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ITERATIVE SOLUTION OF NONLINEAR EQUATIONS OF HAMMERSTEIN TYPE

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ABSTRACT. Suppose X is a real Banach space and $F, K : X \to X$ are accretive maps. Under different continuity assumptions on F and K such that the inclusion $0 = u + KFu$ has a solution, iterative methods are constructed which converge strongly to such a solution. No invertibility assumption is imposed on K and the operators K and F need not be defined on compact subsets of X. Our method of proof is of independent interest.

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1. INTRODUCTION

Let X be a real normed linear space with dual X^*. For $1 < q < \infty$, we denote by J_q the generalized duality mapping from X to 2^{X^*} defined by

$$J_q(x) := \{f^* \in X^* : \langle x, f^* \rangle = ||x|| ||f^*||, ||f^*|| = ||x||^{q-1}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If $q = 2$, $J_q = J_2$ and is denoted by J. If X^* is strictly convex, then J_q is single-valued (see e.g., [25]).

A map A with domain $D(A) \subseteq X$ is said to be *accretive* if for every $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq 0.$$

A is said to be $m-$*accretive* if it is accretive and $\mathcal{R}(I + \lambda A)$ (range of $(I + \lambda A)) = X$, for all $\lambda > 0$, where I is the identity mapping. A is said to be $\phi-$ *strongly accretive* if for every $x, y \in D(A)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$, $\phi(0) = 0$ such that

$$\langle Ax - Ay, j(x-y) \rangle \geq \phi(||x-y||)||x-y||,$$

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and it is *strongly accretive* if for each $x, y \in D(A)$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq k||x-y||^2.$$

Clearly, every strongly accretive map is ϕ-strongly accretive and every ϕ-strongly accretive map is accretive. Closely related to the class of accretive mappings is the class of pseudocontractive mappings. A mapping $T : X \to X$ is said to be *pseudocontractive* if and only if $A := I - T$ is accretive. One can easily show that the fixed point of pseudocontractive mapping T is the zero of accretive mapping $A := I - T$. If X is a Hilbert space, accretive operators are also called *monotone.* The accretive mappings were introduced independently in 1967 by Browder [3] and Kato [20]. Interest in such mappings stems mainly from their firm connection with equations of evolution. It is known (see e.g., [28]) that many physically significant problems can be modelled by initial-value problems of the form

$$x'(t) + Ax(t) = 0,\ x(0) = x_0, \tag{1.1}$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equations. One of the fundamental results in the theory of accretive operators, due to Browder [4], states that if A is locally Lipschitzian and accretive then A is $m-$accretive which immediately implies that the equation $x + Ax = h$ has a solution $x^* \in D(A)$ for any $h \in X$. This result was subsequently generalized by Martin [22] to the continuous accretive operators. If in (1.1), $x(t)$ is independent of t, then (1.1) reduces to

$$Au = 0, \tag{1.2}$$

whose solutions correspond to the equilibrium points of the system (1.1). Consequently, considerable research efforts have been devoted, especially within the past 20 years or so, to methods of finding approximate solutions (when they exist) of equation (1.2). One important generalization of equation (1.2) is the so-called *equation of Hammerstein type* (see e.g., [18]), where a nonlinear integral equation of Hammerstein type is one of the form:

$$u(x) + \int_\Omega K(x,y)f(y,u(y))dy = h(x), \tag{1.3}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel K is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \Re$ and is, in general, nonlinear and h is a given function on Ω. If we now define an operator K by

$$Kv(x) := \int_\Omega K(x,y)v(y)dy;\ x \in \Omega,$$

and the so-called *superposition* or *Nemytskii* operator by $Fu(y) := f(y, u(y))$ then, the integral equation (1.3) can be put in operator theoretic form as follows:

$$u + KFu = 0, \tag{1.4}$$

where, without loss of generality, we have taken $h \equiv 0$. We note that if K is an arbitrary accretive map (not necessarily the identity), then $A := I + KF$ need not be accretive. Interest in equation (1.4) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Greens functions can, as a rule, be transformed into the form (1.4) (see e.g., [23, Chapter IV]). Equations of Hammerstein type play a crucial role in the theory of optimal control systems (see e.g., [17]). Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see e.g., [2, 5, 6, 8, 15]).

For the iterative approximation of solutions of *equation (1.2)*, the *accretivity/ monotonicity* of A is crucial. The Mann iteration scheme (see e.g., [21]) and the Ishikawa iteration scheme (see e.g., [19]) have successfully been employed (see e.g., [7, 10, 11, 12, 13, 14, 16, 19, 21, 24, 27]). Attempts to apply these schemes to equation (1.4) have not provided satisfactory results. In particular, the recursion formulas obtained involved K^{-1} and this is not convenient in applications. Part of the difficulty is, as has already been noted, the fact that the composition of two accretive operators need not be accretive. In the special case in which the operators are defined on subsets D of X which are compact (or more generally, *angle-bounded* (see e.g., [1]), Brèzis and Browder [1] have proved the strong convergence of a suitably defined Galerkin approximation to a solution of (1.4).

