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ASYMPTOTIC BEHAVIOUR OF SOME EQUATIONS IN ORLICZ SPACES

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ABSTRACT. In this paper, we prove an existence and uniqueness result for solutions of some bilateral problems of the form

$$\begin{cases} \langle Au, v-u\rangle \geq \langle f, v-u\rangle,\ \forall v\in K \\ u\in K \end{cases}$$

where A is a standard Leray-Lions operator defined on $W_0^1L_M(\Omega)$, with M an N-function which satisfies the Δ_2-condition, and where K is a convex subset of $W_0^1L_M(\Omega)$ with obstacles depending on some Carathéodory function $g(x,u)$. We consider first, the case $f\in W^{-1}E_{\overline{M}}(\Omega)$ and secondly where $f\in L^1(\Omega)$. Our method deals with the study of the limit of the sequence of solutions u_n of some approximate problem with nonlinearity term of the form $|g(x,u_n)|^{n-1}g(x,u_n)\times M(|\nabla u_n|)$.

Key words and phrases: Strongly nonlinear elliptic equations, Natural growth, Truncations, Variational inequalities, Bilateral problems.

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1. INTRODUCTION

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N\geq 2$, with the segment property. Consider the following obstacle problem:

$$(\mathcal{P})\qquad \begin{cases} \langle Au, v-u\rangle \geq \langle f, v-u\rangle,\ \forall v\in K, \\ u\in K, \end{cases}$$

where $A(u) = -div(a(x,u,\nabla u))$ is a Leray-Lions operator defined on $W_0^1L_M(\Omega)$, with M being an N-function which satisfies the Δ_2-condition and where K is a convex subset of $W_0^1L_M(\Omega)$.

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In the variational case (i.e. where $f \in W^{-1}E_{\overline{M}}(\Omega)$), it is well known that problem ($\mathcal{P}$) has been already studied by Gossez and Mustonen in [10].

In this paper, we consider a recent approach of penalization in order to prove an existence theorem for solutions of some bilateral problems of ($\mathcal{P}$) type.

We recall that L. Boccardo and F. Murat, see [6], have approximated the model variational inequality:

$$\begin{cases} \langle -\Delta_p u, v-u\rangle \geq \langle f, v-u\rangle, \ \forall v \in K \\ u \in K = \{v \in W_0^{1,p}(\Omega) : |v(x)| \leq 1 \text{ a.e. in } \Omega\}, \end{cases}$$

with $f \in W^{-1,p'}(\Omega)$ and $-\Delta_p u = -div(|\nabla u|^{p-2}\nabla u)$, by the sequence of problems:

$$\begin{cases} -\Delta_p u_n + |u_n|^{n-1}u_n = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^{1,p}(\Omega) \cap L^n(\Omega). \end{cases}$$

In [7], A. Dall'aglio and L. Orsina generalized this result by taking increasing powers depending also on some Carathéodory function g satisfying the sign condition and some hypothesis of integrability. Following this idea, we have studied in [5] the sequence of problems:

$$\begin{cases} -\Delta_p u_n + |g(x,u_n)|^{n-1} g(x,u_n)|\nabla u_n|^p = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^{1,p}(\Omega), \ |g(x,u_n)|^n |\nabla u_n|^p \in L^1(\Omega) \end{cases}$$

Here, we introduce the general sequence of equations in the setting of Orlicz-Sobolev spaces

$$\begin{cases} Au_n + |g(x,u_n)|^{n-1} g(x,u_n)M(|\nabla u_n|) = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^1 L_M(\Omega), \ |g(x,u_n)|^n M(|\nabla u_n|) \in L^1(\Omega). \end{cases}$$

We are interested throughout the paper in studying the limit of the sequence u_n. We prove that this limit satisfies some bilateral problem of the ($\mathcal{P}$) form under some conditions on g. In the first we take $f \in W^{-1}E_{\overline{M}}(\Omega)$ and next in $L^1(\Omega)$.

2. PRELIMINARIES

2.1. $N-$**Functions.** Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$ and $\frac{M(t)}{t} \to \infty$ as $t \to \infty$.

Equivalently, M admits the representation: $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t)$ tends to ∞ as $t \to \infty$.

The N-function $\overline{M}$ conjugate to M is defined by $\overline{M}(t) = \int_0^t \bar{a}(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$ (see [1]).

