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SOME COMPANIONS OF THE GRÜSS INEQUALITY IN INNER PRODUCT SPACES

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Abstract

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Abstract

Some companions of Grüss inequality in inner product spaces and applications for integrals are given.

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1. Introduction

The following inequality of Grüss type in real or complex linear spaces is known (see [1]).

Theorem 1.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) and $e \in H$, $\|e\| = 1$. If $\phi, \gamma, \Phi, \Gamma$ are real or complex numbers and x, y are vectors in H such that the condition*

$$(1.1) \quad \operatorname{Re} \langle \Phi e - x, x - \phi e \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle \Gamma e - y, y - \gamma e \rangle \geq 0$$

or, equivalently (see [3]),

$$(1.2) \quad \left\| x - \frac{\phi + \Phi}{2} e \right\| \leq \frac{1}{2} |\Phi - \phi| \quad \text{and} \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

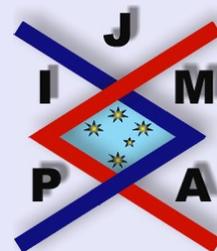
holds, then we have the inequality

$$(1.3) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Remark 1.1. *The case for $\mathbb{K} = \mathbb{R}$ for the above theorem has been published by the author in [2].*

The following particular instances for integrals and means are useful in applications.



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Corollary 1.2. Let $f, g : [a, b] \rightarrow \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) be Lebesgue measurable and such that there exists the constants $\phi, \gamma, \Phi, \Gamma \in \mathbb{K}$ with the property

$$(1.4) \quad \begin{aligned} \operatorname{Re} \left[(\Phi - f(x)) \left(\overline{f(x)} - \overline{\phi} \right) \right] &\geq 0, \\ \operatorname{Re} \left[(\Gamma - g(x)) \left(\overline{g(x)} - \overline{\gamma} \right) \right] &\geq 0 \end{aligned}$$

for a.e. $x \in [a, b]$, or, equivalently

$$(1.5) \quad \left| f(x) - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \quad \text{and} \quad \left| g(x) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|$$

for a.e. $x \in [a, b]$.

Then we have the inequality

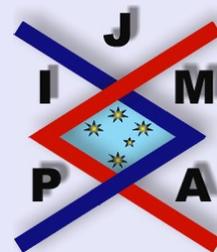
$$(1.6) \quad \left| \frac{1}{b-a} \int_a^b f(x) \overline{g(x)} dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b \overline{g(x)} dx \right| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible.

The discrete case is incorporated in

Corollary 1.3. Let $\mathbf{x}, \mathbf{y} \in \mathbb{K}^n$, with $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ and $\phi, \gamma, \Phi, \Gamma \in \mathbb{K}$ be such that

$$(1.7) \quad \operatorname{Re} \left[(\Phi - x_i) \left(\overline{x_i} - \overline{\phi} \right) \right] \geq 0 \quad \text{and} \quad \operatorname{Re} \left[(\Gamma - y_i) \left(\overline{y_i} - \overline{\gamma} \right) \right] \geq 0,$$



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for each $i \in \{1, \dots, n\}$, or, equivalently,

$$(1.8) \quad \left| x_i - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \quad \text{and} \quad \left| y_i - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma|,$$

for each $i \in \{1, \dots, n\}$.

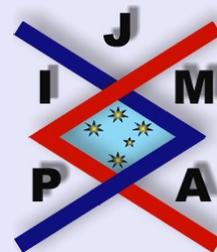
Then we have the inequality

$$(1.9) \quad \left| \frac{1}{n} \sum_{i=1}^n x_i \bar{y}_i - \frac{1}{n} \sum_{i=1}^n x_i \cdot \frac{1}{n} \sum_{i=1}^n \bar{y}_i \right| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|.$$

The constant $\frac{1}{4}$ is best possible in (1.9).

For some recent results related to Grüss type inequalities in inner product spaces, see [3]. More applications of Theorem 1.1 for integral and discrete inequalities may be found in [4].

The main aim of this paper is to provide other inequalities of Grüss type in the general setting of inner product spaces over the real or complex number field \mathbb{K} . Applications for Lebesgue integrals are pointed out as well.



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2. A Grüss Type Inequality

The following Grüss type inequality in inner product spaces holds.

Theorem 2.1. *Let $x, y, e \in H$ with $\|e\| = 1$, and the scalars $a, A, b, B \in \mathbb{K}$ ($\mathbb{K} = \mathbb{C}, \mathbb{R}$) such that $\operatorname{Re}(\bar{a}A) > 0$ and $\operatorname{Re}(\bar{b}B) > 0$. If*

$$(2.1) \quad \operatorname{Re} \langle Ae - x, x - ae \rangle \geq 0 \quad \text{and} \quad \operatorname{Re} \langle Be - y, y - be \rangle \geq 0$$

or, equivalently (see [3]),

$$(2.2) \quad \left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2} |A - a| \quad \text{and} \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2} |B - b|,$$

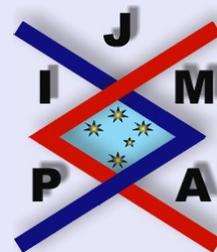
then we have the inequality

$$(2.3) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} \cdot \frac{|A - a| |B - b|}{\sqrt{\operatorname{Re}(\bar{a}A) \operatorname{Re}(\bar{b}B)}} |\langle x, e \rangle \langle e, y \rangle|.$$

The constant $\frac{1}{4}$ is best possible in the sense that it cannot be replaced by a smaller constant.

