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ON THE STABILITY OF A CLASS OF FUNCTIONAL EQUATIONS

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Abstract

In this paper, we study the Baker's superstability for the following functional equation

$$(E(K)) \qquad \sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k) = |\Phi| f(x)f(y), \quad x,y\in G$$

where G is a locally compact group, K is a compact subgroup of G, ω_K is the normalized Haar measure of K, Φ is a finite group of K-invariant morphisms of G and f is a continuous complex-valued function on G satisfying the Kannappan type condition, for all $x,y,z\in G$

$$(*) \quad \int_K\int_K f(zkxk^{-1}hyh^{-1})d\omega_K(k)d\omega_K(h)=\int_K\int_K f(zkyk^{-1}hxh^{-1})d\omega_K(k)d\omega_K(h).$$

We treat examples and give some applications.

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Contents

1. Introduction, Notations and Preliminaries

Let G be a locally compact group. Let K be a compact subgroup of G. Let ω_K be the normalized Haar measure of K. A mapping $\varphi : G \longmapsto G$ is a morphism of G if φ is a homeomorphism of G onto itself which is either a group-homorphism, i.e ($\varphi(xy) = \varphi(x)\varphi(y), x, y \in G$), or a group-antihomorphism, i.e ($\varphi(xy) = \varphi(y)\varphi(x), x, y \in G$). We denote by $Mor(G)$ the group of morphisms of G and Φ a finite subgroup of $Mor(G)$ of a K-invariant morphisms of G (i.e $\varphi(K) \subset K$). The number of elements of a finite group Φ will be designated by $|\Phi|$. The Banach algebra of bounded measures on G with complex values is denoted by $M(G)$ and the Banach space of all complex measurable and essentially bounded functions on G by $L_\infty(G)$. $\mathcal{C}(G)$ designates the Banach space of all continuous complex valued functions on G. We say that a function f is a K-central function on G if

$$f(kx) = f(xk), \quad x \in G,\ k \in K. \tag{1.1}$$

In the case where $G = K$, a function f is central if

$$f(xy) = f(yx) \quad x, y \in G. \tag{1.2}$$

See [2] for more information.

In this note, we are going to generalize the results obtained by J.A. Baker in [8] and [9]. As applications, we discuss the following cases:

a) $K \subset Z(G)$, ($Z(G)$ is the center of G).

b) $f(hxk) = f(x),\ h, k \in K,\ x \in G$ (i.e. f is bi-K-invariant (see [3] and [6])).

c) $f(hxk) = \chi(k)f(x)\chi(h),\ x \in G,\ k,h \in K$ (χ is a unitary character of K) (see [11]).

d) (G,K) is a Gelfand pair (see [3], [6] and [11]).

e) $G = K$ (see [2]).

In the next section, we note some results for later use.

2. General Properties

In what follows, we study general properties. Let G, K and Φ be given as above.

Proposition 2.1. *For an arbitrary fixed $\tau \in \Phi$, the mapping*

$$\begin{aligned} \Phi &\longrightarrow \Phi, \\ \varphi &\longrightarrow \varphi \circ \tau \end{aligned}$$

is a bijection.

Proof. Follows from the fact that Φ is a finite group. □

Proposition 2.2. *Let $\varphi \in \Phi$ and $f \in \mathcal{C}(G)$, then we have:*

i) $\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) = \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k), \quad x, y \in G, \; h \in K.$

ii) *If f satisfy (*), the for all $z, y, x \in G$, we have*

$$\int_K \int_K f(zh\varphi(ykxk^{-1})h^{-1})d\omega_K(h)d\omega_K(k) = \int_K \int_K f(zh\varphi(xkyk^{-1})h^{-1})d\omega_K(h)d\omega_K(k).$$

Proof. i) Let $\varphi \in \Phi$ and let $x, y \in G$, $h \in K$, then we have

Case 1: If φ is a group-homomorphism, we obtain, by replacing k by $k\varphi(h)^{-1}$

$$\begin{aligned}\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) &= \int_K f(xk\varphi(h)\varphi(y)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(y)\varphi(h)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k).\end{aligned}$$

