Novi Sad J. Math.
Vol. 34, No. 2, 2004, 5-15
Proc. Novi Sad Algebraic Conf. 2003
(eds. I. Dolinka, A. Tepavčević)

NECKS OF AUTOMATA

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Abstract. Directable automata, known also as synchronizable, cofinal and reset automata, are a significant type of automata with very interesting algebraic properties and important applications in various branches of Computer Science. They have been a subject of interest of many eminent authors since 1964, when they were introduced by J. Černý in [4], whereas various specializations and generalizations of directable automata have appeared recently, in a paper by T. Petković, M. Ćirić and S. Bogdanović [7].

The purpose of this paper is to study directable, monogenically and generalized directable automata from another point of view, using the notions of a neck and a local neck that we introduce here.

AMS Mathematics Subject Classification (2000): 68Q70

Key words and phrases: directable automata, necks, local necks, monogenically directable automata, generalized directable automata

1. Introduction and preliminaries

Directable automata, known also as synchronizable, cofinal and reset automata, are a significant type of automata with very interesting algebraic properties and important applications in various branches of Computer Science (synchronization of binary messages, symbolic dynamics, verification of software, etc.). They have been a subject of interest of many eminent authors since 1964, when they were introduced by J. Černý in [4], although some of their special types were investigated even several years earlier.

Various specializations and generalizations of directable automata have appeared recently. T. Petković, M. Ćirić and S. Bogdanović in [7] introduced and studied trap-directable, trapped, monogenically, locally and generalized directable automata, as well as other related kinds of automata. These automata

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have been also studied by Ž. Popović, S. Bogdanović, T. Petković and M. Ćirić in [8] and [9]. We also refer to the survey paper by S. Bogdanović, B. Imreh, M. Ćirić and T. Petković [2], devoted to directable automata, their generalizations and specializations.

The purpose of this paper is to study directable, monogenically and generalized directable automata from another point of view, using the notions of a neck and a local neck that we introduce here. We describe basic properties of necks and local necks of automata, introduce some new types of automata, such as strongly directable and monogenically strongly directable automata, and give new structural characterizations of directable, monogenically and generalized directable automata.

Automata considered throughout this paper will be automata without outputs in the sense of the definition from the book [5] by F. Gécseg and I. Peák. It is well known that automata without outputs, with the input alphabet X, can be considered as unary algebras of a type indexed by X, so the notions such as a *congruence*, *homomorphism*, *generating set* etc., as well as *subautomaton*, will have their usual algebraic meanings (see, for example, [3]). The state set and the input set of an automaton are not necessarily finite. In order to simplify notation, an automaton with the state set A is also denoted by the same letter A. For any considered automaton A, its input alphabet is denoted by X, and the free monoid over X, the input monoid of A, is denoted by X^*. Under the action of an input word $u \in X^*$, the automaton A goes from a state a into the state denoted by au.

A state $a \in A$ is *reversible* if for every word $v \in X^*$ there exists a word $u \in X^*$ such that $avu = a$, and the set of all reversible states of A, called the *reversible part* of A, is denoted by $R(A)$. If it is non-empty, $R(A)$ is a subautomaton of A. An automaton A is *reversible* if all of its states are reversible. If for every $a, b \in A$ there exists $u \in X^*$ such that $b = au$, then the automaton A is *strongly connected*. Equivalently, A is strongly connected if it has no proper subautomata. A state $a \in A$ is called a *trap* of A if $au = a$ for every word $u \in X^*$.

For any non-empty $H \subseteq A$, the subautomaton of A generated by H, i.e., the least subautomaton A containing H, is denoted by $\langle H\rangle$. In the case $H = \{a\}$, $\langle H\rangle$ is called a *monogenic subautomaton* of A generated by a, and we write just $\langle a\rangle$ instead of $\langle\{a\}\rangle$. If $H = \{a_1, \dots, a_n\}$ is a finite set, then $\langle H\rangle$ is a *finitely generated subautomaton* of A, and we write $\langle a_1, \dots, a_n\rangle$ instead of $\langle\{a_1, \dots, a_n\}\rangle$. It is obvious that $\langle H\rangle = \{aw \mid a \in H, w \in X^*\}$, $\langle a\rangle = \{au \mid u \in X^*\}$ and $\langle a_1, \dots, a_n\rangle = \langle a_1\rangle \cup \dots \cup \langle a_n\rangle$. The least non-empty subautomaton of an automaton A, if it exists, is called the *kernel* of A, and in this case, it is the unique strongly connected subautomaton of A.