The N-function is said to satisfy the Δ_2 condition, denoted by $M \in \Delta_2$, if for some $k > 0$:

$$M(2t) \leq kM(t) \ \ \forall t \geq 0; \tag{2.1}$$

when (2.1) holds only for $t \geq$ some $t_0 > 0$ then M is said to satisfy the Δ_2 condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, i.e. for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t\to\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

2.2. **Orlicz spaces.** Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < +\infty \quad \left(\text{resp. } \int_\Omega M(\frac{u(x)}{\lambda})dx < +\infty \text{ for some } \lambda > 0\right).$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_{M,\Omega} = \inf\left\{\lambda > 0 : \int_\Omega M(\frac{u(x)}{\lambda})dx \leq 1\right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if only if M satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$, and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. **Orlicz-Sobolev spaces.** We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ and $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwarz space $D(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma\left(\prod L_M, \prod E_{\overline{M}}\right)$ closure of $D(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \to 0 \text{ for all } |\alpha| \leq 1.$$

This implies convergence for $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$, then modular convergence coincides with norm convergence.

2.4. **The spaces $W^{-1}L_{\bar{M}}(\Omega)$ and $W^{-1}E_{\bar{M}}(\Omega)$.** Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property then the space $D(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma\left(\prod L_M, \prod L_{\overline{M}}\right)$ (cf. [8, 9]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined.

2.5. **Lemmas related to the Nemytskii operators in Orlicz spaces.** We recall some lemmas introduced in [3] which will be used in this paper.

Lemma 2.1. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let M be an $N-$function and let $u\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then $F(u)\in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u)=\begin{cases} F'(u)\frac{\partial}{\partial x_i}u & \text{a.e. in } \{x\in\Omega : u(x)\notin D\}, \\ 0 & \text{a.e. in } \{x\in\Omega : u(x)\notin D\}. \end{cases}$$

Lemma 2.2. *Let $F:\mathbb{R}\to\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N-function, then the mapping $F:W^1L_M(\Omega)\to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma\left(\prod L_M,\prod E_{\overline{M}}\right)$.*

2.6. **Abstract lemma applied to the truncation operators.** We now give the following lemma which concerns operators of the Nemytskii type in Orlicz spaces (see [3]).

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure.*

Let M,P and Q be N-functions such that $Q\ll P$, and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that a.e. $x\in\Omega$ and all $s\in\mathbb{R}$:

$$|f(x,s)|\le c(x)+k_1P^{-1}M(k_2|s|),$$

where k_1,k_2 are real constants and $c(x)\in E_Q(\Omega)$.

Then the Nemytskii operator N_f defined by $N_f(u)(x)=f(x,u(x))$ is strongly continuous from

$$\mathcal{P}\left(E_M(\Omega),\frac{1}{k_2}\right)=\left\{u\in L_M(\Omega): d(u,E_M(\Omega))<\frac{1}{k_2}\right\}$$

into $E_Q(\Omega)$.

3. THE MAIN RESULT

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N\ge 2$, with the segment property.

Let M be an N-function satisfying the Δ_2-condition near infinity.

Let $A(u)=-div(a(x,\nabla u))$ be a Leray-Lions operator defined on $W_0^1L_M(\Omega)$ into $W^{-1}L_{\overline{M}}(\Omega)$, where $a:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x\in\Omega$ and for all $\zeta,\zeta'\in\mathbb{R}^N,(\zeta\neq\zeta')$:

(3.1) $$|a(x,\zeta)|\le h(x)+\overline{M}^{-1}M(k_1|\zeta|)$$

(3.2) $$(a(x,\zeta)-a(x,\zeta'))(\zeta-\zeta')>0$$

(3.3) $$a(x,\zeta)\zeta\ge\alpha M\left(\frac{|\zeta|}{\lambda}\right)$$

with $\alpha,\lambda>0,\ k_1\ge 0,\ h\in E_{\overline{M}}(\Omega)$.