Case 2: if φ is a group-antihomomorphism, we have, by replacing k by $k\varphi(h)$

$$\begin{aligned}\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) &= \int_K f(xk\varphi(y)\varphi(h)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(h)\varphi(y)k^{-1})d\omega_K(k)\\ &= \int_K f(xk\varphi(yh)k^{-1})d\omega_K(k).\end{aligned}$$

ii) Follows by simple computation.

□

Proposition 2.3. *For each $\tau \in \Phi$ and $x, y \in G$, we have*

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(\tau(y))k^{-1})d\omega_K(k) = \sum_{\psi\in\Phi}\int_K f(xk\psi(y)k^{-1})d\omega_K(k). \tag{2.1}$$

Proof. By applying Proposition 2.1, we get that when φ is iterated over Φ, the morphism of the form $\varphi \circ \tau$ annihilates all the elements of Φ. $\square$

3. The Main Results

Theorem 3.1. *Let G be a locally compact group; let K be a compact subgroup of G with the normalized Haar measure ω_K and let Φ given as above.*

Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality

$$\left| \sum_{\varphi \in \Phi} \int_K f(xk\varphi(y)k^{-1}) d\omega_K(k) - |\Phi| f(x) f(y) \right| \leq \delta, \quad x, y \in G. \tag{3.1}$$

Then one of the assertions is satisfied:

(a) If f is bounded, then

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}. \tag{3.2}$$

(b) If f is unbounded, then

i) f is K-central,

ii) $f \circ \tau = f$, for all $\tau \in \Phi$,

iii) $\int_K f(xkyk^{-1}) d\omega_K(k) = \int_K f(ykxk^{-1}) d\omega_K(k), \quad x, y \in G$.

Proof.

a) Let $X = \sup |f|$, then we get for all $x \in G$

$$|\Phi||f(x)f(x)| \leq |\Phi|X + \delta,$$

from which we obtain that

$$|\Phi|X^2 - |\Phi|X - \delta \leq 0,$$

such that

$$X \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}.$$

b) i) Let $x, y \in G,\ h \in K$, then by using Proposition 2.2, we find

$$\begin{aligned}
&|\Phi||f(x)||f(hy) - f(yh)| \\
&\qquad = ||\Phi|f(x)f(hy) - |\Phi|f(x)f(yh)| \\
&\qquad \leq \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(hy)k^{-1})d\omega_K(k) - |\Phi|f(x)f(hy)\right| \\
&\qquad\qquad + \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(yh)k^{-1})d\omega_K(k) - |\Phi|f(x)f(yh)\right| \\
&\qquad \leq 2\delta.
\end{aligned}$$

Since f is unbounded it follows that $f(yh) = f(hy)$, for all $h \in K, y \in G$.

ii) Let $\tau \in \Phi$, by using Proposition 2.3, we get for all $x, y \in G$

$$\begin{aligned}
&|\Phi||f(x)||f\circ\tau(y) - f(y)| \\
&\qquad = ||\Phi|f(x)f(\tau(y)) - |\Phi|f(x)f(y)|
\end{aligned}$$

$$\leq \left| \sum_{\varphi\in\Phi} \int_K f(xk\varphi(\tau(y))k^{-1})d\omega_K(k) - |\Phi|\, f(x)f(\tau(y)) \right| + \left| \sum_{\psi\in\Phi} \int_K f(xk\psi(y)k^{-1})d\omega_K(k) - |\Phi| f(x)f(y) \right|$$
$$\leq 2\delta.$$

Since f is unbounded it follows that $f \circ \tau = f$, for all $\tau \in \Phi$.

