

# Journal of Inequalities in Pure and Applied Mathematics

KANTOROVICH TYPE INEQUALITIES FOR  $1 > p > 0$

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©2000 Victoria University  
ISSN (electronic): 1443-5756  
089-03



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volume 4, issue 5, article 105,  
2003.

*Received 24 May, 2003;  
accepted 28 June, 2003.*

*Communicated by: T. Furuta*

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## Abstract

We shall discuss operator inequalities for  $1 > p > 0$  associated with Hölder-McCarthy and Kantorovich inequalities.

*2000 Mathematics Subject Classification:* 47A63

*Key words:* Kantorovich type inequality, Order preserving inequality, Concave function.

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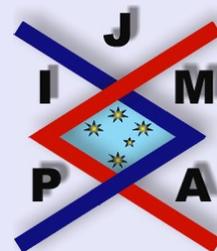
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# 1. Introduction

In this paper, an operator is taken to be a bounded linear operator on a Hilbert space  $H$ . An operator  $T$  is said to be positive (denoted by  $T \geq 0$ ) if  $(Tx, x) \geq 0$ , also  $T$  is said to be strictly positive (denoted by  $T > 0$ ) if  $T$  is positive and invertible. The celebrated Kantorovich inequality asserts that if  $T$  is a strictly positive operator such that  $MI \geq T \geq mI > 0$ , then  $(T^{-1}x, x)(Tx, x) \leq \frac{(m+M)^2}{4mM}$  holds for every unit vector  $x$  in  $H$ . There have been many papers published on Kantorovich type inequalities, some of them are the papers of B. Mond and J. Pečarić [9], [10], and [11]. Other examples of Kantorovich type inequalities can be found in the work of Furuta [4] and the extended work [8]. More general results may be seen in the work of Li and Mathias in [7]. We shall discuss operator inequalities for  $1 > p > 0$  associated with the Hölder-McCarthy and Kantorovich inequalities as a complementary result of [6].



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## 2. Operator Inequalities for $1 > p > 0$ Associated with Hölder-McCarthy and Kantorovich Inequalities

**Theorem 2.1.** Let  $T$  be a strictly positive operator on a Hilbert space  $H$  such that  $MI \geq T \geq mI > 0$ , where  $M > m > 0$ . Also, let  $f(t)$  be a real valued continuous concave function on  $[m, M]$  and let  $1 > q > 0$ .

Then the following inequality holds for every unit vector  $x$ :

$$(2.1) \quad f((Tx, x)) \geq (f(T)x, x) \geq K(m, M, f, q)(Tx, x)^q,$$

where  $K(m, M, f, q)$  is defined by

$$K(m, M, f, q)$$

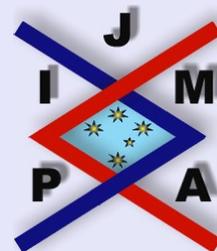
$$= \begin{cases} B_1 = \frac{(mf(M)-Mf(m))}{(q-1)(M-m)} \left( \frac{(q-1)(f(M)-f(m))}{qmf(M)-Mf(m)} \right)^q & \text{if Case 1 holds;} \\ B_2 = \frac{f(m)}{m^q} & \text{if Case 2 holds;} \\ B_3 = \frac{f(M)}{M^q} & \text{if Case 3 holds,} \end{cases}$$

where Case 1, Case 2 and Case 3 are as follows:

$$\text{Case 1: } f(M) > f(m), \quad \frac{f(M)}{M} < \frac{f(m)}{m} \quad \text{and} \quad \frac{f(m)}{m} q \geq \frac{f(M) - f(m)}{M - m} \geq \frac{f(M)}{M} q,$$

$$\text{Case 2: } f(M) > f(m), \quad \frac{f(M)}{M} < \frac{f(m)}{m} \quad \text{and} \quad \frac{f(m)}{m} q < \frac{f(M) - f(m)}{M - m},$$

$$\text{Case 3: } f(M) > f(m), \quad \frac{f(M)}{M} < \frac{f(m)}{m} \quad \text{and} \quad \frac{f(M)}{M} q > \frac{f(M) - f(m)}{M - m}.$$



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Theorem 2.1 easily implies the following result.