Furthermore, let $g:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$:

(3.4) $$g(x,s)s\ge 0$$

(3.5) $$|g(x,s)| \leq b(|s|)$$

(3.6) $$\begin{cases} \text{for almost } x \in \Omega \backslash \Omega_+^\infty \text{ there exists } \epsilon = \epsilon(x) > 0 \text{ such that:} \\ g(x,s) > 1, \ \forall s \in]q_+(x), q_+(x) + \epsilon[; \\ \text{for almost } x \in \Omega \backslash \Omega_-^\infty \text{ there exists } \epsilon = \epsilon(x) > 0 \text{ such that:} \\ g(x,s) < -1, \ \forall s \in]q_-(x) - \epsilon, q_-(x)[, \end{cases}$$

where $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous and nondecreasing function, with $b(0) = 0$ and where

$$q_+(x) = \inf\{s > 0 : g(x,s) \geq 1\}$$
$$q_-(x) = \sup\{s < 0 : g(x,s) \leq -1\}$$
$$\Omega_+^\infty = \{x \in \Omega : q_+(x) = +\infty\}$$
$$\Omega_-^\infty = \{x \in \Omega : q_-(x) = -\infty\}.$$

We define for s and k in $\mathbb{R}$, $k \geq 0$, $T_k(s) = \max(-k, \min(k,s))$.

Theorem 3.1. *Let $f \in W^{-1}E_{\overline{M}}(\Omega)$. Assume that (3.1) – (3.6) hold true and that the function $s \to g(x,s)$ is nondecreasing for a.e. $x \in \Omega$. Then, for any real number $\mu > 0$, the problem*

(P_n) $$\begin{cases} A(u_n) + |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) = f \text{ in } \mathcal{D}'(\Omega) \\ u_n \in W_0^1 L_M(\Omega), |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) \in L^1(\Omega) \end{cases}$$

admits at least one solution u_n such that:

(3.7) $$\forall k > 0 \qquad T_k(u_n) \to T_k(u) \text{ for modular convergence in } W_0^1 L_M(\Omega)$$

where u is the unique solution of the following bilateral problem

(P) $$\begin{cases} \langle Au, v-u \rangle \geq \langle f, v-u \rangle, \ \forall v \in K \\ u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e.}\}, \end{cases}$$

Remark 3.2. If the function $s \to g(x,s)$ is strictly nondecreasing for a.e. $x \in \Omega$ then the assumption (3.6) holds true.

Proof. **Step 1:** A priori estimates.

The existence of u_n is given by Theorem 3.1 of [3]. Choosing $v = u_n$ as a test function in (P_n), and using the sign condition (3.4), we get

$$\langle Au_n, u_n \rangle \leq \langle f, u_n \rangle.$$

By Proposition 5 of [11] one has:

(3.8) $$\int_\Omega M\left(\frac{|\nabla u_n|}{\lambda}\right) dx \leq C, \text{ and } \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n dx \leq C,$$

(3.9) $$(a(x,u_n,\nabla u_n)) \text{ is bounded in } (L_{\overline{M}}(\Omega))^N,$$

(3.10) $$\int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) u_n dx \leq C.$$

We then deduce

$$\int_{\{|u_n|>k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le C, \quad \text{for all } \ k>0.$$

Since b is continuous and since $b(0)=0$ there exists $\delta>0$ such that

$$b(|s|) \le 1 \text{ for all } |s| \le \delta.$$

On the other hand, by the Δ_2 condition there exist two positive constants K and K' such that

$$M\left(\frac{t}{\mu}\right) \le KM\left(\frac{t}{\lambda}\right) + K' \ \text{ for all } \ t \ge 0,$$

which implies

$$\int_{\{|u_n|\le\delta\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le \int_{\{|u_n|\le\delta\}} \left(K' + KM\left(\frac{|\nabla u_n|}{\lambda}\right)\right) dx.$$

Consequently from (3.8)

(3.11) $$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le C, \quad \text{for all } \ n.$$

Step 2: Almost everywhere convergence of the gradients.
Since (u_n) is a bounded sequence in $W_0^1 L_M(\Omega)$ there exist some $u \in W_0^1 L_M(\Omega)$ such that (for a subsequence still denoted by u_n)

(3.12) $$u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma\left(\prod L_M, \prod E_{\overline{M}}\right), \text{ strongly in } E_M(\Omega),$$
$$\text{and a.e. in } \Omega.$$

Furthermore, if we have

$$Au_n = f - |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)$$

with $|g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)$ being bounded in $L^1(\Omega)$ then as in [2], one can show that