iii) Let f be an unbounded solution of the functional inequality (3.1), such that f satisfies the condition (*), then, for all $x, y \in G$, we obtain, by using Part i) of Proposition 2.2:

$$\begin{aligned}
|\Phi||f(z)| &\left| \int_K f(xkyk^{-1})d\omega_K(k) - \int_K f(ykxk^{-1})d\omega_K(k) \right| \\
&= \left| |\Phi| \int_K f(z)f(xkyk^{-1})d\omega_K(k) - |\Phi| \int_K f(z)f(ykxk^{-1})d\omega_K(k) \right| \\
&\leq \left| \int_K \Sigma_{\varphi\in\Phi} \int_K f(zh\varphi(xkyk^{-1})h^{-1})d\omega_K(h)d\omega_K(k) - |\Phi| \int_K f(z)f(xkyk^{-1})d\omega_K(k) \right|
\end{aligned}$$

$$+\left|\int_K \Sigma_{\varphi\in\Phi}\int_K f(zh\varphi(ykxk^{-1})h^{-1})d\omega_K(h)d\omega_K(k)\right.$$
$$\left.-|\Phi|\int_K f(z)f(ykxk^{-1})d\omega_K(k)\right|$$
$$\leq 2\delta.$$

Since f is unbounded we get

$$\int_K f(xkyk^{-1})d\omega_K(k)=\int_K f(ykxk^{-1})d\omega_K(k),\quad x,y\in G.$$

□

The main result is the following theorem.

Theorem 3.2. *Let $\delta>0$ and let $f\in\mathcal{C}(G)$ such that f satisfies the condition (*) and the functional inequality*

$$\left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k)-|\Phi|f(x)f(y)\right|\leq\delta,\ \ x,y\in G. \tag{3.3}$$

Then either

$$|f(x)|\leq\frac{|\Phi|+\sqrt{|\Phi|^2+4\delta|\Phi|}}{2|\Phi|},\quad x\in G, \tag{3.4}$$

or

$$\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y)k^{-1})d\omega_K(k)=|\Phi|f(x)f(y),\ \ x,y\in G. \tag{$E(K)$}$$

Proof. The idea is inspired by the paper [1].

If f is bounded, by using Theorem 3.1, we obtain the first case of the theorem.

Now let f be an unbounded solution of the functional inequality (3.3), then there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in G such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$.

For the second case we will use the following lemma.

Lemma 3.3. *Let f be an unbounded solution of the functional inequality (3.3) satisfying the condition (*) and let $(z_n)_{n\in\mathbb{N}}$ be a sequence in G such that $f(z_n) \neq 0$ and $\lim_n |f(z_n)| = +\infty$. It follows that the convergence of the sequences of functions:*

i)

$$x \longmapsto \frac{\sum_{\varphi\in\Phi}\int_K f(z_n k\varphi(x)k^{-1})d\omega_K(k)}{f(z_n)}, \quad n \in \mathbb{N}, \tag{3.5}$$

to the function

$$x \longmapsto |\Phi| f(x).$$

ii)

$$x \longmapsto \frac{\sum_{\varphi\in\Phi}\int_K f(z_n h\varphi(xk\varphi(\tau(y))k^{-1})h^{-1})d\omega_K(h)}{f(z_n)}, \quad n \in \mathbb{N},\ \tau \in \Phi,\ k \in K,\ y \in G \tag{3.6}$$

to the function

$$x \longmapsto |\Phi| f(xk\tau(y)k^{-1}) \quad \tau \in \Phi,\ k \in K,\ y \in G,$$

is uniform.

