Journal of Inequalities in Pure and Applied Mathematics

KANTOROVICH TYPE INEQUALITIES FOR $1 > p > 0$

MARIKO GIGA

Department of Mathematics
Nippon Medical School
2-297-2 Kosugi Nakahara-ku
Kawasaki 211-0063 Japan.
EMail: mariko@nms.ac.jp

ISSN (electronic): 1443-5756
089-03

volume 4, issue 5, article 105, 2003.

Received 24 May, 2003; accepted 28 June, 2003.

Communicated by: T. Furuta

Abstract

We shall discuss operator inequalities for $1 > p > 0$ associated with Hölder-McCarthy and Kantorovich inequalities.

2000 Mathematics Subject Classification: 47A63
Key words: Kantorovich type inequality, Order preserving inequality, Concave function.

Contents

1. Introduction

In this paper, an operator is taken to be a bounded linear operator on a Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$, also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The celebrated Kantorovich inequality asserts that if T is a strictly positive operator such that $MI \geq T \geq mI > 0$, then $(T^{-1}x, x)(Tx, x) \leq \frac{(m+M)^2}{4mM}$ holds for every unit vector x in H. There have been many papers published on Kantorovich type inequalities, some of them are the papers of B. Mond and J. Pečarić [9], [10], and [11]. Other examples of Kantorovich type inequalities can be found in the work of Furuta [4] and the extended work [8]. More general results may be seen in the work of Li and Mathias in [7]. We shall discuss operator inequalities for $1 > p > 0$ associated with the Hölder-McCarthy and Kantorovich inequalities as a complementary result of [6].

2. Operator Inequalities for $1 > p > 0$ Associated with Hölder-McCarthy and Kantorovich Inequalities

Theorem 2.1. *Let T be a strictly positive operator on a Hilbert space H such that $MI \geq T \geq mI > 0$, where $M > m > 0$. Also, let $f(t)$ be a real valued continuous concave function on $[m, M]$ and let $1 > q > 0$.*

Then the following inequality holds for every unit vector x:

$$f((Tx,x)) \geq (f(T)x,x) \geq K(m,M,f,q)(Tx,x)^q, \tag{2.1}$$

where $K(m,M,f,q)$ is defined by

$$K(m,M,f,q) = \begin{cases} B_1 = \frac{(mf(M)-Mf(m))}{(q-1)(M-m)} \left(\frac{(q-1)(f(M)-f(m))}{q(mf(M)-Mf(m))}\right)^q & \text{if Case 1 holds;} \\ B_2 = \frac{f(m)}{m^q} & \text{if Case 2 holds;} \\ B_3 = \frac{f(M)}{M^q} & \text{if Case 3 holds,} \end{cases}$$

where Case 1, Case 2 and Case 3 are as follows:

Case 1: $f(M) > f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and* $\frac{f(m)}{m}q \geq \frac{f(M)-f(m)}{M-m} \geq \frac{f(M)}{M}q$,

Case 2: $f(M) > f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and* $\frac{f(m)}{m}q < \frac{f(M)-f(m)}{M-m}$,

Case 3: $f(M) > f(m)$, $\frac{f(M)}{M} < \frac{f(m)}{m}$ *and* $\frac{f(M)}{M}q > \frac{f(M)-f(m)}{M-m}$.

Theorem 2.1 easily implies the following result.

Corollary 2.2. *Let T be a strictly positive operator on a Hilbert space H such that $MI \geq T \geq mI > 0$, where $M > m > 0$. Also let $1 > p > 0$ and $1 > q > 0$, then we have*

$$(Tx,x)^p \geq (T^p x,x) \geq K(m,M,p,q)(Tx,x)^q, \tag{2.2}$$

where $K(m,M,p,q)$ is defined by

$$K(m,M,p,q) = \begin{cases} K^{(1)}(m,M,p,q) & \text{if } m^{p-1}q \geq \dfrac{M^p - m^p}{M-m} \geq M^{p-1}q; \\ m^{p-q} & \text{if } m^{p-1}q < \dfrac{M^p - m^p}{M-m}; \\ M^{p-q} & \text{if } M^{p-1}q > \dfrac{M^p - m^p}{M-m}, \end{cases}$$

where $K^{(1)}(m,M,p,q)$ is defined by

$$K^{(1)}(m,M,p,q) = \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q. \tag{2.3}$$

3. Proofs of the Results in §2

We state the following fundamental lemma before giving proofs of the results in §2.