(3.13) $$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

Step 3: $u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \le v \le q_+ \ \text{ a.e. in } \Omega\}$.
Since $s \to g(x,s)$ is nondecreasing, then in view of (3.6), we have:

$$\{s \in \mathbb{R} : |g(x,s)| \le 1 \text{ a.e. in } \Omega\} = \{s \in \mathbb{R} : q_- \le s \le q_+ \text{ a.e. in } \Omega\}.$$

It suffices to verify that $|g(x,u)| \le 1$ a.e.
We have

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le C,$$

which gives

$$\int_{\{|g(x,u_n)|>k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) |dx \le C$$

and

$$\int_{\{|g(x,u_n)|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le \frac{C}{k^n}$$

where $k > 1$. Letting $n \to +\infty$ for k fixed, we deduce by using Fatou's lemma

$$\int_{\{|g(x,u)|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx = 0$$

and so that,

$$|g(x,u)| \leq 1 \text{ a.e. in } \Omega.$$

Step 4: Strong convergence of the truncations.
Let $\phi(s) = s\exp(\gamma s^2)$, where γ is chosen such that $\gamma \geq \left(\frac{1}{\alpha}\right)^2$.
It is well known that $\phi'(s) - \frac{2K}{\alpha}|\phi(s)| \geq \frac{1}{2}, \forall s \in \mathbb{R}$, where K is a constant which will be used later. The use of the test function $v_n = \phi(z_n)$ in (P_n) where $z_n = T_k(u_n) - T_k(u)$ gives

$$\langle Au_n, \phi(z_n)\rangle + \int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\phi(z_n)dx = \langle f, \phi(z_n)\rangle$$

which implies, by using the fact that $g(x,u_n)\phi(z_n) \geq 0$ on $\{x \in \Omega : |u_n| > k\}$,

$$\begin{aligned}\langle Au_n, \phi(z_n)\rangle + \int_{\{0\leq u_n\leq T_k(u)\}\cap\{|u_n|\leq k\}} |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\phi(z_n)dx \\ + \int_{\{T_k(u)\leq u_n\leq 0\}\cap\{|u_n|\leq k\}} |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)\phi(z_n)dx \leq \langle f, \phi(z_n)\rangle.\end{aligned}$$

The second and the third terms of the last inequality will be denoted respectively by $I^1_{n,k}$ and $I^2_{n,k}$ and $\epsilon_i(n)$ denote various sequences of real numbers which tend to 0 as $n \to +\infty$.
On the one hand we have

$$\begin{aligned}\left|I^1_{n,k}\right| &\leq \int_{\{0\leq u_n\leq T_k(u)\}\cap\{|u_n|\leq k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx \\ &\leq \int_{\{0\leq u_n\leq u\}\cap\{|u_n|\leq k\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx,\end{aligned}$$

but since $|g(x,u_n)| \leq 1$ on $\{x \in \Omega : 0 \leq u_n \leq u\}$, then we have

$$\left|I^1_{n,k}\right| \leq \int_{\{|u_n|\leq k\}} M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx.$$

By using the fact that

$$M\left(\frac{|\nabla u_n|}{\mu}\right) \leq K' + KM\left(\frac{|\nabla u_n|}{\lambda}\right)$$

we obtain

$$\left|I^1_{n,k}\right| \leq \int_\Omega K'|\phi(z_n)|dx + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)|\phi(z_n)|dx,$$

which gives

$$\left|I^1_{n,k}\right| \leq \epsilon_1(n) + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)|\phi(z_n)|dx. \tag{3.14}$$

Similarly,

$$\left|I_{n,k}^2\right| \le \int_{\{|u_n|\le k\}} M\left(\frac{|\nabla u_n|}{\mu}\right)|\phi(z_n)|dx \le \epsilon_1(n) + \frac{K}{\alpha}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)|\phi(z_n)|dx. \tag{3.15}$$

The first term on the left hand side of the last inequality can be written as:

$$\int_\Omega a(x,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(z_n)dx = \int_{\{|u_n|\le k\}} a(x,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(z_n)dx - \int_{\{|u_n|>k\}} a(x,\nabla u_n)\nabla T_k(u)\phi'(z_n)dx. \tag{3.16}$$