It is our purpose in this paper to use the method introduced in [12] which contains an auxiliary operator, defined in terms of K and F in an arbitrary real Banach space which, under certain conditions, is accretive whenever K and F are, and whose zeros are solutions of equation (1.4). Moreover, the operators K and F need not be defined on a compact or angle-bounded subset of X. Furthermore, our method which does not involve K^{-1} provides an explicit algorithm for the computation of solutions of equation (1.4).

2. PRELIMINARIES

Let X be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of X is defined by:

$$\rho_X(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

It is well known that $\rho_X(\tau) \leq \tau \ \forall\tau > 0$ (see e.g., [26]). If $\rho_X(\tau) > 0 \ \ \forall\tau > 0$, then X is said to be *smooth*. If there exist a constant $c > 0$ and a real number $1 < q < \infty$, such that $\rho_X(\tau) \leq c\tau^q$, then X is said to be q*-uniformly smooth*. A Banach space X is called *uniformly smooth* if $\lim_{\tau\to 0} \frac{\rho_X(\tau)}{\tau} = 0$. If E is a real uniformly smooth Banach space, then

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + D\max\left\{||x|| + ||y||, \frac{c}{2}\right\}\rho_X(||y||), \tag{2.1}$$

for every $x, y \in X$, where D and c are positive constants (see e.g., [26]). Typical examples of such uniformly smooth spaces are the Lebesgue L_p, the sequence ℓ_p and the Sobolev W_p^m spaces for $1 < p < \infty$. Moreover, we have

$$\rho_{l_p}(\tau) = \rho_{L^p}(\tau) = \rho_{W_m^p}(\tau) \leq \begin{cases} \frac{1}{p}\tau^p, & \text{if } \ 1 < 2 < p; \\ \frac{p-1}{2}\tau^2, & \text{if } \ p \geq 2, \end{cases} \tag{2.2}$$

$\forall\tau > 0$ (see e.g., [26]).

In the sequel we shall need the following results.

Theorem 2.1. [25]. *Let $q > 1$ and X be a real smooth Banach space. Then the following are equivalent.*

(1) *X is uniformly smooth.*
(2) *There exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^+$, such that for every $x, y \in B_r$ for some $r > 0$ we get*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + g(||y||). \tag{2.3}$$

Lemma 2.2. *(see, e.g.,* [13]*). Let X be a normed linear space and J be the normalized duality map on E. Then for any given $x, y \in X$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle, \ \ \forall j(x+y) \in J(x+y).$$

Theorem 2.3. [9]. *Let X be a real Banach space, $A : X \to X$ be a Lipschitz and strongly accretive map with Lipschitz constant $L > 0$ and strong accretivity constant $\lambda \in (0,1)$. Assume that $Ax = 0$ has a solution $x^* \in X$. Define $A_\varepsilon : X \to X$ by $A_\varepsilon x := x - \varepsilon Ax$ for $x \in X$ where $\varepsilon := \frac{1}{2}\left\{\frac{\lambda}{1+L(3+L-\lambda)}\right\}$. For arbitrary $x_0 \in X$, define the Picard sequence $\{x_n\}$ in X by $x_{n+1} = A_\varepsilon x_n, n \geq 0$. Then, $\{x_n\}$ converges strongly to x^* with $||x_{n+1} - x^*|| \leq \delta^n ||x_1 - x^*||$ where $\delta := \left(1 - \frac{1}{2}\lambda\varepsilon\right) \in (0,1)$. Moreover, x^* is unique.*

Theorem 2.4. [13] *Let X be a real normed linear space. Let $A : X \to X$ be uniformly continuous $\phi-$ strongly accretive mapping. Assume $0 = Ax$ has a solution $x^* \in X$. Then, there exists a real number $\gamma_0 > 0$ such that if the real sequence $\{\alpha_n\} \subset [0, \gamma_0]$ satisfies the following conditions:*

(i) $\lim \alpha_n = 0$;
(ii) $\sum \alpha_n = \infty$,

then for arbitrary $x_0 \in X$ the sequence $\{x_n\}$, defined by

$$x_{n+1} := x_n - \alpha_n A x_n, \ \ n \geq 0,$$

converges strongly to x^*, the unique solution of $0 = Ax$.