Let B be a subautomaton of an automaton A. The *Rees congruence* on A determined by B is a congruence relation ϱ_B on A defined by: For $a, b \in A$ we say that $(a, b) \in \varrho_B$ if and only if either $a = b$ or $a, b \in B$ holds. The factor automaton A/ϱ_B is usually denoted by A/B, and it is called a *Rees* *factor automaton* of A with respect to B. We say that an automaton A is an *extension* of an automaton B by an automaton C (with a trap) if B is a subautomaton of A and the Rees factor automaton A/B is isomorphic to C. Clearly, the automaton C can be viewed as an automaton obtained from A by contraction of B into a single state. An automaton A is a *direct sum* of its subautomata A_α, $\alpha \in Y$, if $A = \bigcup_{\alpha\in Y} A_\alpha$ and $A_\alpha \cap A_\beta = \varnothing$, for every $\alpha, \beta \in Y$ such that $\alpha \neq \beta$.

2. Directable automata and their necks

In this section we describe some elementary properties of directable automata and their necks. We start with definitions.

Let A be an automaton. A state $a \in A$ is called a *neck* of A if there exists $u \in X^*$ such that $bu = a$, for every $b \in A$. In that case a is also said to be a *u-neck* of A, the word u is called a *directing word* of A, and A is called a *directable automaton*. The set of all necks of A is denoted by $N(A)$, and the set of all directing words of A is denoted by $DW(A)$. If A has a single neck, then it is a trap of A and A is called a *trap-directable* automaton.

The next three lemmas describe basic properties of the sets of necks of directable automata.

Lemma 1. *Let A be an automaton. If $N(A) \neq \varnothing$, then $N(A)$ is a subautomaton of A.*

Proof. Let $a \in N(A)$ and $v \in X^*$. Assume that a is a u-neck of A, for some $u \in X^*$. Then for every $b \in A$ we have that $buv = av$, which means that av is a uv-neck of A, and hence, $av \in N(A)$. Therefore, $N(A)$ is a subautomaton of A. □

Lemma 2. *Let A be a directable automaton. Then $N(A)$ is the kernel of A and $N(A) = R(A)$.*

Proof. Let $a \in N(A)$ and $b \in A$. Then $a = bu$, for every $u \in DW(A)$, and hence $a \in \langle b\rangle$. Therefore, $N(A) \subseteq \langle b\rangle$, for every $b \in A$. This means that $N(A)$ is a subautomaton contained in every other subautomaton of A, that is, $N(A)$ is the kernel of A.

On the other hand, we have that $R(A)$ is the union of all minimal subautomata of A, and since A has a unique minimal subautomaton $N(A)$, we conclude that $R(A) = N(A)$. □

Lemma 3. *Let B be a subautomaton of a directable automaton A. Then B is also directable and $N(B) = N(A)$.*

Proof. By Theorem 9 of [7], the class of directable automata is closed under subautomata. Further, by Lemma 2 we have that $N(A)$ is the kernel of A, so $N(A) \subseteq N(B) \subseteq B$. On the other hand, $N(B)$ is the kernel of B, so $N(A) \subseteq B$ implies $N(B) \subseteq N(A)$. Thus, $N(B) = N(A)$. □

We define an automaton A to be *strongly directable* if $A = N(A)$. These automata can be characterized as follows:

Theorem 1. *An automaton A is strongly directable if and only if it is strongly connected and directable.*

Proof. Let A be a strongly directable automaton. It is clear that it is directable. On the other hand, let $a, b \in A$. Since $b \in N(A)$, it follows that there exists $u \in X^*$ such that $au = b$. Therefore, A is strongly connected.

Conversely, let A be strongly connected and directable. Then $N(A) \neq \varnothing$ and by Lemma 1, $N(A)$ is a subautomaton of A. But since A is strongly connected, it follows that $A = N(A)$. Thus, A is strongly directable. □

Now we can characterize directable automata in terms of strongly directable and trap-directable automata.