By inequality (3.1), we have

$$\left| \frac{\sum_{\varphi\in\Phi} \int_K f(z_n k\varphi(y)k^{-1}) d\omega_K(k)}{f(z_n)} - |\Phi| f(y) \right| \le \frac{\delta}{|f(z_n)|},$$

then we have, by letting $n \longmapsto +\infty$, that

$$\lim_n \frac{\sum_{\varphi\in\Phi} \int_K f(z_n k\varphi(y)k^{-1}) d\omega_K(k)}{f(z_n)} = |\Phi| f(y),$$

and

$$\lim_n \frac{\sum_{\varphi\in\Phi} \int_K f\left(z_n h\varphi\left(xk\varphi\left(\tau(y)\right)k^{-1}\right)h^{-1}\right) d\omega_K(h)}{f(z_n)} = |\Phi| f(xk\tau(y)k^{-1}).$$

Since by Proposition 2.3, we have

$$\sum_{\tau\in\Phi} \int_K \frac{\sum_{\varphi\in\Phi} \int_K f(z_n h\varphi(x)k\varphi(\tau(y))k^{-1}h^{-1}) d\omega_K(h)}{f(z_n)} d\omega_K(k)$$
$$= \sum_{\psi\in\Phi} \int_K \frac{\sum_{\varphi\in\Phi} \int_K f(z_n h\varphi(x)k\psi(y)k^{-1}h^{-1}) d\omega_K(h)}{f(z_n)} d\omega_K(k),$$

combining this and the fact that f satisfies the condition (*), we obtain

$$\left| \sum_{\tau\in\Phi} \int_K \frac{\sum_{\varphi\in\Phi}\int_K f(z_n h\varphi(x)k\varphi(\tau(y))k^{-1}h^{-1})d\omega_K(h)}{f(z_n)} d\omega_K(k) \right.$$
$$\left. -|\Phi| f(x) \frac{\sum_{\psi\in\Phi}\int_K f(z_n k\psi(y)k^{-1})d\omega_K(k)}{f(z_n)} \right| \le \frac{\delta}{|f(z_n)|}.$$

Since the convergence is uniform, we have

$$\left| |\Phi| \sum_{\varphi\in\Phi} \int_K f(xk\varphi(y)k^{-1})d\omega_K(k) - |\Phi|^2 f(x)f(y) \right| \le 0,$$

thus ($E\,(K)$) holds and the proof is complete. □

4. Applications

If $K \subset Z(G)$, we obtain the following corollary.

Corollary 4.1. *Let $\delta > 0$ and let f be a complex-valued function on G satisfying the Kannappan condition (see [10])*

(*) $$f(zxy) = f(zyx), \quad x, y \in G,$$

and the functional inequality

(4.1) $$\left| \sum_{\varphi \in \Phi} f(x\varphi(y)) - |\Phi| f(x) f(y) \right| \leq \delta, \quad x, y \in G.$$

Then either

(4.2) $$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G,$$

or

(4.3) $$\sum_{\varphi \in \Phi} f(x\varphi(y)) = |\Phi| f(x) f(y), \quad x, y \in G.$$

If G is abelian, then the condition (*) holds and we have the following:

If $\Phi = \{i\}$ (resp. $\Phi = \{i, \sigma\}$), where $i(x) = x$ and $\sigma(x) = -x$, we find the Baker's stability see [8] (resp. [9]).

If $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$ and $x \in G$, where χ is a character of K (see [11]), then we have the following corollary.

Corollary 4.2. *Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ such that $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$,*

$$(*)\quad \int_K \int_K f(zkxhy)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h) = \int_K \int_K f(zkyhx)\overline{\chi}(k)\overline{\chi}(h)d\omega_K(k)d\omega_K(h)$$

and

$$(4.4)\quad \left| \sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k) - |\Phi|f(x)f(y) \right| \le \delta, \quad x, y \in G.$$

Then either

$$(4.5)\quad |f(x)| \le \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G,$$

or

$$(4.6)\quad \sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))\overline{\chi}(k)d\omega_K(k) = |\Phi|f(x)f(y), \quad x, y \in G.$$

Proposition 4.3. *If the algebra $\overline{\chi}\omega_K \star M(G) \star \overline{\chi}\omega_K$ is commutative then the condition (*) holds.*