We note that Theorem 2.4 is Theorem 3.6 of [13] with A ϕ-strongly accretive mapping.

3. Main Results

Lemma 3.1. *For $q > 1$, let X be a real uniformly smooth Banach space. Let $E := X \times X$ with norm*

$$||z||_E := \left(||u||_X^q + ||v||_X^q\right)^{\frac{1}{q}},$$

for arbitrary $z = [u, v] \in E$. Let $E^* := X^* \times X^*$ denote the dual space of E. For arbitrary $x = [x_1, x_2] \in E$ define the map $j_q^E : E \to E^*$ by

$$j_q^E(x) = j_q^E[x_1, x_2] := [j_q^X(x_1), j_q^X(x_2)],$$

so that for arbitrary $z_1 = [u_1, v_1], z_2 = [u_2, v_2]$ in E the duality pairing $\langle \cdot, \cdot \rangle$ is given by

$$\left\langle z_1, j_q^E(z_2)\right\rangle = \left\langle u_1, j_q^X(u_2)\right\rangle + \left\langle v_1, j_q^X(v_2)\right\rangle.$$

Then

(a) *E is uniformly smooth;*
(b) *j_q^E is a single-valued duality mapping on E.*

Proof. (a) Let $x = [x_1, x_2],\ y = [y_1, y_2]$ be arbitrary elements of E. It suffices to show that x and y satisfy condition (2) of Theorem 2.1. We compute as follows:

$$\begin{aligned} ||x+y||_E^q &= ||[x_1 + y_1, x_2 + y_2]||_E^q \\ &= ||x_1 + y_1||_X^q + ||x_2 + y_2||_X^q \\ &\leq ||x_1||_X^q + ||x_2||_X^q + g(||y_1||) + g(||y_2||) \\ &\qquad + q\Big\{ \left\langle y_1, j_q^X(x_1)\right\rangle + \left\langle y_2, j_q^X(x_2)\right\rangle \Big\}, \end{aligned}$$

where g is continuous, strictly increasing and a convex function (using (2) of Theorem 2.1, since X is uniformly smooth). It follows that

$$||x+y||_E^q \leq ||x||_E^q + q\left\langle y, j_q^E(x)\right\rangle + g'(||y||),$$

where $g'(||y||) := g(||y_1||) + g(||y_2||)$. So, the result follows from Theorem 2.1.

(b) For arbitrary $x = [x_1, x_2] \in E$, let $j_q^E(x) = j_q^E[x_1, x_2] = \psi_q$. Then $\psi_q = [j_q^X(x_1), j_q^X(x_2)]$ in E^*. Observe that for $p > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}||\psi_q||_{E^*} &= \Big(||[j_q^X(x_1), j_q^X(x_2)]||\Big)^{\frac{1}{p}} \\ &= \Big(||j_q(x_1)||_{X^*}^p + ||j_q(x_2)||_{X^*}^p\Big)^{\frac{1}{p}} \\ &= \Big(||x_1||_X^{(q-1)p} + ||x_2||_X^{(q-1)p}\Big)^{\frac{1}{p}} \\ &= \Big(||x_1||_X^q + ||x_2||_X^q\Big)^{\frac{q-1}{q}} \\ &= ||x||_X^{q-1}.\end{aligned}$$

Hence, $||\psi_q||_{E^*} = ||x||_E^{q-1}$. Furthermore,

$$\begin{aligned}\langle x, \psi_q\rangle &= \big\langle [x_1, x_2], [j_q^X(x_1), j_q^X(x_2)]\big\rangle \\ &= \big\langle x_1, j_q^X(x_1)\big\rangle + \big\langle x_2, j_q^X(x_2)\big\rangle \\ &= ||x_1||_X^q + ||x_2||_X^q \\ &= (||x_1||_X^q + ||x_2||_X^q)^{\frac{1}{q}}\Big(||x_1||_X^q + ||x_2||_X^q\Big)^{\frac{q-1}{q}} \\ &= ||x||_E \cdot ||\psi||_{E^*}^{q-1}.\end{aligned}$$

Hence, j_q^E is a single-valued (since E is uniformly smooth) duality mapping on E. □

Lemma 3.2. *Suppose X is a real normed linear space. Let $F, K : X \to X$ be maps such that the following conditions hold:*

(i) *For each $u_1, u_2 \in X$ there exist $j(u_1 - u_2) \in J(u_1 - u_2)$ and a strictly increasing function $\phi_1 : [0, \infty) \to [0, \infty)$, $\phi_1(0) = 0$ such that*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \phi_1(||u_1 - u_2||)||u_1 - u_2||;$$