For the second term on the right hand side of the last equality, we have

$$\left|\int_{\{|u_n|>k\}} a(x,\nabla u_n)\nabla T_k(u)\phi'(z_n)dx\right| \le C_k\int_\Omega |a(x,\nabla u_n)||\nabla T_k(u)|\chi_{\{|u_n|>k\}}dx.$$

The right hand side of the last inequality tends to 0 as n tends to infinity. Indeed, the sequence $(a(x,\nabla u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$ while $\nabla T_k(u)\chi_{\{|u_n|>k\}}$ tends to 0 strongly in $(E_M(\Omega))^N$.
We define for every $s>0$, $\Omega_s=\{x\in\Omega : |\nabla T_k(u(x))|\le s\}$ and we denote by χ_s its characteristic function. For the first term of the right hand side of (3.16), we can write

$$\begin{aligned}\int_{\{|u_n|\le k\}} & a(x,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(z_n)dx \\ &= \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)dx \\ &\quad + \int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)dx \\ &\quad - \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}\phi'(z_n)dx.\end{aligned} \tag{3.17}$$

The second term of the right hand side of (3.17) tends to 0 since

$$a(x,\nabla T_k(u_n)\chi_s)\phi'(z_n) \to a(x,\nabla T_k(u)\chi_s) \text{ strongly in } (E_{\overline{M}}(\Omega))^N$$

by Lemma 2.3 and

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ weakly in } (L_M(\Omega))^N \text{ for } \sigma\left(\prod L_M(\Omega), \prod E_{\overline{M}}(\Omega)\right).$$

The third term of (3.17) tends to $-\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx$ as $n\to\infty$ since

$$a(x,\nabla T_k(u_n)) \rightharpoonup a(x,\nabla T_k(u)) \text{ weakly for } \sigma\left(\prod E_{\overline{M}}(\Omega), \prod L_M(\Omega)\right).$$

Consequently, from (3.16) we have

$$\begin{aligned} \int_\Omega a(x,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(z_n)dx \\ = \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)] \\ \times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\phi'(z_n)dx + \epsilon_2(n). \end{aligned} \tag{3.18}$$

We deduce that, in view of (3.17) and (3.18),

$$\begin{aligned} \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)] \\ \times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]\left(\phi'(z_n) - \frac{2K}{\alpha}|\phi(z_n)|\right)dx \\ \leq \epsilon_3(n) + \int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx, \end{aligned}$$

and so

$$\begin{aligned} \int_\Omega [a(x,\nabla T_k(u_n)) - a(x,\nabla T_k(u)\chi_s)][\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx \\ \leq 2\epsilon_3(n) + 2\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx. \end{aligned}$$

Hence

$$\begin{aligned} \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)dx \\ \leq \int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)\chi_s dx + \int_\Omega a(x,\nabla T_k(u)\chi_s)[\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx \\ + 2\epsilon_3(n) + 2\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx. \end{aligned}$$

Now considering the limit sup over n, one has

$$\begin{aligned} \limsup_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u_n)dx \\ \leq \limsup_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u_n))\nabla T_k(u)\chi_s dx + \limsup_{n\to+\infty}\int_\Omega a(x,\nabla T_k(u)\chi_s) \\ \times [\nabla T_k(u_n)-\nabla T_k(u)\chi_s]dx + 2\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_{\Omega\setminus\Omega_s}dx. \end{aligned} \tag{3.19}$$

The second term of the right hand side of the inequality (3.19) tends to 0, since

$$a(x,\nabla T_k(u_n)\chi_s) \to a(x,\nabla T_k(u)\chi_s) \text{ strongly in } E_{\overline{M}}(\Omega),$$

while $\nabla T_k(u_n)$ tends weakly to $\nabla T_k(u)$.
The first term of the right hand side of (3.19) tends to $\int_\Omega a(x,\nabla T_k(u))\nabla T_k(u)\chi_s dx$ since

$$a(x,\nabla T_k(u_n)) \rightharpoonup a(x,\nabla T_k(u)) \text{ weakly in } (L_{\overline{M}}(\Omega))^N$$

for $\sigma\left(\prod L_{\overline{M}}, \prod E_M\right)$ while $\nabla T_k(u)\chi_s \in E_M(\Omega)$. We deduce then