Theorem 2. *An automaton A is directable if and only if it is an extension of a strongly directable automaton B by a trap-directable automaton C.*

In that case we have:

(a) $N(A) = B$;

(b) $DW(C) \cdot DW(B) \subseteq DW(A) \subseteq DW(C) \cap DW(B)$.

Proof. Let A be a directable automaton. Then $N(A)$ is non-empty, and by Lemma 1, it is a subautomaton of A. The Rees factor automaton $A/N(A)$ is also directable, since it is a homomorphic image of a directable automaton [7]. On the other hand, every Rees factor automaton has a trap. Therefore, $A/N(A)$ is a trap-directable automaton, and hence, A is an extension of a strongly directable automaton $N(A)$ by a trap-directable automaton $A/N(A)$.

Conversely, let A be an extension of a strongly directable automaton B by a trap-directable automaton C. Let $u \in DW(C)$ and $v \in DW(B)$. Then for all $a, b \in A$ we have that $au, bu \in B$, whence $auv = buv$. Thus, $uv \in DW(A)$, and hence, A is a directable automaton. We have also proved that $DW(C) \cdot DW(B) \subseteq DW(A)$. It is clear that $DW(A) \subseteq DW(C) \cap DW(B)$. Thus, (b) holds.

By Lemma 2, $N(A)$ is the kernel of A, so $N(A) \subseteq B$. Conversely, assume that $b \in B$. Bearing in mind that B is strongly directable, we conclude that there exists $v \in DW(B)$ such that $cv = b$ holds for each $c \in B$. Hence, for every $a \in A$ and $u \in DW(C)$, $au \in B$ and $auv = (au)v = b$. Therefore, $b \in N(A)$, and hence, we have proved that $N(A) = B$. □

3. Local necks of automata

We define a state a of an automaton A to be a *local neck* of A if it is a neck of some directable subautomaton of A. The set of all local necks of A is denoted by $LN(A)$.

The next three lemmas describe basic properties of local necks.

Lemma 4. *Let a be a state of an automaton A. Then the following conditions are equivalent:*

(i) *a is a local neck;*

(ii) *$\langle a \rangle$ is a strongly directable automaton;*

(iii) *there exists $u \in X^*$ such that for every $v \in X^*$, $avu = a$.*

Proof. (i)⇒(ii). Let a be a local neck of A. Then there exists a directable subautomaton B of A such that $a \in N(B)$, and by the proof of Theorem 2, $N(B)$ is a strongly directable automaton. We also have that $\langle a \rangle \subseteq N(B)$, and since $N(B)$ is strongly connected, then $\langle a \rangle = N(B)$, so we have that $\langle a \rangle$ is a strongly directable automaton.

(ii)⇒(iii). Let $\langle a \rangle$ be a strongly directable automaton. Then a is a u-neck of $\langle a \rangle$ for some $u \in X^*$, and for every $v \in X^*$ we have $av \in \langle a \rangle$, implying $avu = a$.

(iii)⇒(i). The condition (iii) clearly means that a is a u-neck of $\langle a \rangle$, and hence, it is a local neck of A. □

Lemma 5. *Let A be an automaton. If $LN(A) \neq \varnothing$, then $LN(A)$ is a subautomaton of A.*

Proof. Let $a \in LN(A)$ and $x \in X$. By Lemma 4, the monogenic subautomaton $\langle a \rangle$ of A is strongly directable. Moreover, $\langle ax \rangle \subseteq \langle a \rangle$, and since $\langle a \rangle$ is strongly connected, we have $\langle ax \rangle = \langle a \rangle$, so ax is also a local neck of A, i.e., $ax \in LN(A)$. Hence, $LN(A)$ is a subautomaton of A. □

Lemma 6. *Let B be a subautomaton of an automaton A. Then*

$$LN(B) = LN(A) \cap B.$$

Proof. Let $a \in LN(B)$. Then there exists a subautomaton B' of B such that $a \in N(B')$. But B' is also a subautomaton of A, so $a \in LN(A)$. Therefore, $LN(B) \subseteq LN(A) \cap B$.