(ii) *For each $u_1, u_2 \in X$ there exist $j(u_1 - u_2) \in J(u_1 - u_2)$ and a strictly increasing function $\phi_2 : [0, \infty) \to [0, \infty)$, $\phi_2(0) = 0$ such that*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \phi_2(||u_1 - u_2||)||u_1 - u_2||;$$

(iii) *$\phi_i(t) \geq (2 + r_i)t$ for all $t \in (0, \infty)$ and for some $r_i > 0$, $i = 1, 2$.*

Let $E := X \times X$ with norm $||z||_E^2 = ||u||_X^2 + ||v||_X^2$ for $z = (u, v) \in E$ and define a map $T : E \to E$ by $Tz := T(u, v) = (Fu - v, u + Kv)$. Then for each $z_1, z_2 \in E$ there exist $j^E(z_1 - z_2) \in J^E(z_1 - z_2)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\big\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\big\rangle \geq \phi(||z_1 - z_2||)||z_1 - z_2||.$$

Proof. Define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(t) := \min\{r_1, r_2\}t$ for each $t \in [0, \infty)$. Clearly, ϕ is a strictly increasing function with $\phi(0) = 0$. Furthermore, observe that for $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ arbitrary elements in E we have $\big\langle z_1, j^E(z_2)\big\rangle = \langle u_1, j(u_2)\rangle + \langle v_1, j(v_2)\rangle$. Thus we have the following estimates:

$$\begin{aligned}\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle &= \langle Fu_1 - Fu_2 - (v_1 - v_2), j(u_1 - u_2)\rangle \\ &\qquad + \langle Kv_1 - Kv_2 + (u_1 - u_2), j(v_1 - v_2)\rangle \\ &= \langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle - \langle v_1 - v_2, j(u_1 - u_2)\rangle \\ &\qquad + \langle Kv_1 - Kv_2, j(v_1 - v_2)\rangle + \langle u_1 - u_2, j(v_1 - v_2)\rangle \\ &\geq \phi_1(||u_1 - u_2||)||u_1 - u_2|| + \phi_2(||v_1 - v_2||)||v_1 - v_2|| \\ &\qquad - \langle v_1 - v_2, j(u_1 - u_2)\rangle + \langle u_1 - u_2, j(v_1 - v_2)\rangle . \qquad (3.1)\end{aligned}$$

Since X is an arbitrary real normed linear space, for each $x, y \in X$ and $j(x+y) \in J(x+y)$ (by Lemma 2.2) we have that

$$\begin{aligned}||x+y||^2 &\leq ||x||^2 + 2\langle y, j(x+y) - j(x)\rangle + 2\langle y, j(x)\rangle \\ &\leq ||x||^2 + 2||y||||j(x+y) - j(x)|| + 2\langle y, j(x)\rangle \\ &\leq ||x||^2 + 2||y||\Big(||x+y|| + ||x||\Big) + 2\langle y, j(x)\rangle \\ &\leq ||x||^2 + 2\left(\frac{||y||^2}{2} + \frac{||x+y||^2}{2} + \frac{||y||^2}{2} + \frac{||x||^2}{2}\right) + 2\langle y, j(x)\rangle \\ &= 2||x||^2 + 2||y||^2 + ||x+y||^2 + 2\langle y, j(x)\rangle .\end{aligned}$$

Thus we get $\langle y, j(x)\rangle \geq -||x||^2 - ||y||^2$.

Replacing y by $-y$ we obtain $-\langle y, j(x)\rangle \geq -||x||^2 - ||y||^2$. Therefore,

$$\begin{aligned}-\langle v_1 - v_2, j(u_1 - u_2)\rangle &\geq -||u_1 - u_2||^2 - ||v_1 - v_2||^2 \quad \text{and} \\ \langle u_1 - u_2, j(v_1 - v_2)\rangle &\geq -||v_1 - v_2||^2 - ||u_1 - u_2||^2.\end{aligned}$$

Thus (3.1) and the above estimates give that

$$\begin{aligned}\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle &\geq \phi_1(||u_1 - u_2||)||u_1 - u_2|| + \phi_2(||v_1 - v_2||)||v_1 - v_2|| \\ &\qquad - 2||u_1 - u_2||^2 - 2||v_1 - v_2||^2 \\ &\geq \Big(\phi_1(||u_1 - u_2||) - 2||u_1 - u_2||\Big)||u_1 - u_2|| \\ &\qquad + \Big(\phi_2(||v_1 - v_2||) - 2||v_1 - v_2||\Big)||v_1 - v_2|| \\ &\geq r_1||u_1 - u_2||^2 + r_2||v_1 - v_2||^2 \\ &\geq \min\{r_1, r_2\}\Big\{||u_1 - u_2||^2 + ||v_1 - v_2||^2\Big\} \\ &= \min\{r_1, r_2\}||z_1 - z_2||^2 \\ &= \phi(||z_1 - z_2||)||z_1 - z_2||,\end{aligned}$$

completing the proof of Lemma 3.2. □

Lemma 3.3. *Suppose X is a real uniformly smooth Banach space. Let $F, K : X \to X$ be maps such that the following conditions hold:*