$$\limsup_{n\to+\infty} \int_\Omega a(x, \nabla T_k(u_n))\nabla T_k(u_n)dx \le \int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)\chi_s dx + 2\int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)\chi_{\Omega\backslash\Omega_s}dx,$$

by using the fact that $a(x, \nabla T_k(u))\nabla T_k(u) \in L^1(\Omega)$ and letting $s \to \infty$ we get, since $meas(\Omega\backslash\Omega_s) \to 0$

$$\limsup_{n\to+\infty} \int_\Omega a(x, \nabla T_k(u_n))\nabla T_k(u_n)dx \le \int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)dx$$

which gives, by using Fatou's lemma,

$$\lim_{n\to+\infty} \int_\Omega a(x, \nabla T_k(u_n))\nabla T_k(u_n)dx = \int_\Omega a(x, \nabla T_k(u))\nabla T_k(u)dx. \tag{3.20}$$

On the other hand, we have

$$M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right) \le K' + \frac{K}{\alpha}\int_\Omega a(x, \nabla T_k(u_n))\nabla T_k(u_n)dx,$$

then by using (3.20) and Vitali's theorem, one easily has

$$M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right) \to M\left(\frac{|\nabla T_k(u)|}{\mu}\right) \text{ strongly in } L^1(\Omega). \tag{3.21}$$

By writing

$$M\left(\frac{|\nabla T_k(u_n) - \nabla T_k(u)|}{2\mu}\right) \le \frac{M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right)}{2} + \frac{M\left(\frac{|\nabla T_k(u_n)|}{\mu}\right)}{2} \tag{3.22}$$

one has, by (3.21) and Vitali's theorem again,

$$T_k(u_n) \to T_k(u) \text{ for modular convergence in } W_0^1L_M(\Omega). \tag{3.23}$$

Step 5: u is the solution of the variational inequality (P).
Choosing $w = T_k(u_n - \theta T_m(v))$ as a test function in (P_n), where $v \in K$ and $0 < \theta < 1$, gives

$$\langle Au_n, T_k(u_n - \theta T_m(v))\rangle + \int_\Omega |g(x, u_n)|^{n-1} g(x, u_n)M\left(\frac{|\nabla u_n|}{\mu}\right)T_k(u_n - \theta T_m(v))dx = \langle f, T_k(u_n - \theta T_m(v))\rangle,$$

since $g(x, u_n)T_k(u_n - \theta T_m(v)) \ge 0$ on

$$\{x \in \Omega : u_n \ge 0 \text{ and } u_n \ge \theta T_m(v)\} \cup \{x \in \Omega : u_n \le 0 \text{ and } u_n \le \theta T_m(v)\}$$

we have

$$\begin{aligned}\int_\Omega |g(x, u_n)|^{n-1} g(x, u_n)M\left(\frac{|\nabla u_n|}{\mu}\right)T_k(u_n - \theta T_m(v))dx \\ \ge \int_{\{0\le u_n\le\theta T_m(v)\}} |g(x, u_n)|^{n-1} g(x, u_n)M\left(\frac{|\nabla u_n|}{\mu}\right)T_k(u_n - \theta T_m(v))dx \\ + \int_{\{\theta T_m(v)\le u_n\le 0\}} |g(x, u_n)|^{n-1} g(x, u_n)M\left(\frac{|\nabla u_n|}{\mu}\right)T_k(u_n - \theta T_m(v))dx.\end{aligned}$$

The first and the second terms in the right hand side of the last inequality will be denoted respectively by $J^1_{n,m}$ and $J^2_{n,m}$.
Defining

$$\delta_{1,m}(x) = \sup_{0 \le s \le \theta T_m(v)} g(x,s)$$

we get $0 \le \delta_{1,m}(x) < 1$ a.e. and

$$|J^1_{n,m}| \le k \int_{\{0 \le u_n \le \theta T_m(v)\}} (\delta_{1,m}(x))^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx.$$