Lemma 3.1. *Let $h(t)$ be defined by (3.1) on $(0,\infty)$ for any real number q such that $q \in (0,1)$ and any real numbers K and k, and $M > m > 0$*

$$h(t) = \frac{1}{t^q}\left(k + \frac{K-k}{M-m}(t-m)\right). \tag{3.1}$$

Then $h(t)$ has the following lower bound $BD(m,M,k,K,q)$ on $[m,M]$:

$$BD(m,M,k,K,q) = \begin{cases} B_1 = \frac{(mK-Mk)}{(q-1)(M-m)}\left(\frac{(q-1)(K-k)}{q(mK-Mk)}\right)^q & \text{if Case 1 holds;} \\ B_2 = \frac{k}{m^q} & \text{if Case 2 holds;} \\ B_3 = \frac{K}{M^q} & \text{if Case 3 holds,} \end{cases}$$

where Case 1, Case 2 and Case 3 are as follows:

$$\text{Case 1: } K > k,\ \frac{K}{M} < \frac{k}{m} \text{ and } \frac{k}{m}q \geq \frac{K-k}{M-m} \geq \frac{K}{M}q;$$

$$\text{Case 2: } K > k,\ \frac{K}{M} < \frac{k}{m} \text{ and } \frac{k}{m}q < \frac{K-k}{M-m};$$

$$\text{Case 3: } K > k,\ \frac{K}{M} < \frac{k}{m} \text{ and } \frac{K}{M}q > \frac{K-k}{M-m}.$$

Proof. We have that $h'(t_1) = 0$ when

$$t_1 = \frac{q}{(q-1)} \cdot \frac{(mK - Mk)}{(K-k)} \quad \text{and} \quad h''(t_1) = \frac{-q(mK-Mk)}{(M-m)t_1^{q+2}},$$

and the conditions in Case 1 ensure that $m \le t_1 \le M$, $h''(t_1) > 0$ and $h(t)$ has the lower bound $B_1 = h(t_1)$ on $[m, M]$. By the geometric properties of $h(t)$, the conditions in Case 2 ensure that $0 < t_1 < m$ and $h(t)$ has the lower bound $B_2 = h(m)$ on $[m, M]$. Also the conditions in Case 3 ensure that $t_1 > M$ and $h(t)$ has the lower bound $B_3 = h(M)$ on $[m, M]$. □

Proof of Theorem 2.1. As $f(t)$ is a real valued continuous concave function on $[m, M]$, we have

$$f(t) \ge f(m) + \frac{f(M) - f(m)}{M - m}(t - m) \quad \text{for any } t \in [m, M]. \tag{3.2}$$

By applying the standard operational calculus of positive operator T to (3.1), since $M \ge (Tx, x) \ge m$, we obtain for every unit vector x

$$(f(T)x, x) \ge f(m) + \frac{f(M) - f(m)}{M - m}((Tx, x) - m). \tag{3.3}$$

Multiplying by $(Tx, x)^{-q}$ on both sides of (3.2), we have

$$(Tx, x)^{-q}(f(T)x, x) \ge h((Tx, x)), \tag{3.4}$$

where

$$h(t) = t^{-q}\left(f(m) + \frac{f(M) - f(m)}{M - m}(t - m)\right).$$

Then we obtain

$$(f(T)x,x) \geq \left[\min_{m\leq t\leq M} h(t)\right](Tx,x)^q. \tag{3.5}$$

Putting $K = f(M)$ and $k = f(m)$ in Lemma 3.1, so that the latter inequality of (2.1) follows by (3.5) and Lemma 3.1 and the former inequality in (2.1) follows by the Jensen inequality (for examples, see [1], [2], [3] and [7]) since $f(t)$ is a concave function. Whence the proof is complete by Lemma 3.1. □