(i) *For each $u_1, u_2 \in X$ there exists a strictly increasing function $\phi_1 : [0, \infty) \to [0, \infty),\ \phi_1(0) = 0$ such that*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \phi_1(||u_1 - u_2||)||u_1 - u_2||;$$

(ii) *For each* $u_1, u_2 \in X$ *there exists a strictly increasing function* $\phi_2 : [0,\infty) \to [0,\infty),\ \phi_2(0) = 0$ *such that*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \phi_2(||u_1 - u_2||)||u_1 - u_2||;$$

(iii) $\phi_i(t) \geq (D + r_i)t + \frac{acD}{4}t^{q-1}$, $\rho_X(t) \leq at^q$ *for all* $t \in (0,\infty)$ *and for some* $q > 1$, $a > 0$ *and* $r_i > 0$, $i = 1, 2$, *where* c *and* D *are the constants appearing in inequality (2.1).*

Let E *and* T *be defined as in Lemma 3.2. Then for each* $z_1, z_2 \in E$ *there exists a strictly increasing function* $\phi : [0,\infty) \to [0,\infty)$ *with* $\phi(0) = 0$ *such that*

$$\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle \geq \phi(||z_1 - z_2||)||z_1 - z_2||.$$

Proof. Define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) := \min\{r_1, r_2\}t$ for each $t \in [0,\infty)$. Thus as in the proof of Lemma 3.2 we have that ϕ is a strictly increasing function with $\phi(0) = 0$ and for $z_1 = (u_1, v_1)$ and $z_2 = (u_2, v_2)$ arbitrary elements in E we have the following estimate:

$$\begin{aligned}\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle &= \phi_1(||u_1 - u_2||)||u_1 - u_2|| + \phi_2(||v_1 - v_2||)||v_1 - v_2|| \\ &\quad - \langle v_1 - v_2, j(u_1 - u_2)\rangle + \langle u_1 - u_2, j(v_1 - v_2)\rangle .\end{aligned} \tag{3.2}$$

Since X is uniformly smooth for each $x, y \in X$ by (2.1) we have that

$$\begin{aligned}||x + y||^2 &\leq ||x||^2 + 2\langle y, j(x)\rangle + D\max\{||x|| + ||y||, \frac{c}{2}\}\rho_X(||y||) \\ &\leq ||x||^2 + 2\langle y, j(x)\rangle + D\{||x|| + ||y|| + \frac{c}{2}\}\rho_X(||y||) \\ &\leq ||x||^2 + 2\langle y, j(x)\rangle + D\left(||x||||y|| + ||y||^2 + \frac{c}{2}\rho_X(||y||)\right) \\ &\text{(since } \rho_X(||y||) \leq ||y||) \\ &\leq ||x||^2 + 2\langle y, j(x)\rangle + D\left(\frac{||x|^2}{2} + \frac{||y||^2}{2} + ||y||^2 + \frac{ac}{2}||y||^q\right) \\ &\text{(since } \rho_X(||y||) \leq a||y||^q \text{ by assumption for } q > 1 \text{ and } a > 0) \\ &\leq \left(1 + \frac{D}{2}\right)||x||^2 + \frac{3D}{2}||y||^2 + \frac{acD}{2}||y||^q + 2\langle y, j(x)\rangle ,\end{aligned}$$

and hence

$$\langle y, j(x)\rangle \geq \frac{1}{2}||x + y||^2 - \frac{1}{2}\left(\left(1 + \frac{D}{2}\right)||x||^2 + \frac{3D}{2}||y||^2 + \frac{acD}{2}||y||^q\right).$$

Replacing y by $-y$ we obtain

$$-\langle y, j(x)\rangle \geq \frac{1}{2}||x - y||^2 - \frac{1}{2}\left(\left(1 + \frac{D}{2}\right)||x||^2 + \frac{3D}{2}||y||^2 + \frac{acD}{2}||y||^q\right).$$

