 1 is the unique common fixed point of f, g, F *and* G*.*
On the other hand, it is clear to see that maps f, g, F *and* G *are discontinuous at* $t = 1$*.*
Further, we have

$$F(\mathcal{X}) = \{0,1\} \subset f(\mathcal{X}) = [0,1] \cup \{2\} \text{ but } G(\mathcal{X}) =]\frac{1}{2}, 1] \not\subseteq g(\mathcal{X}) = \{1,2\}.$$

So, this example illustrate the generality of our result.

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