Proof. Since $f(kxh) = \chi(k)f(x)\chi(h)$, $k, h \in K$, $x \in G$, then we have $\chi\omega_K \star f \star \chi\omega_K = f$. Suppose that the algebra $\overline{\chi}\omega_K \star M(G) \star \overline{\chi}\omega_K$ is commutative, then we get:

$$\begin{aligned}
&\int_K\int_K f(xkyk^{-1}hzh^{-1})d\omega_K(k)d\omega_K(h)\\
&\qquad = \int_K\int_K f(xkyhzh^{-1}k^{-1})d\omega_K(k)d\omega_K(h)\\
&\qquad = \langle \delta_z \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K \star \delta_x, f\rangle\\
&\qquad = \langle \delta_z \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K \star \delta_x, \chi\omega_K \star f \star \chi\omega_K\rangle\\
&\qquad = \langle \overline{\chi}\omega_K \star \delta_z \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K \star \delta_x \star \overline{\chi}\omega_K, f\rangle\\
&\qquad = \langle \overline{\chi}\omega_K \star \delta_z \star \overline{\chi}\omega_K \star \delta_x \star \overline{\chi}\omega_K \star \delta_y \star \overline{\chi}\omega_K, f\rangle\\
&\qquad = \int_K\int_K f(ykxk^{-1}hzh^{-1})d\omega_K(k)d\omega_K(h).
\end{aligned}$$

□

Let f be bi-K-invariant (i.e $f(hxk) = f(x), h, k \in K, x \in G$), then we have:

Corollary 4.4. *Let $\delta > 0$ and let $f \in \mathcal{C}(G)$ be bi-K-invariant such that for all $x, y, z \in G$,*

$$\text{(*)}\qquad \int_K\int_K f(zkxhy)d\omega_K(k)d\omega_K(h) = \int_K\int_K f(zkyhx)d\omega_K(k)d\omega_K(h),$$

and

$$\text{(4.7)}\qquad \left|\sum_{\varphi\in\Phi}\int_K f(xk\varphi(y))d\omega_K(k) - |\Phi|f(x)f(y)\right| \le \delta, \quad x, y \in G.$$

Then either

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G, \tag{4.8}$$

or

$$\sum_{\varphi\in\Phi} \int_K f(xk\varphi(y))d\omega_K(k) = |\Phi| f(x)f(y), \;\; x, y \in G. \tag{4.9}$$

Proposition 4.5. *If the pair (G, K) is a Gelfand pair (i.e $\omega_K \star M(G) \star \omega_K$ is commutative), then the condition (*) holds.*

Proof. We take $\chi = 1$ (unit character of K) in Proposition 4.3 (see [3] and [6]). ☐

In the next corollary, we assume that $G = K$ is a compact group.

Lemma 4.6. *If f is central, then f satisfies the condition (*). Consequently, we have*

$$\int_G f(xtyt^{-1})dt = \int_G f(ytxt^{-1})dt, \quad x, y \in G. \tag{4.10}$$

Corollary 4.7. *Let $\delta > 0$ and let f be a complex measurable and essentially bounded function on G such that*

$$\left| \sum_{\varphi\in\Phi} \int_G f(xt\varphi(y)t^{-1})dt - |\Phi| f(x)f(y) \right| \leq \delta, \;\; x, y \in G. \tag{4.11}$$

Then

$$|f(x)| \leq \frac{|\Phi| + \sqrt{|\Phi|^2 + 4\delta|\Phi|}}{2|\Phi|}, \quad x \in G. \tag{4.12}$$

Proof. Let $f \in L_\infty(G)$ be a solution of the inequality (4.11), then f is bounded, if not, then f satisfies the second case of Theorem 3.2 which implies that f is central (i.e the condition (*) holds) and f is a solution of the following functional equation

$$\sum_{\varphi\in\Phi} \int_G f(xt\varphi(y)t^{-1})dt = |\Phi| f(x)f(y), \quad x, y \in G. \tag{4.13}$$

In view of the proposition in [5], we have that f is continuous. Since G is compact, then the proof is accomplished. □

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