Proof of Corollary 2.2. Put $f(t) = t^p$ for $p \in (0,1)$ in Theorem 2.1. As $f(t)$ is a real valued continuous concave function on $[m,M]$, $M^p > m^p$ and $M^{p-1} < m^{p-1}$ hold for any $p \in (0,1)$, that is, $f(M) > f(m)$ and $\frac{f(M)}{M} < \frac{f(m)}{m}$ for any $p \in (0,1)$.

Whence the proof of Corollary 2.2 is complete by Theorem 2.1. □

4. Application of Corollary 2.2 to Kantorovich Type Operator Inequalities

Theorem 4.1. *Let A and B be two strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$, where $M_1 > m_1 > 0$ and $M_2 > m_2 > 0$ and $A \geq B$.*

(a) *If $p > 1$ and $q > 1$, then the following inequality holds:*

$$K(m_2, M_2, p, q) A^q \geq B^p,$$

where $K(m_1, M_1, p, q)$ is defined by

$$K(m_2, M_2, p, q) = \begin{cases} K^{(1)}(m_2, M_2, p, q) & \text{if } m_2^{p-1} q \leq \frac{M_2^p - m_2^p}{M_2 - m_2} \leq M_2^{p-1} q; \\ m_2^{p-q} & \text{if } m_2^{p-1} q > \frac{M_2^p - m_2^p}{M_2 - m_2}; \\ M_2^{p-q} & \text{if } M_2^{p-1} q < \frac{M_2^p - m_2^p}{M_2 - m_2}. \end{cases}$$

(b) *If $p < 0$ and $q < 0$, then the following inequality holds:*

$$K(m_1, M_1, p, q) B^q \geq A^p,$$

where $K(m_1, M_1, p, q)$ is defined by

$$K(m_1, M_1, p, q) = \begin{cases} K^{(1)}(m_1, M_1, p, q) & \text{if } m_1^{p-1} q \leq \frac{M_1^p - m_1^p}{M_1 - m_1} \leq M_1^{p-1} q; \\ m_1^{p-q} & \text{if } m_1^{p-1} q > \frac{M_1^p - m_1^p}{M_1 - m_1}; \\ M_1^{p-q} & \text{if } M_1^{p-1} q < \frac{M_1^p - m_1^p}{M_1 - m_1}. \end{cases}$$

(c) If $1 > p > 0$ and $1 > q > 0$, then the following inequality holds:

$$A^p \geq K(m_1, M_1, p, q) B^q, \tag{4.1}$$

$$K(m_1, M_1, p, q) = \begin{cases} K^{(1)}(m_1, M_1, p, q) & \text{if } m_1^{p-1} q \geq \frac{M_1^p - m_1^p}{M_1 - m_1} \geq M_1^{p-1} q; \\ m_1^{p-q} & \text{if } m_1^{p-1} q < \frac{M_1^p - m_1^p}{M_1 - m_1}; \\ M_1^{p-q} & \text{if } M_1^{p-1} q > \frac{M_1^p - m_1^p}{M_1 - m_1}, \end{cases}$$

where $K^{(1)}(m, M, p, q)$ in (a), (b) and (c) is defined in (2.3).

Proof. We have only to prove (c) since (a) and (b) are both shown in [6].