Conversely, let $a \in LN(A) \cap B$. Then $a \in N(A')$, for some directable subautomaton A' of A. Let $B' = B \cap A'$. Then B' is a subautomaton of B, and by Lemma 3 we have that B' is directable and $N(B') = N(A')$. By this it follows that $a \in N(B')$, which means that $a \in LN(B)$. Thus, we have proved that $LN(A) \cap B \subseteq LN(B)$. □

Following the terminology from [1] (which is slightly different from the one used in [7], [2] and [9]), an automaton A is called *monogenically directable* if every monogenic subautomaton of A is directable, and similarly, it is called *monogenically strongly directable* if every monogenic subautomaton of A is strongly directable. On the other hand, A is defined to be *locally directable* if every finitely generated subautomaton of A is directable.

Theorem 3. *The following conditions on an automaton A are equivalent:*

(i) *Every state of* A *is a local neck;*

(ii) A *is monogenically strongly directable;*

(iii) A *is monogenically directable and reversible;*

(iv) A *is a direct sum of strongly directable automata;*

(v) $(\forall a \in A)(\exists u \in X^*)(\forall v \in X^*)\ avu = a$.

Proof. (i)⇒(ii). If every state $a \in A$ is a local neck of A, then by Lemma 4 we have that for every $a \in A$ the monogenic subautomaton $\langle a \rangle$ of A is strongly directable. Hence, A is monogenically strongly directable.

(ii)⇒(iii). If A is monogenically strongly directable, then it is clear that it is monogenically directable. On the other hand, every monogenic subautomaton of A is strongly connected, whence it follows that A is reversible.

(iii)⇒(iv). As was proved by Thierrin in [10], if A is reversible then it is a direct sum of strongly connected automata A_α, $\alpha \in Y$. Let $\alpha \in Y$ and $a \in A_\alpha$. Then $\langle a \rangle = A_\alpha$, since A_α is strongly connected, and by the monogenic directability of A we have that $A_\alpha = \langle a \rangle$ is directable. Therefore, A_α is strongly directable, for any $\alpha \in Y$.

(iv)⇒(i). Let A be a direct sum of strongly directable automata A_α, $\alpha \in Y$. Then for each state $a \in A$ there exists $\alpha \in Y$ such that $a \in A_\alpha$, that is $a \in A_\alpha = N(A_\alpha)$, so a is a local neck of A.

(i)⇔(v). This is an immediate consequence of Lemma 4. □

Let us observe that the word u appearing in (v) of Theorem 3, as well as in (iii) of Lemma 4, is a directing word of the monogenic subautomaton $\langle a \rangle$, and it depends on a. In the general case, different monogenic subautomata of A do not necessarily have a common directing word. Now we will consider automata whose all monogenic subautomata have common directing words.

Let A be an arbitrary automaton. We define a word $u \in X^*$ to be a *local directing word* of A if u is a directing word of every monogenic subautomaton of A, i.e., if $u \in DW(\langle a \rangle)$, for every $a \in A$. The set of all local directing words of A will be denoted by $LDW(A)$. In other words,

$$LDW(A) = \bigcap_{a \in A} DW(\langle a \rangle). \tag{1}$$

If the set $LDW(A)$ is non-empty, then A is said to be a *uniformly monogenically directable* automaton. Thus, A is a uniformly monogenically directable automaton if all monogenic subautomata of A are directable and have at least one common directing word. Similarly, A will be called a *uniformly monogenically strongly directable* automaton if all monogenic subautomata of A are strongly directable and have at least one common directing word.

Uniformly monogenically directable automata will be considered in the next section. Here we give a characterization of uniformly monogenically strongly directable automata.

Theorem 4. *The following conditions on an automaton A are equivalent:*

(i) *A is uniformly monogenically strongly directable;*

(ii) $(\exists u \in X^*)(\forall a \in A)(\forall v \in X^*)\ avu = a$.

Proof. This is an immediate consequence of Theorem 3. □

Finally, note that in the case of finite automata there are no differences between monogenically and uniformly monogenically strongly directable automata, as well as between monogenically and uniformly monogenically directable automata. This is an immediate consequence of (1) and the fact that the intersection of every finite family of ideals of a semigroup is non-empty, which does not hold for infinite families.

4. Monogenically and generalized directable automata

Let us introduce a relation η on an arbitrary automaton A as follows:

$$a\,\eta\,b \;\Leftrightarrow\; N(\langle a\rangle) = N(\langle b\rangle).$$

This relation is clearly an equivalence relation. We prove the following result, which describes its classes.