**Corollary 2.2.** *Let  $T$  be a strictly positive operator on a Hilbert space  $H$  such that  $MI \geq T \geq mI > 0$ , where  $M > m > 0$ . Also let  $1 > p > 0$  and  $1 > q > 0$ , then we have*

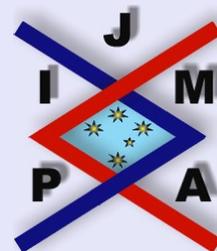
$$(2.2) \quad (Tx, x)^p \geq (T^p x, x) \geq K(m, M, p, q)(Tx, x)^q,$$

where  $K(m, M, p, q)$  is defined by

$$K(m, M, p, q) = \begin{cases} K^{(1)}(m, M, p, q) & \text{if } m^{p-1}q \geq \frac{M^p - m^p}{M - m} \geq M^{p-1}q; \\ m^{p-q} & \text{if } m^{p-1}q < \frac{M^p - m^p}{M - m}; \\ M^{p-q} & \text{if } M^{p-1}q > \frac{M^p - m^p}{M - m}, \end{cases}$$

where  $K^{(1)}(m, M, p, q)$  is defined by

$$(2.3) \quad K^{(1)}(m, M, p, q) = \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left( \frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q.$$




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### 3. Proofs of the Results in §2

We state the following fundamental lemma before giving proofs of the results in §2.

**Lemma 3.1.** *Let  $h(t)$  be defined by (3.1) on  $(0, \infty)$  for any real number  $q$  such that  $q \in (0, 1)$  and any real numbers  $K$  and  $k$ , and  $M > m > 0$*

$$(3.1) \quad h(t) = \frac{1}{t^q} \left( k + \frac{K - k}{M - m} (t - m) \right).$$

Then  $h(t)$  has the following lower bound  $BD(m, M, k, K, q)$  on  $[m, M]$ :

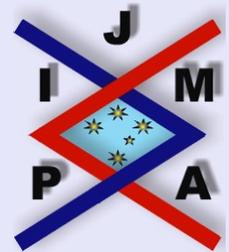
$$BD(m, M, k, K, q) = \begin{cases} B_1 = \frac{(mK - Mk)}{(q-1)(M-m)} \left( \frac{(q-1)(K-k)}{q(mK - Mk)} \right)^q & \text{if Case 1 holds;} \\ B_2 = \frac{k}{m^q} & \text{if Case 2 holds;} \\ B_3 = \frac{K}{M^q} & \text{if Case 3 holds,} \end{cases}$$

where Case 1, Case 2 and Case 3 are as follows:

$$\text{Case 1: } K > k, \frac{K}{M} < \frac{k}{m} \text{ and } \frac{k}{m} q \geq \frac{K - k}{M - m} \geq \frac{K}{M} q;$$

$$\text{Case 2: } K > k, \frac{K}{M} < \frac{k}{m} \text{ and } \frac{k}{m} q < \frac{K - k}{M - m};$$

$$\text{Case 3: } K > k, \frac{K}{M} < \frac{k}{m} \text{ and } \frac{K}{M} q > \frac{K - k}{M - m}.$$



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*Proof.* We have that  $h'(t_1) = 0$  when

$$t_1 = \frac{q}{(q-1)} \cdot \frac{(mK - Mk)}{(K - k)} \quad \text{and} \quad h''(t_1) = \frac{-q(mK - Mk)}{(M - m)t_1^{q+2}},$$

and the conditions in Case 1 ensure that  $m \leq t_1 \leq M$ ,  $h''(t_1) > 0$  and  $h(t)$  has the lower bound  $B_1 = h(t_1)$  on  $[m, M]$ . By the geometric properties of  $h(t)$ , the conditions in Case 2 ensure that  $0 < t_1 < m$  and  $h(t)$  has the lower bound  $B_2 = h(m)$  on  $[m, M]$ . Also the conditions in Case 3 ensure that  $t_1 > M$  and  $h(t)$  has the lower bound  $B_3 = h(M)$  on  $[m, M]$ .  $\square$

*Proof of Theorem 2.1.* As  $f(t)$  is a real valued continuous concave function on  $[m, M]$ , we have

$$(3.2) \quad f(t) \geq f(m) + \frac{f(M) - f(m)}{M - m}(t - m) \quad \text{for any } t \in [m, M].$$