Thus (3.2) and the above estimates give that

$$\begin{aligned}
&\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle \\
&\qquad \geq \phi_1(||u_1-u_2||)||u_1-u_2|| + \phi_2(||v_1-v_2||)||v_1-v_2|| \\
&\qquad\qquad + \frac{1}{2}\Bigg(||u_1-u_2-(v_1-v_2)||^2 - \left(1+\frac{D}{2}\right)||u_1-u_2||^2 \\
&\qquad\qquad - \frac{3D}{2}||v_1-v_2||^2 - \frac{acD}{2}||v_1-v_2||^q\Bigg) \\
&\qquad\qquad + \frac{1}{2}\Bigg(||u_1-u_2+v_1-v_2||^2 - \left(1+\frac{D}{2}\right)||v_1-v_2||^2 \\
&\qquad\qquad - \frac{3D}{2}||u_1-u_2||^2 - \frac{acD}{2}||u_1-u_2||^q\Bigg) \\
&\qquad \geq \phi_1(||u_1-u_2||)||u_1-u_2|| + \phi_2(||v_1-v_2||)||v_1-v_2|| \\
&\qquad\qquad + \frac{1}{2}\left(||u_1-u_2-(v_1-v_2)||^2 + ||u_1-u_2+v_1-v_2||^2\right) \\
&\qquad\qquad - \frac{1}{2}\left((1+2D)||u_1-u_2||^2 + \frac{acD}{2}||u_1-u_2||^q\right) \\
&\qquad\qquad - \frac{1}{2}\left((1+2D)||v_1-v_2||^2 + \frac{acD}{2}||v_1-v_2||^q\right). \qquad (3.3)
\end{aligned}$$

Since for all $x, y \in X, x \neq y$,

$$\left\|\frac{x+y}{2}\right\|^2 \leq \frac{1}{2}(||x||^2 + ||y||^2)$$

we have that

$$||(u_1-u_2)-(v_1-v_2)||^2 + ||(u_1-u_2)+(v_1-v_2)||^2 \geq ||u_1-u_2||^2 + ||v_1-v_2||^2.$$

Then (3.3) becomes

$$\begin{aligned}
\left\langle Tz_1 - Tz_2, j^E(z_1 - z_2)\right\rangle &\geq \phi_1(||u_1-u_2||)||u_1-u_2|| - \Bigg(D||u_1-u_2||^2 \\
&\quad + \frac{acD}{4}||u_1-u_2||^q\Bigg) + \phi_2(||v_1-v_2||)||v_1-v_2|| \\
&\quad - \left(D||v_1-v_2||^2 + \frac{acD}{4}||v_1-v_2||^q\right) \\
&\geq r_1||u_1-u_2||^2 + r_2||v_1-v_2||^2 \\
&\geq \min\{r_1, r_2\}\left\{||u_1-u_2|^2 + ||v_1-v_2||^2\right\} \\
&= \min\{r_1, r_2\}||z_1-z_2||^2 \\
&= \phi(||z_1-z_2||)||z_1-z_2||,
\end{aligned}$$

completing the proof of Lemma 3.3. □

3.1. **Convergence Theorems for Lipschitz Maps.**

Remark 3.4. If K and F are Lipschitz single-valued maps with Lipschitz constants L_K and L_F respectively, then T is a Lipschitz map with constant $L := \left(d \ \max\{L_F^2+1, L_K^2+1\}\right)^{\frac{1}{2}}$ for some constant $d > 0$. Indeed, if $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2)$ in E then we have that

$$
\begin{aligned}
||Tz_1 - Tz_2||^2 &= ||(Fu_1 - Fu_2) - (v_1 - v_2)||^2 + ||u_1 - u_2 + Kv_1 - Kv_2||^2 \\
&\leq \Big(L_F||u_1 - u_2|| + ||v_1 - v_2||\Big)^2 + \Big(||u_1 - u_2|| + L_K||v_1 - v_2||\Big)^2 \\
&\leq d\Big(L_F^2||u_1 - u_2||^2 + ||v_1 - v_2||^2 + ||u_1 - u_2||^2 + L_K^2||v_1 - v_2||^2\Big) \\
&\quad \text{for some } d > 0 \\
&\leq d\max\{L_F^2 + 1, L_K^2 + 1\}\Big(||u_1 - u_2||^2 + ||v_1 - v_2||^2\Big) \\
&= d\max\{L_F^2 + 1, L_K^2 + 1\}||z_1 - z_2||^2.
\end{aligned}
$$

Thus $||Tz_1 - Tz_2|| \leq L||z_1 - z_2||$. Consequently, we have the following theorem.

