

AN ELEMENTARY PROOF OF THE PRESERVATION OF LIPSCHITZ CONSTANTS BY THE MEYER-KÖNIG AND ZELLER OPERATORS

TIBERIU TRIF

UNIVERSITATEA BABEȘ-BOLYAI,

FACULTATEA DE MATEMATICĂ ȘI INFORMATICĂ,

STR. M. KOGĂLNICEANU, 1,

3400 CLUJ-NAPOCA, ROMANIA.

EMail: trif@math.ubbcluj.ro



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Given the real numbers $A \geq 0$ and $0 < \alpha \leq 1$, we denote by $\text{Lip}_A \alpha$ the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$, satisfying

$$|f(x_2) - f(x_1)| \leq A|x_2 - x_1|^\alpha \quad \text{for all } x_1, x_2 \in [0, 1].$$

The main purpose of this note is to present an elementary proof of the following result:

Given the continuous function $f : [0, 1] \rightarrow \mathbb{R}$, it holds that

$$(1) \quad f \in \text{Lip}_A \alpha$$

if and only if

$$(2) \quad M_n f \in \text{Lip}_A \alpha \quad \text{for all } n \geq 1,$$

where $(M_n)_{n \geq 1}$ is the sequence of Meyer-König and Zeller operators.

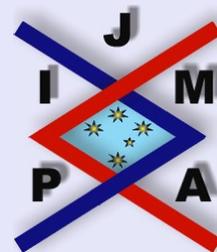
It should be mentioned that similar proofs for other operators are to be found in [2] and [3]. On the other hand, the equivalence (1) \Leftrightarrow (2) is a special case of a much more general result [1, Theorem 1]. However, the proof presented in [1] is completely different and does not have an elementary character.

Proof. Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function and let n be a positive integer. Recall that the n th Meyer-König and Zeller power series associated to f is defined by (see [4])

$$M_n f(1) = f(1),$$

$$M_n f(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \quad x \in [0, 1],$$

$$m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}, \quad k = 0, 1, 2, \dots$$



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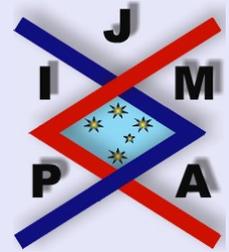
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That (2) implies (1) follows from the fact that the sequence $(M_n f)_{n \geq 1}$ converges uniformly to f on $[0, 1]$. Thus it remains to prove that (1) implies (2). To this end, let n be an arbitrary positive integer and let $0 \leq x_1 < x_2 < 1$ (since $M_n f$ is continuous at 1, it suffices to consider only the case $x_2 < 1$). Then we have

$$\begin{aligned}
 & M_n f(x_2) \\
 &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right) \binom{n+j}{j} x_2^j (1-x_2)^{n+1} \\
 &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right) \binom{n+j}{j} (1-x_2)^{n+1} \left(\frac{x_2 - x_1 + x_1 - x_1 x_2}{1-x_1}\right)^j \\
 &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right) \binom{n+j}{j} \frac{(1-x_2)^{n+1}}{(1-x_1)^j} \sum_{k=0}^j \binom{j}{k} x_1^k (1-x_2)^k (x_2 - x_1)^{j-k} \\
 &= \sum_{j=0}^{\infty} \sum_{k=0}^j f\left(\frac{j}{n+j}\right) \frac{(n+j)!}{n!k!(j-k)!} \cdot \frac{x_1^k (x_2 - x_1)^{j-k} (1-x_2)^{n+k+1}}{(1-x_1)^j} \\
 &= \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} f\left(\frac{j}{n+j}\right) \frac{(n+j)!}{n!k!(j-k)!} \cdot \frac{x_1^k (x_2 - x_1)^{j-k} (1-x_2)^{n+k+1}}{(1-x_1)^j} \\
 &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} f\left(\frac{k+\ell}{n+k+\ell}\right) \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2 - x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}},
 \end{aligned}$$

where the change of index $j - k = \ell$ was used for the last equality. We have



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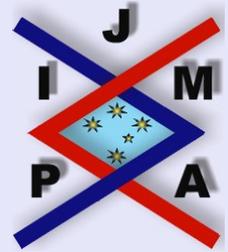
$$\begin{aligned}
 & M_n f(x_1) \\
 &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x_1^k (1-x_1)^{n+1} \\
 &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x_1^k \cdot \frac{(1-x_2)^{n+k+1}}{(1-x_1)^k} \cdot \frac{1}{\left(1 - \frac{x_2-x_1}{1-x_1}\right)^{n+k+1}} \\
 &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} \frac{x_1^k (1-x_2)^{n+k+1}}{(1-x_1)^k} \sum_{\ell=0}^{\infty} \binom{n+k+\ell}{\ell} \left(\frac{x_2-x_1}{1-x_1}\right)^\ell \\
 &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} f\left(\frac{k}{n+k}\right) \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}}.
 \end{aligned}$$

In particular, the above equalities show that

$$(3) \quad \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = 1,$$

$$(4) \quad \sum_{k,\ell=0}^{\infty} \frac{k+\ell}{n+k+\ell} \cdot \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_2,$$

$$(5) \quad \sum_{k,\ell=0}^{\infty} \frac{k}{n+k} \cdot \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k (x_2-x_1)^\ell (1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_1.$$



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Since $f \in \text{Lip}_A \alpha$, we have

$$\begin{aligned} & |M_n f(x_2) - M_n f(x_1)| \\ & \leq \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left| f\left(\frac{k+\ell}{n+k+\ell}\right) - f\left(\frac{k}{n+k}\right) \right| \\ & \leq A \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k} \right)^\alpha. \end{aligned}$$

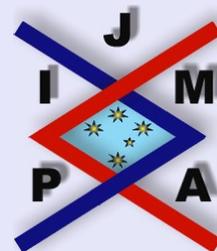
Taking into account (3) and the fact that the function $t \in [0, \infty[\mapsto t^\alpha \in [0, \infty[$ is concave, we deduce that

$$\begin{aligned} & |M_n f(x_2) - M_n f(x_1)| \\ & \leq A \left[\sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k} \right) \right]^\alpha. \end{aligned}$$

Using now (4) and (5) we get

$$|M_n f(x_2) - M_n f(x_1)| \leq A(x_2 - x_1)^\alpha,$$

i.e., $M_n f \in \text{Lip}_A \alpha$. This completes the proof. \square



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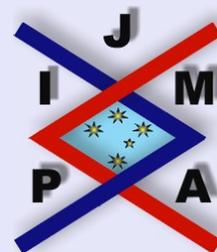
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