By applying the standard operational calculus of positive operator  $T$  to (3.1), since  $M \geq (Tx, x) \geq m$ , we obtain for every unit vector  $x$

$$(3.3) \quad (f(T)x, x) \geq f(m) + \frac{f(M) - f(m)}{M - m}((Tx, x) - m).$$

Multiplying by  $(Tx, x)^{-q}$  on both sides of (3.2), we have

$$(3.4) \quad (Tx, x)^{-q}(f(T)x, x) \geq h((Tx, x)),$$

where

$$h(t) = t^{-q} \left( f(m) + \frac{f(M) - f(m)}{M - m}(t - m) \right).$$



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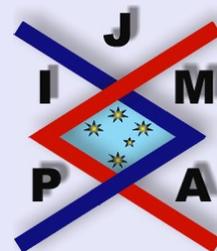
Then we obtain

$$(3.5) \quad (f(T)x, x) \geq \left[ \min_{m \leq t \leq M} h(t) \right] (Tx, x)^q.$$

Putting  $K = f(M)$  and  $k = f(m)$  in Lemma 3.1, so that the latter inequality of (2.1) follows by (3.5) and Lemma 3.1 and the former inequality in (2.1) follows by the Jensen inequality (for examples, see [1], [2], [3] and [7]) since  $f(t)$  is a concave function. Whence the proof is complete by Lemma 3.1.  $\square$

*Proof of Corollary 2.2.* Put  $f(t) = t^p$  for  $p \in (0, 1)$  in Theorem 2.1. As  $f(t)$  is a real valued continuous concave function on  $[m, M]$ ,  $M^p > m^p$  and  $M^{p-1} < m^{p-1}$  hold for any  $p \in (0, 1)$ , that is,  $f(M) > f(m)$  and  $\frac{f(M)}{M} < \frac{f(m)}{m}$  for any  $p \in (0, 1)$ .

Whence the proof of Corollary 2.2 is complete by Theorem 2.1.  $\square$



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## 4. Application of Corollary 2.2 to Kantorovich Type Operator Inequalities

**Theorem 4.1.** Let  $A$  and  $B$  be two strictly positive operators on a Hilbert space  $H$  such that  $M_1 I \geq A \geq m_1 I > 0$  and  $M_2 I \geq B \geq m_2 I > 0$ , where  $M_1 > m_1 > 0$  and  $M_2 > m_2 > 0$  and  $A \geq B$ .

(a) If  $p > 1$  and  $q > 1$ , then the following inequality holds:

$$K(m_2, M_2, p, q)A^q \geq B^p,$$

where  $K(m_1, M_1, p, q)$  is defined by

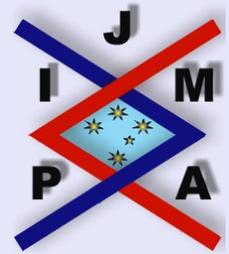
$$K(m_2, M_2, p, q) = \begin{cases} K^{(1)}(m_2, M_2, p, q) & \text{if } m_2^{p-1}q \leq \frac{M_2^p - m_2^p}{M_2 - m_2} \leq M_2^{p-1}q; \\ m_2^{p-q} & \text{if } m_2^{p-1}q > \frac{M_2^p - m_2^p}{M_2 - m_2}; \\ M_2^{p-q} & \text{if } M_2^{p-1}q < \frac{M_2^p - m_2^p}{M_2 - m_2}. \end{cases}$$

(b) If  $p < 0$  and  $q < 0$ , then the following inequality holds:

$$K(m_1, M_1, p, q)B^q \geq A^p,$$

where  $K(m_1, M_1, p, q)$  is defined by

$$K(m_1, M_1, p, q) = \begin{cases} K^{(1)}(m_1, M_1, p, q) & \text{if } m_1^{p-1}q \leq \frac{M_1^p - m_1^p}{M_1 - m_1} \leq M_1^{p-1}q; \\ m_1^{p-q} & \text{if } m_1^{p-1}q > \frac{M_1^p - m_1^p}{M_1 - m_1}; \\ M_1^{p-q} & \text{if } M_1^{p-1}q < \frac{M_1^p - m_1^p}{M_1 - m_1}. \end{cases}$$



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(c) If  $1 > p > 0$  and  $1 > q > 0$ , then the following inequality holds:

$$(4.1) \quad A^p \geq K(m_1, M_1, p, q)B^q,$$

$$K(m_1, M_1, p, q) = \begin{cases} K^{(1)}(m_1, M_1, p, q) & \text{if } m_1^{p-1}q \geq \frac{M_1^p - m_1^p}{M_1 - m_1} \geq M_1^{p-1}q; \\ m_1^{p-q} & \text{if } m_1^{p-1}q < \frac{M_1^p - m_1^p}{M_1 - m_1}; \\ M_1^{p-q} & \text{if } M_1^{p-1}q > \frac{M_1^p - m_1^p}{M_1 - m_1}, \end{cases}$$

where  $K^{(1)}(m, M, p, q)$  in (a), (b) and (c) is defined in (2.3).