Theorem 3.5. *Let X be real Banach space. Let $F, K : X \to X$ be Lipschitzian maps with Lipschitz constants L_K and L_F, respectively such that the following conditions hold:*

(i) *For each $u_1, u_2 \in X$ there exist $j(u_1 - u_2) \in J(u_1 - u_2)$ and a strictly increasing function $\phi_1 : [0, \infty) \to [0, \infty),\ \phi_1(0) = 0$ such that*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \phi_1(||u_1 - u_2||)||u_1 - u_2||;$$

(ii) *For each $u_1, u_2 \in X$ there exist $j(u_1 - u_2) \in J(u_1 - u_2)$ and a strictly increasing function $\phi_2 : [0, \infty) \to [0, \infty),\ \phi_2(0) = 0$ such that*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \phi_2(||u_1 - u_2||)||u_1 - u_2||;$$

(iii) *$\phi_i(t) \geq (2 + r_i)t$ for all $t \in (0, \infty)$ and for some $r_i > 0$, $i = 1, 2$ and let $\gamma := \min\{r_1, r_2\}$.*

Assume that $u + KFu = 0$ has a solution u^* in X and let $E := X \times X$ and $||z||_E^2 = ||u||_X^2 + ||v||_X^2$ for $z = (u, v) \in E$ and define the map $T : E \to E$ by $Tz := T(u, v) = (Fu - v, Kv + u)$. Let L denote the Lipschitz constant of T and $\varepsilon := \frac{1}{2}\left(\frac{\gamma}{1+L(3+L-\gamma)}\right)$. Define the map $A_\varepsilon : E \to E$ by $A_\varepsilon z := z - \varepsilon Tz$ for each $z \in E$. For arbitrary $z_0 \in E$, define the Picard sequence $\{z_n\}$ in E by $z_{n+1} := A_\varepsilon z_n, n \geq 0$. Then $\{z_n\}$ converges strongly to $z^* = [u^*, v^*]$ the unique solution of the equation $Tz = 0$ with $||z_{n+1} - z^*|| \leq \delta^n||z_1 - z^*||$, where $\delta := \left(1 - \frac{1}{2}\gamma\varepsilon\right) \in (0, 1)$.

Proof. Observe that u^* is a solution of $u + KFu = 0$ if and only if $z^* = [u^*, v^*]$ is a solution of $Tz = 0$. Hence $Tz = 0$ has a solution $z^* = [u^*, v^*]$ in E. Since T is Lipschitz and by Lemma 3.2 it is strongly accretive with constant γ (which, without loss of generality, we may assume is in $(0, 1)$), the conclusion follows from Theorem 2.3. □

Following the method of the proof of Theorem 3.5 and making use of Lemma 3.3 instead of Lemma 3.2 we obtain the following theorem.

Theorem 3.6. *Let X be a real uniformly smooth Banach space. Let $F, K : X \to X$ be Lipschitzian maps with Lipschitz constants L_K and L_F, respectively such that conditions (i)-(iii) of Lemma 3.3 are satisfied and let $\gamma := \min\{r_1, r_2\}$. Assume that $u + KFu = 0$ has the solution u^* and set E and T as in Theorem 3.5. Let $L, \varepsilon, A_\varepsilon$, and $\{z_n\}$ be defined as in Theorem 3.5. Then the conclusion of Theorem 3.5 holds.*

3.2. **Convergence Theorems for Uniformly Continuous ϕ-Strongly Accretive Maps.**

Theorem 3.7. *Let X be a real normed linear space. Let $F, K : X \to X$ be uniformly continuous maps such that the following conditions hold:*

(i) *For each* $u_1, u_2 \in X$ *there exist* $j(u_1 - u_2) \in J(u_1 - u_2)$ *and a strictly increasing function* $\phi_1 : [0, \infty) \to [0, \infty),\ \phi_1(0) = 0$ *such that*

$$\langle Fu_1 - Fu_2, j(u_1 - u_2)\rangle \geq \phi_1(||u_1 - u_2||)||u_1 - u_2||;$$

(ii) *For each* $u_1, u_2 \in X$ *there exist* $j(u_1 - u_2) \in J(u_1 - u_2)$ *and a strictly increasing function* $\phi_2 : [0, \infty) \to [0, \infty),\ \phi_2(0) = 0$ *such that*

$$\langle Ku_1 - Ku_2, j(u_1 - u_2)\rangle \geq \phi_2(||u_1 - u_2||)||u_1 - u_2||;$$

(iii) $\phi_i(t) \geq (2 + r_i)t$ *for all* $t \in (0, \infty)$ *and for some* $r_i > 0, i = 1, 2.$

Assume that $0 = u + KFu$ *has a solution* u^* *in* X. *Let* $E := X \times X$ *and* $||z||_E^2 = ||u||_X^2 + ||v||_X^2$ *for* $z = (u, v) \in E$ *and define the map* $T : E \to E$ *by* $Tz := T(u, v) = (Fu - v, u + Kv)$. *Then there exists a real number* $\gamma_0 > 0$ *such that if the real sequence* $\{\alpha_n\} \subset [0, \gamma_0]$ *satisfies the following conditions*

(a) $\lim_{n\to\infty} \alpha_n = 0$*;*
(b) $\sum \alpha_n = \infty$,

then for arbitrary $z_0 \in E$ *the sequence* $\{z_n\}$*, defined by*

$$z_{n+1} := z_n - \alpha_n T z_n,\ n \geq 0,$$

converges strongly to $z^* = [u^*, v^*]$*, where* u^* *is the unique solution of* $0 = u + KFu$.