Theorem 5. *Let B be an arbitrary η-class of an automaton A. Then one of the following conditions hold:*

(a) $B = \{a \in A \mid N(\langle a\rangle) = \varnothing\}$;

(b) *B is a locally directable subautomaton of A.*

Proof. Suppose that (a) does not hold. Then there exists a strongly directable subautomaton N of A such that $N(\langle a\rangle) = N$, for every $a \in B$.

Consider an arbitrary $a \in B$. Then $N(\langle a\rangle) \neq \varnothing$, which means that $\langle a\rangle$ is a directable automaton. Moreover, for every $x \in X$, by Lemma 3 it follows that $\langle ax\rangle$ is directable and $N(\langle ax\rangle) = N(\langle a\rangle)$, that is $ax \in B$. Therefore, B is a subautomaton of A. It remains to prove that B is a locally directable automaton.

Let $a_1, a_2, \ldots, a_n \in B$ and $d \in N$. For every $i \in [1, n]$, $N = N(\langle a_i\rangle)$, so there exists $w_i \in DW(\langle a_i\rangle)$ such that $aw_i = d$, for every $a \in \langle a_i\rangle$. Now set $w = w_1w_2\cdots w_n$ and consider arbitrary $i \in [1, n]$ and $a \in \langle a_i\rangle$. Since $DW(\langle a_i\rangle)$ is an ideal of X^*, then $w \in DW(\langle a_i\rangle)$, so

$$\begin{aligned} aw &= (aw_1\cdots w_{i-1})w_iw_{i+1}\cdots w_n \\ &= dw_{i+1}\cdots w_n && \text{since } aw_1\cdots w_{i-1} \in \langle a_i\rangle \text{ and } w_i \in DW(\langle a_i\rangle) \\ &= d && \text{since } d \in N = N(\langle a_{i+1}\rangle) = \cdots = N(\langle a_n\rangle). \end{aligned}$$

Since $\langle a_1, a_2, \ldots, a_n\rangle = \bigcup_{i=1}^{n}\langle a_i\rangle$, we conclude that $w \in DW(\langle a_1, \ldots, a_n\rangle)$. Thus, $\langle a_1, a_2, \ldots, a_n\rangle$ is a directable automaton with $N(\langle a_1, a_2, \ldots, a_n\rangle) = N$, so we have proved that B is a locally directable automaton. □

Let A and B be automata. By a *relation* ξ from A into B, denoted by $\xi : A \to B$, we mean a mapping from A into the power automaton $\mathcal{P}(B)$. The *graph* of the relation ξ, in notation graph(ξ), is a subset of $A \times B$ defined by graph$(\xi) = \{(a, b) \mid b \in a\xi\}$. We say that ξ is a relation from A *onto* B, or that it is a *surjective relation*, if for every $b \in B$ there exists $a \in A$ such that $b \in a\xi$. A *relational morphism* from A into B is defined to be a relation $\xi : A \to B$ having the following properties:

(i) $a\xi \neq \varnothing$, for every $a \in A$;

(ii) $(a\xi)x \subseteq (ax)\xi$, for every $a \in A$.

Now we are ready to state and prove a theorem which characterizes monogenicaly directable automata in a different way than it was done in [7].

Theorem 6. *The following conditions on an automaton A are equivalent:*

(i) *A is a monogenically directable automaton;*

(ii) *A is a direct sum of locally directable automata;*

(iii) *the mapping $\xi : a \mapsto N(\langle a\rangle)$ is a relational morphism of A onto $LN(A)$.*

Proof. (i)⇒(ii). By the monogenic directability of A it follows that $N(\langle a\rangle) \neq \varnothing$, for every $a \in A$, and by Theorem 5 we have that η is a direct sum congruence on A and every η-class is a locally directable automaton.

(ii)⇒(i). This implication is obvious.

(i)⇒(iii). For an arbitrary $a \in A$ we have that $\langle a\rangle$ is a directable automaton, so $a\xi \neq \varnothing$. Moreover, if $b \in LN(A)$ then $b \in b\xi$, so ξ is a relation from A onto $LN(A)$.

