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# AN ELEMENTARY PROOF OF THE PRESERVATION OF LIPSCHITZ CONSTANTS BY THE MEYER-KÖNIG AND ZELLER OPERATORS

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ABSTRACT. An elementary proof of the preservation of Lipschitz constants by the Meyer-König and Zeller operators is presented.

*Key words and phrases:* Meyer-König and Zeller operators, Preservation of Lipschitz constants.

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Given the real numbers $A \geq 0$ and $0 < \alpha \leq 1$, we denote by $\mathrm{Lip}_A\alpha$ the set of all functions $f : [0,1] \to \mathbb{R}$, satisfying

$$|f(x_2) - f(x_1)| \leq A|x_2 - x_1|^\alpha \qquad \text{for all} \quad x_1, x_2 \in [0,1].$$

The main purpose of this note is to present an elementary proof of the following result:
*Given the continuous function* $f : [0,1] \to \mathbb{R}$*, it holds that*

$$f \in \mathrm{Lip}_A\alpha \tag{1}$$

*if and only if*

$$M_n f \in \mathrm{Lip}_A\alpha \qquad \text{for all} \quad n \geq 1, \tag{2}$$

*where* $(M_n)_{n\geq 1}$ *is the sequence of Meyer-König and Zeller operators.*

It should be mentioned that similar proofs for other operators are to be found in [2] and [3]. On the other hand, the equivalence (1) ⇔ (2) is a special case of a much more general result [1, Theorem 1]. However, the proof presented in [1] is completely different and does not have an elementary character.

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*Proof.* Let $f : [0,1] \to \mathbb{R}$ be a continuous function and let $n$ be a positive integer. Recall that the $n$th Meyer-König and Zeller power series associated to $f$ is defined by (see [4])

$$\begin{aligned} M_n f(1) &= f(1), \\ M_n f(x) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \qquad x \in [0,1[, \\ m_{n,k}(x) &= \binom{n+k}{k} x^k (1-x)^{n+1}, \qquad k = 0,1,2,\dots. \end{aligned}$$

That (2) implies (1) follows from the fact that the sequence $(M_n f)_{n\geq 1}$ converges uniformly to $f$ on $[0,1]$. Thus it remains to prove that (1) implies (2). To this end, let $n$ be an arbitrary positive integer and let $0 \leq x_1 < x_2 < 1$ (since $M_n f$ is continuous at $1$, it suffices to consider only the case $x_2 < 1$). Then we have

$$\begin{aligned} M_n f(x_2) &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j} x_2^j (1-x_2)^{n+1} \\ &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j}(1-x_2)^{n+1}\left(\frac{x_2 - x_1 + x_1 - x_1 x_2}{1-x_1}\right)^j \\ &= \sum_{j=0}^{\infty} f\left(\frac{j}{n+j}\right)\binom{n+j}{j}\frac{(1-x_2)^{n+1}}{(1-x_1)^j}\sum_{k=0}^{j}\binom{j}{k} x_1^k (1-x_2)^k (x_2-x_1)^{j-k} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} f\left(\frac{j}{n+j}\right)\frac{(n+j)!}{n!k!(j-k)!}\cdot\frac{x_1^k (x_2-x_1)^{j-k}(1-x_2)^{n+k+1}}{(1-x_1)^j} \\ &= \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} f\left(\frac{j}{n+j}\right)\frac{(n+j)!}{n!k!(j-k)!}\cdot\frac{x_1^k (x_2-x_1)^{j-k}(1-x_2)^{n+k+1}}{(1-x_1)^j} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} f\left(\frac{k+\ell}{n+k+\ell}\right)\frac{(n+k+\ell)!}{n!k!\ell!}\cdot\frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}}, \end{aligned}$$

where the change of index $j - k = \ell$ was used for the last equality. We have also

$$\begin{aligned} M_n f(x_1) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x_1^k (1-x_1)^{n+1} \\ &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k} x_1^k \cdot \frac{(1-x_2)^{n+k+1}}{(1-x_1)^k}\cdot\frac{1}{\left(1-\frac{x_2-x_1}{1-x_1}\right)^{n+k+1}} \\ &= \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\binom{n+k}{k}\frac{x_1^k(1-x_2)^{n+k+1}}{(1-x_1)^k}\sum_{\ell=0}^{\infty}\binom{n+k+\ell}{\ell}\left(\frac{x_2-x_1}{1-x_1}\right)^{\ell} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} f\left(\frac{k}{n+k}\right)\frac{(n+k+\ell)!}{n!k!\ell!}\cdot\frac{x_1^k (x_2-x_1)^{\ell}(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}}. \end{aligned}$$

In particular, the above equalities show that

$$\sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = 1, \tag{3}$$

$$\sum_{k,\ell=0}^{\infty} \frac{k+\ell}{n+k+\ell} \cdot \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_2, \tag{4}$$

$$\sum_{k,\ell=0}^{\infty} \frac{k}{n+k} \cdot \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} = x_1. \tag{5}$$

Since $f \in \mathrm{Lip}_A\alpha$, we have

$$\begin{aligned}
&|M_nf(x_2) - M_nf(x_1)| \\
&\le \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left| f\left(\frac{k+\ell}{n+k+\ell}\right) - f\left(\frac{k}{n+k}\right)\right| \\
&\le A \sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k}\right)^\alpha.
\end{aligned}$$

Taking into account (3) and the fact that the function $t \in [0,\infty[ \longmapsto t^\alpha \in [0,\infty[$ is concave, we deduce that

$$\begin{aligned}
&|M_nf(x_2) - M_nf(x_1)| \\
&\qquad \le A \left[\sum_{k,\ell=0}^{\infty} \frac{(n+k+\ell)!}{n!k!\ell!} \cdot \frac{x_1^k(x_2-x_1)^\ell(1-x_2)^{n+k+1}}{(1-x_1)^{k+\ell}} \left(\frac{k+\ell}{n+k+\ell} - \frac{k}{n+k}\right)\right]^\alpha.
\end{aligned}$$

Using now (4) and (5) we get

$$|M_nf(x_2) - M_nf(x_1)| \le A(x_2-x_1)^\alpha,$$

i.e., $M_nf \in \mathrm{Lip}_A\alpha$. This completes the proof. □

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