Proof. Since K and F are uniformly continuous maps we have that T is a uniformly continuous map. Observe also that u^* is the solution of $0 = u + KFu$ in X if and only if $z^* = [u^*, v^*]$ is a solution of $0 = Tz$ in E. Thus we obtain that $N(T)$ (null space of T) $\neq \emptyset$. Also by Lemma 3.2, T is $\phi-$strongly accretive. Therefore the conclusion follows from Theorem 2.4. □

Following the method of proof of Theorem 3.7 and making use of Lemma 3.3 instead of Lemma 3.2 we obtain the following theorem.

Theorem 3.8. *Let* X *be a real uniformly smooth Banach space. Let* $F, K : X \to X$ *be uniformly continuous maps such that conditions (i)-(iii) of Theorem 3.6 are satisfied. Assume that* $0 = u + KFu$ *has a solution* u^* *in* X. *Let* E, T *and* $\{z_n\}$ *be defined as in Theorem 3.7. Then, the conclusion of Theorem 3.7 holds.*

Remark 3.9. We note that for the special case in which the real Banach space X is $q-$uniformly smooth using the above method, the author and Chidume [12] proved the following theorem.

Theorem 3.10. [12]. *Let* X *be a real* q*-uniformly smooth Banach space. Let* $F, K : X \to X$ *be Lipschitzian maps with positive constants* L_K *and* L_F *respectively with the following conditions:*

(i) *There exists* $\alpha > 0$ *such that*

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2)\rangle \geq \alpha||u_1 - u_2||^q,\ \forall\, u_1, u_2 \in D(F);$$

(ii) *There exists* $\beta > 0$ *such that*

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2)\rangle \geq \beta||u_1 - u_2||^q,\ \forall\, u_1, u_2 \in D(K);$$

(iii) $\alpha, \beta > d := q^{-1}(1 + d_q - c^{-1}2^{q-1})$ *and* $\gamma := \min\{\alpha - d, \beta - d\}$ *where* d_q *and* c *are as in (3.2) and (2.1) of* [12]*, respectively.*

Assume that $u + KFu = 0$ *has solution* u^* *and set* E *and* T *as in Theorem 3.5. Let* L *be a Lipschitz constant of* T *and* ε, A_ε *and* z_n *be defined as in Theorem 3.5. Then* $\{z_n\}$ *converges strongly to* $z^* = [u^*, v^*]$ *the unique solution of the equation* $Tz = 0$ *with* $||z_{n+1} - z^*|| \leq \delta^n||z_1 - z^*||$*, where* u^* *is the solution of the equation* $u + KFu = 0$ *and* $\delta := \left(1 - \frac{1}{2}\gamma\varepsilon\right) \in (0, 1)$.

The cases for Hilbert spaces and L_p spaces $(1 < p < \infty)$ are easily deduced from Theorem 3.10. *The theorems proved in this paper are analogues of the theorems* in [12] for the more general real *Banach spaces considered here*.

3.3. **Explicit Algorithms.**

The method of our proofs provides the following explicit algorithms for computing the solution of the inclusion $0 = u + KFu$ *in the space* X.

(a) For Lipschitz operators (Theorem 3.5 and Theorem 3.6) with initial values $u_0, v_0 \in X$, define the sequences $\{u_n\}$ and $\{v_n\}$ in X as follows:

$$u_{n+1} = u_n - \varepsilon\Big(Fu_n - v_n\Big);$$
$$v_{n+1} = v_n - \varepsilon\Big(Kv_n + u_n\Big).$$

Then $u_n \to u^*$ in X, the unique solution u^* of $0 = u + KFu,$ where ε is as defined in Theorem 3.5.

(b) For uniformly continuous operators (Theorem 3.7 and Theorem 3.8) with initial values $u_0, v_0 \in X$, define the sequences $\{u_n\}$ and $\{v_n\}$ in X as follows:

$$u_{n+1} = u_n - \alpha_n(Fu_n - v_n);$$
$$v_{n+1} = v_n - \alpha_n(Kv_n + u_n).$$

Then $u_n \to u^*$ in X, the unique solution u^* of $0 = u + KFu,$ where α_n is as defined in Theorem 3.7.

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