Since

$$\left|(\delta_{1,m}(x))^n M\left(\frac{|\nabla u_n|}{\mu}\right)\chi_{\{|u_n| \le m\}}\right| \le M\left(\frac{|\nabla T_m(u_n)|}{\mu}\right),$$

we have then by using (3.23) and Lebesgue's theorem

$$J^1_{n,m} \longrightarrow 0 \text{ as } n \to +\infty.$$

Similarly

$$|J^2_{n,m}| \le k \int_{\{|u_n| \le m\}} |\delta_{2,m}(x)|^n M\left(\frac{|\nabla T_m(u_n)|}{\mu}\right) dx \to 0 \text{ as } n \to +\infty,$$

where

$$\delta_{2,m}(x) = \inf_{\theta T_m(v) \le s \le 0} g(x,s).$$

On the other hand , by using Fatou's lemma and the fact that

$$a(x,\nabla u_n) \to a(x,\nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \text{ for } \sigma(\Pi L_{\overline{M}}, \Pi E_M),$$

one easily has

$$\liminf_{n\to+\infty} \langle Au_n, T_k(u_n - \theta T_m(v))\rangle \le \langle Au, T_k(u - \theta T_m(v))\rangle.$$

Consequently

$$\langle Au, T_k(u - \theta T_m(v))\rangle \le \langle f, T_k(u - \theta T_m(v))\rangle,$$

this implies that by letting $k \to +\infty$, since $T_k(u - \theta T_m(v)) \to u - \theta T_m(v)$ for modular convergence in $W^1_0 L_M(\Omega)$,

$$\langle Au, u - \theta T_m(v)\rangle \le \langle f, u - \theta T_m(v)\rangle,$$

in which we can easily pass to the limit as $\theta \to 1$ and $m \to +\infty$ to obtain

$$\langle Au, u - v\rangle\rangle \le \langle f, u - v\rangle.$$

□

4. THE L^1 CASE

In this section, we study the same problems as before but we assume that q_- and q_+ are bounded.

Theorem 4.1. *Let $f \in L^1(\Omega)$. Assume that the hypotheses are as in Theorem 3.1, q_- and q_+ belong to $L^\infty(\Omega)$. Then the problem (P_n) admits at least one solution u_n such that:*

$$u_n \to u \textit{ for modular convergence in } W^1_0 L_M(\Omega),$$

where u is the unique solution of the bilateral problem:

$$(Q) \qquad \begin{cases} \langle Au, v-u\rangle \geq \int_\Omega f(v-u)dx, \forall v \in K \\ u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e.}\}. \end{cases}$$

Proof. We sketch the proof since the steps are similar to those in Section 3.
The existence of u_n is given by Theorem 1 of [4]. Indeed, it is easy to see that $|g(x,s)| \geq 1$ on $\{|s| \geq \gamma\}$, where $\gamma = \max\{\operatorname{supess} q_+, -\operatorname{infess} q_-\}$ and so that

$$|g(x,s)|^n M\left(\frac{|\zeta|}{\mu}\right) \geq M\left(\frac{|\zeta|}{\mu}\right) \text{ for } |s| \geq \gamma.$$

Step 1: A priori estimates.
Choosing $v = T_\gamma(u_n)$, as a test function in (P_n), and using the sign condition (3.4), we obtain

$$\alpha \int_\Omega M\left(\frac{|\nabla T_\gamma(u_n)|}{\lambda}\right) dx \leq \gamma \|f\|_1 \tag{4.1}$$

and

$$\int_{\{|u_n|>\gamma\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq \|f\|_1,$$

which gives

$$\int_{\{|u_n|>\gamma\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C$$

and finally

$$\int_\Omega M\left(\frac{|\nabla u_n|}{\max\{\lambda,\mu\}}\right) dx \leq C. \tag{4.2}$$

On the other hand, as in Section 3, we have

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C. \tag{4.3}$$

Step 2: Almost everywhere convergence of the gradients.
Due to (4.2), there exists some $u \in W_0^1 L_M(\Omega)$ such that (for a subsequence)

$$u_n \rightharpoonup u \text{ weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma(\Pi L_M, \Pi E_{\overline{M}}).$$

Write

$$Au_n = f - |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right)$$

and remark that, by (4.2), the second hand side is uniformly bounded in $L^1(\Omega)$. Then as in Section 3

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

Step 3: $u \in K = \{v \in W_0^1 L_M(\Omega) : q_- \leq v \leq q_+ \text{ a.e. in } \Omega\}$.
Similarly, as in the proof of Theorem 3.1, one can prove this step with the aid of property (4.3).

Step 4: Strong convergence of the truncations.
It is easy to see that the proof is the same as in Section 3.