*Proof.* We have only to prove (c) since (a) and (b) are both shown in [6].

*Proof of (c).* For every unit vector  $x$ ,  $1 > p > 0$  and  $1 > q > 0$ , we have

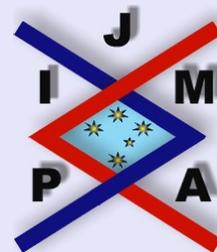
$$\begin{aligned} (A^p x, x) &\geq K(m_1, M_1, p, q)(Ax, x)^q && \text{by Corollary 2.2} \\ &\geq K(m_1, M_1, p, q)(Bx, x)^q && \text{since } A \geq B > 0 \text{ and } 1 > q > 0 \\ &\geq K(m_1, M_1, p, q)(B^q x, x) && \text{by the Hölder-McCarthy inequality,} \\ &&& \text{since } 1 > q > 0 \end{aligned}$$

so that (4.1) is shown and the proof is complete.  $\square$

**Corollary 4.2.** Let  $A$  and  $B$  be two strictly positive operators on a Hilbert space  $H$  such that  $M_1 I \geq A \geq m_1 I > 0$  and  $M_2 I \geq B \geq m_2 I > 0$ , where  $M_1 > m_1 > 0$ ,  $M_2 > m_2 > 0$  and  $A \geq B$ .

(i) If  $p > 1$ , then the following inequality holds

$$K^{(1)}(m_2, M_2, p)A^p \geq B^p.$$



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(ii) If  $p < 0$ , then the following inequality holds

$$K^{(1)}(m_1, M_1, p)B^p \geq A^p,$$

where

$$K^{(1)}(m, M, p) = \frac{(mM^p - Mm^p)}{(p-1)(M-m)} \left( \frac{(p-1)(M^p - m^p)}{p(mM^p - Mm^p)} \right)^p.$$

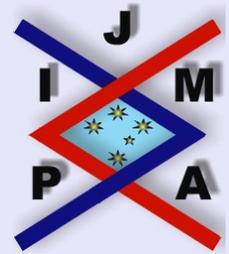
*Proof of Corollary 4.2.* Since  $t^p$  is a convex function for  $p > 1$  or  $p < 0$ , and  $t^p$  is a concave function for  $1 > p > 0$ , we have only to put  $p = q$  in Theorem 4.1.  $\square$

**Remark 4.1.** We remark that (i) of Corollary 4.2 is shown in [4, Theorem 2.1] and Theorem 1 in §3.6.2 of [5]. In the case  $p = q \in (0, 1)$ , the result (4.1) may be given as follows:  $A \geq B > 0$  ensures that  $A^p \geq B^p \geq K(m_1, M_1, p, p)B^p$  for all  $p \in (0, 1)$ . In fact, the first inequality follows by the Löwner-Heinz inequality and the second one holds since  $K(m_1, M_1, p, p) \leq 1$  which is derived from (2.2).

**Remark 4.2.** We remark that for  $p > 1$  and  $q > 1$ ,  $K^{(1)}(m, M, p, q)$  can be rewritten as

$$\begin{aligned} K^{(1)}(m, M, p, q) &= \frac{(mM^p - Mm^p)}{(q-1)(M-m)} \left( \frac{(q-1)(M^p - m^p)}{q(mM^p - Mm^p)} \right)^q \\ &= \frac{(q-1)^{q-1}}{q^q} \frac{(M^p - m^p)^q}{(M-m)(mM^p - m^p)^{q-1}} \end{aligned}$$

and in fact this latter simple form is in [6].



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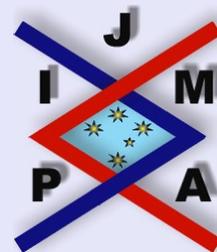
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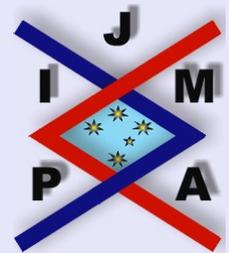
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