Proof of (c). For every unit vector x, $1 > p > 0$ and $1 > q > 0$, we have

$$\begin{aligned} (A^p x, x) &\geq K(m_1, M_1, p, q)(Ax, x)^q && \text{by Corollary 2.2} \\ &\geq K(m_1, M_1, p, q)(Bx, x)^q && \text{since } A \geq B > 0 \text{ and } 1 > q > 0 \\ &\geq K(m_1, M_1, p, q)(B^q x, x) && \text{by the Hölder-McCarthy inequality,} \\ & && \text{since } 1 > q > 0 \end{aligned}$$

so that (4.1) is shown and the proof is complete. □

Corollary 4.2. *Let A and B be two strictly positive operators on a Hilbert space H such that $M_1 I \geq A \geq m_1 I > 0$ and $M_2 I \geq B \geq m_2 I > 0$, where $M_1 > m_1 > 0$, $M_2 > m_2 > 0$ and $A \geq B$.*

(i) If $p > 1$, then the following inequality holds

$$K^{(1)}(m_2, M_2, p) A^p \geq B^p.$$

(ii) If $p < 0$, then then the following inequality holds

$$K^{(1)}(m_1, M_1, p)B^p \geq A^p,$$

where

$$K^{(1)}(m, M, p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left(\frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p .$$

Proof of Corollary 4.2. Since t^p is a convex function for $p > 1$ or $p < 0$, and t^p is a concave function for $1 > p > 0$, we have only to put $p = q$ in Theorem 4.1. □

Remark 4.1. *We remark that (i) of Corollary 4.2 is shown in [4, Theorem 2.1] and Theorem 1 in §3.6.2 of [5]. In the case $p = q \in (0, 1)$, the result (4.1) may be given as follows: $A \geq B > 0$ ensures that $A^p \geq B^p \geq K(m_1, M_1, p, p)B^p$ for all $p \in (0, 1)$. In fact, the first inequality follows by the Löwner-Heinz inequality and the second one holds since $K(m_1, M_1, p, p) \leq 1$ which is derived from (2.2).*

Remark 4.2. *We remark that for $p > 1$ and $q > 1$, $K^{(1)}(m, M, p, q)$ can be rewritten as*

$$\begin{aligned} K^{(1)}(m, M, p, q) &= \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left(\frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q \\ &= \frac{(q-1)^{q-1}}{q^q} \frac{(M^p - m^p)^q}{(M-m)(mM^p - m^p)^{q-1}} \end{aligned}$$

and in fact this latter simple form is in [6].

References

[1] T. ANDO, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Linear Alg. and Appl.*, **26** (1979), 203–241.

[2] M.D. CHOI, A Schwarz inequality for positive linear maps on C^*-algebras, *Illinois J. Math*, **18** (1974), 565–574.

[3] C. DAVIS, A Schwarz inequality for convex operator functions, *Proc. Amer. Math. Soc.*, **8** (1957), 42–44.

[4] T. FURUTA, Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities, *J. Inequal. and Appl.*, **2** (1998),137–148.

[5] T. FURUTA, *Invitation to Linear Operators*, Taylor & Francis, London, 2001.

[6] T. FURUTA AND M. GIGA, A complementary result of Kantorovich type order preserving inequalities by J.Mićić-J.Pečarić-Seo, to appear in *Linear Alg. and Appl.*

[7] C.-K. LI AND R. MATHIAS, Matrix inequalities involving positive linear map, *Linear and Multilinear Alg.*, **41** (1996), 221–231.

[8] J. MIĆIĆ, J. PEČARIĆ AND Y. SEO, Function order of positive operators based on the Mond-Pečarić method, *Linear Alg. and Appl.*, **360** (2003), 15–34.

[9] B. MOND AND J. PEČARIĆ, Convex inequalities in Hilbert space, *Houston J. Math.*, **19** (1993), 405–420.

[10] B. MOND AND J. PEČARIĆ, A matrix version of Ky Fan Generalization of the Kantorovich inequality, *Linear and Multilinear Algebra*, **36** (1994), 217–221.

[11] B. MOND AND J. PEČARIĆ, Bound for Jensen's inequality for several operators, *Houston J. Math.*, **20** (1994), 645–651.