Let $a \in A$ and $x \in X$. Then $\langle a\rangle$ is directable, and by Lemma 3, $\langle ax\rangle$ is also directable and $N(\langle ax\rangle) = N(\langle a\rangle)$, so $(ax)\xi = a\xi$. Furthermore,

$$(a\xi)x = N(\langle a\rangle)x \subseteq N(\langle a\rangle) = N(\langle ax\rangle) = (ax)\xi,$$

so we have proved that ξ is a relational morphism of A onto B.

(iii)⇒(i). By the fact that $a\xi \neq \varnothing$, for every $a \in A$, it follows that any monogenic subautomaton of A is directable, i.e., that A is monogenically directable. □

In the general case, we have that at most one η-class consists of all states satisfying $N(\langle a\rangle) = \varnothing$ (if such states exist at all), while all remaining classes (if they exist) are subautomata of A and their union is the largest monogenically directable subautomaton of A. Therefore, every automaton having at least one local neck has a monogenically directable part. An interesting example of automata having both of two types of η-classes considered in Theorem 5 are generalized directable automata, which will be discussed at the end of this section.

In the previous section we defined uniformly monogenically directable automata as automata whose all monogenic subautomata are directable and have the same directing word. These automata have been first studied and characterized by T. Petković, M. Ćirić and S. Bogdanović in [7]. Here we characterize them in another way.

Theorem 7. *The following conditions on an automaton A are equivalent:*

(i) *A is uniformly monogenically directable;*

(ii) *A is a direct sum of directable automata with a common directing word;*

(iii) *A is an extension of a uniformly monogenically strongly directable automaton B by a trap-directable automaton C and there exists a relational morphism ξ of A onto B such that*

$$aw = bw,$$

for every $a \in A$, $b \in a\xi$ and $w \in DW(C) \cdot LDW(B)$.

Proof. (i)⇔(ii). This was proved in [7].

(i)⇒(iii). Let A be a uniformly monogenically directable automaton, i.e., assume that there exists a word $u \in X^*$ such that for every state $a \in A$, the monogenic subautomaton $\langle a\rangle$ of A is directable with $u \in DW(\langle a\rangle)$. Set $B = LN(A)$. Then $B = LN(A) = \bigcup\{N(\langle a\rangle) \,|\, a \in A\}$, and by Lemma 1, B is a subautomaton of A. On the other hand, by Lemma 4 and Theorem 3 we have that B is monogenically strongly directable. Finally, it is clear that u is a common directing word of all monogenic subautomata of B. Therefore, B is a uniformly monogenically strongly directable automaton.

Next, consider an arbitrary state $a \in A$. Then for every $b \in N(\langle a\rangle)$ we have that $au = bu$, since $u \in DW(\langle a\rangle)$, and $b \in B$ implies $au = bu \in B$. Therefore, $C = A/B$ is a trap-directable automaton with $u \in DW(C)$.

Consider again the relational morphism ξ defined by $a\xi = N(\langle a\rangle)$, for every $a \in A$. Let $w \in DW(C) \cdot LDW(B)$, let $a \in A$ and $b \in a\xi = N(\langle a\rangle)$. Then $w = uv$, for some $u \in DW(C)$ and $v \in LDW(B)$, so it follows that $au \in B \cap \langle a\rangle = N(\langle a\rangle)$ and $bu \in N(\langle a\rangle)$. Therefore, $auv = buv$, since $v \in LDW(B)$, i.e., $aw = bw$, which was to be proved.

(iii)⇒(ii). Suppose that (iii) holds. By Theorem 6, B is a direct sum of locally directable automata B_α, $\alpha \in Y$. First we prove that for every $a \in A$ there exists a unique $\alpha \in Y$ such that $a\xi \subseteq B_\alpha$. Let $b, b' \in a\xi$, $b \neq b'$, and let $b \in B_\alpha$ and $b' \in B_{\alpha'}$, for some $\alpha, \alpha' \in Y$. For an arbitrary $w \in DW(C) \cdot LDW(B)$, by the hypothesis we have that $bw = aw = b'w$. But $bw \in B_\alpha$ and $b'w \in B_{\alpha'}$, whence it follows that $\alpha = \alpha'$. Thus, $a\xi \subseteq B_\alpha$.