Step 5: u is the solution of the bilateral problem (Q).
Let $v \in K$ and $0 < \theta < 1$. Taking $v_n = T_k(u_n - \theta v), k > 0$ as a test function in (P_n), one can see that the proof is the same by replacing $T_m(v)$ with v in Section 3. We remark that $K \subset L^\infty(\Omega)$.

Step 6: $u_n \to u$ for modular convergence in $W_0^1 L_M(\Omega)$.
We shall prove that $\nabla u_n \to \nabla u$ in $(L_M(\Omega))^N$ for the modular convergence by using Vitali's theorem.
Let E be a measurable subset of Ω, we have for any $k > 0$

$$\int_E M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le \int_{E\cap\{|u_n|\le k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx + \int_{E\cap\{|u_n|>k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx.$$

Let $\epsilon > 0$. By virtue of the modular convergence of the truncates, there exists some $\eta(\epsilon, k)$ such that for any E measurable

$$|E| < \eta(\epsilon,k) \Rightarrow \int_{E\cap\{|u_n|\le k\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \ \ \forall n. \tag{4.4}$$

Choosing $T_1(u_n - T_k(u_n))$, with $k > 0$ a test function in (P_n) we obtain:

$$\langle Au_n, T_1(u_n - T_k(u_n))\rangle + \int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) T_1(u_n - T_k(u_n))dx$$
$$= \int_\Omega f T_1(u_n - T_k(u_n))dx,$$

which implies

$$\int_{\{|u_n|>k+1\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \le \int_{\{|u_n|>k\}} |f| dx.$$

Note that $meas\{x \in \Omega : |u_n(x)| > k\} \to 0$ uniformly on n when $k \to \infty$. We deduce then that there exists $k = k(\epsilon)$ such that

$$\int_{\{|u_n|>k\}} |f| dx < \frac{\epsilon}{2}, \ \ \forall n,$$

which gives

$$\int_{\{|u_n|>k+1\}} |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \ \ \forall n.$$

By setting $t(\epsilon) = \max\{k+1, \gamma\}$ we obtain

$$\int_{\{|u_n|>t(\epsilon)\}} M\left(\frac{|\nabla u_n|}{\mu}\right) dx < \frac{\epsilon}{2}, \ \ \forall n. \tag{4.5}$$

Combining (4.4) and (4.5) we deduce that there exists $\eta > 0$ such that

$$\int_E M\left(\frac{|\nabla u_n|}{\mu}\right) < \epsilon, \ \ \forall n \text{ when } |E| < \eta, \ E \text{ measurable},$$

which shows the equi-integrability of $M\left(\frac{|\nabla u_n|}{\mu}\right)$ in $L^1(\Omega)$, and therefore we have

$$M\left(\frac{|\nabla u_n|}{\mu}\right) \to M\left(\frac{|\nabla u|}{\mu}\right) \text{ strongly in } L^1(\Omega).$$

By remarking that

$$M\left(\frac{|\nabla u_n - \nabla u|}{2\mu}\right) \leq \frac{1}{2}\left[M\left(\frac{|\nabla u_n|}{\mu}\right) + M\left(\frac{|\nabla u|}{\mu}\right)\right]$$

one easily has, by using the Lebesgue theorem

$$\int_\Omega M\left(\frac{|\nabla u_n - \nabla u|}{2\mu}\right) dx \to 0 \text{ as } n \to +\infty,$$

which completes the proof.

□

Remark 4.2. The condition $b(0) = 0$ is not necessary. Indeed, taking $\theta_h(u_n),\ h > 0$, as a test function in (P_n) with

$$\theta_h(s) = \begin{cases} hs & \text{if} \quad |s| \leq \frac{1}{h} \\ \operatorname{sgn}(s) & \text{if} \quad |s| \geq \frac{1}{h}, \end{cases}$$

we obtain

$$\int_\Omega |g(x,u_n)|^{n-1} g(x,u_n) M\left(\frac{|\nabla u_n|}{\mu}\right) \theta_h(u_n) dx \leq \int_\Omega f\theta_h(u_n) dx.$$

and then, by letting $h \to +\infty$,

$$\int_\Omega |g(x,u_n)|^n M\left(\frac{|\nabla u_n|}{\mu}\right) dx \leq C.$$

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