For $\alpha \in Y$, set $A_\alpha = \{a \in A \,|\, a\xi \subseteq B_\alpha\}$. Evidently, $A_\alpha \cap A_\beta = \varnothing$, whenever $\alpha \neq \beta$, and $A = \bigcup\{A_\alpha \,|\, \alpha \in Y\}$. Let $\alpha \in Y$. First we prove that A_α is a subautomaton of A. Indeed, let $a \in A_\alpha$ and $x \in X$. Suppose that $ax \in A_\beta$, for some $\beta \in Y$, i.e., $(ax)\xi \subseteq B_\beta$. Then $(a\xi)x \subseteq (ax)\xi \subseteq B_\beta$ and $(a\xi)x \subseteq B_\alpha x \subseteq B_\alpha$, so we conclude that $\alpha = \beta$. Thus, $ax \in A_\alpha$, so we have proved that A_α is a subautomaton of A.

Next we prove that for every $w \in DW(C) \cdot LDW(B)$, A_α is a directable automaton with $w \in DW(A_\alpha)$. Let $a, a' \in A_\alpha$, $b \in a\xi$, $b' \in a'\xi$, and let $w = uv$, where $u \in DW(C)$ and $v \in LDW(B)$. By the hypothesis we have that $aw = bw$ and $a'w = b'w$. On the other hand, $b, b' \in B_\alpha$, so $bu, b'u \in B_\alpha$, and since $v \in DW(B_\alpha)$, then $bw = (bu)v = (b'u)v = b'w$. Hence, $aw = a'w$, which was to be proved. □

An automaton A is said to be *generalized directable* if there exists a word $u \in X^*$ such that $auvu = au$, for every state $a \in A$ and every word $v \in X^*$. In this case u is called a *generalized directing word* of A, and the set of all generalized directing words of A is denoted by $GDW(A)$.

Generalized directable automata were introduced by T. Petković, M. Ćirić and S. Bogdanović in [7], where it was proved that an automaton is generalized directable if and only if it is an extension of a uniformly monogenically directable automaton by a trap-directable automaton.

A more precise structural characterization of these automata is given by the following theorem.

Theorem 8. *An automaton A is generalized directable if and only if it is an extension of a uniformly monogenically strongly directable automaton B by a trap-directable automaton C.*

In that case we have

(a) $LN(A) = B$;

(b) $DW(C) \cdot LDW(B) \subseteq GDW(A) \subseteq DW(C) \cap LDW(B)$.

Proof. Let A be a generalized directable automaton. Set

$$B = \{au \mid a \in A,\, u \in GDW(A)\}.$$

We have that B is a subautomaton of A, since $GDW(A)$ is an ideal of X^*. We shall prove that B is a uniformly monogenically strongly directable automaton.

Let $b \in B$, i.e., assume that $b = au$, for some $a \in A$ and $u \in GDW(A)$. Then for every $v \in X^*$ we have that $bvu = auvu = au = b$, and by Lemma 4, $\langle b\rangle$ is a strongly directable automaton and $u \in DW(\langle b\rangle)$. Therefore, B is a uniformly monogenically strongly directable automaton and $GDW(A) \subseteq LDW(B)$. On the other hand, for each $a \in A$ we have that $au \in B$, for every $u \in GDW(A)$, which implies that $C = A/B$ is a trap-directable automaton with $GDW(A) \subseteq DW(C)$. This fact, taken together with $GDW(A) \subseteq LDW(B)$, proves (b).

Conversely, let A be an extension of a uniformly monogenically strongly directable automaton B by a trap-directable automaton C. Let $u \in DW(C)$ and $v \in LDW(B)$. For every $a \in A$ we have $au \in B$, so $\langle au\rangle$ is a strongly directable automaton with $v \in DW(\langle au \rangle)$. Now, for an arbitrary $w \in X^*$, $au, auvwu \in \langle au \rangle$ and $v \in DW(\langle au \rangle)$ yield $auv = auvwuv$. Therefore, we have proved that A is a generalized directable automaton with $uv \in GDW(A)$, and hence, $DW(C) \cdot LDW(B) \subseteq GDW(A)$.

Further, $B = LN(B) \subseteq LN(A)$, by Lemma 6. Conversely, if $a \in LN(A)$, then by Lemma 4 it follows that there exists $u \in X^*$ such that $avu = a$, for every $v \in X^*$, and if we assume that $v \in DW(C)$, then $av \in B$, so $a = avu \in B$. Therefore, $LN(A) \subseteq B$, and hence, we have proved (a). □

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