Proof. Apply Schwartz's inequality in $(H; \langle \cdot, \cdot \rangle)$ for the vectors $x - \langle x, e \rangle e$ and $y - \langle y, e \rangle e$, to get (see also [1])

$$(2.4) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \leq (\|x\|^2 - |\langle x, e \rangle|^2) (\|y\|^2 - |\langle y, e \rangle|^2).$$



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Now, assume that $u, v \in H$, and $c, C \in \mathbb{K}$ with $\operatorname{Re}(\bar{c}C) > 0$ and $\operatorname{Re}\langle Cv - u, u - cv \rangle \geq 0$. This last inequality is equivalent to

$$(2.5) \quad \begin{aligned} \|u\|^2 + \operatorname{Re}(\bar{c}C) \|v\|^2 &\leq \operatorname{Re} \left[C \overline{\langle u, v \rangle} + \bar{c} \langle u, v \rangle \right] \\ &= \operatorname{Re} \left[(\bar{C} + \bar{c}) \langle u, v \rangle \right], \end{aligned}$$

since

$$\operatorname{Re} \left[C \overline{\langle u, v \rangle} \right] = \operatorname{Re} \left[\bar{C} \langle u, v \rangle \right].$$

Dividing this inequality by $[\operatorname{Re}(C\bar{c})]^{\frac{1}{2}} > 0$, we deduce

$$(2.6) \quad \frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} \|v\|^2 \leq \frac{\operatorname{Re} \left[(\bar{C} + \bar{c}) \langle u, v \rangle \right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}.$$

On the other hand, by the elementary inequality

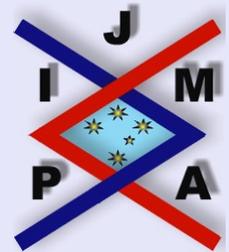
$$\alpha p^2 + \frac{1}{\alpha} q^2 \geq 2pq, \quad \alpha > 0, \quad p, q \geq 0,$$

we deduce

$$(2.7) \quad 2 \|u\| \|v\| \leq \frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}} \|v\|^2.$$

Making use of (2.6) and (2.7) and the fact that for any $z \in \mathbb{C}$, $\operatorname{Re}(z) \leq |z|$, we get

$$\|u\| \|v\| \leq \frac{\operatorname{Re} \left[(\bar{C} + \bar{c}) \langle u, v \rangle \right]}{2 [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \leq \frac{|C + c|}{2 [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} |\langle u, v \rangle|.$$



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Consequently

$$(2.8) \quad \|u\|^2 \|v\|^2 - |\langle u, v \rangle|^2 \leq \left[\frac{|C + c|^2}{4 [\operatorname{Re}(\bar{c}C)]} - 1 \right] |\langle u, v \rangle|^2 \\ = \frac{1}{4} \cdot \frac{|C - c|^2}{\operatorname{Re}(\bar{c}C)} |\langle u, v \rangle|^2.$$

Now, if we write (2.8) for the choices $u = x$, $v = e$ and $u = y$, $v = e$ respectively and use (2.4), we deduce the desired result (2.2). The sharpness of the constant will be proved in the case where H is a real inner product space. \square

The following corollary which provides a simpler Grüss type inequality for real constants (and in particular, for real inner product spaces) holds.

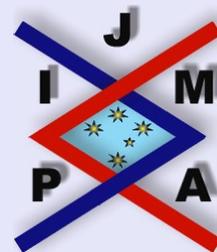
Corollary 2.2. *With the assumptions of Theorem 2.1 and if $a, b, A, B \in \mathbb{R}$ are such that $A > a > 0$, $B > b > 0$ and*

$$(2.9) \quad \left\| x - \frac{a + A}{2} e \right\| \leq \frac{1}{2} (A - a) \quad \text{and} \quad \left\| y - \frac{b + B}{2} e \right\| \leq \frac{1}{2} (B - b),$$

then we have the inequality

$$(2.10) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}} |\langle x, e \rangle \langle e, y \rangle|.$$

The constant $\frac{1}{4}$ is best possible.



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Proof. To prove the sharpness of the constant $\frac{1}{4}$ assume that the inequality (2.10) holds in real inner product spaces with $x = y$ and for a constant $k > 0$, i.e.,

$$(2.11) \quad \|x\|^2 - |\langle x, e \rangle|^2 \leq k \cdot \frac{(A - a)^2}{aA} |\langle x, e \rangle|^2 \quad (A > a > 0),$$

provided $\|x - \frac{a+A}{2}e\| \leq \frac{1}{2}(A - a)$, or equivalently, $\langle Ae - x, x - ae \rangle \geq 0$.

We choose $H = \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then we have

$$\begin{aligned} \|x\|^2 - |\langle x, e \rangle|^2 &= x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{(x_1 - x_2)^2}{2}, \\ |\langle x, e \rangle|^2 &= \frac{(x_1 + x_2)^2}{2}, \end{aligned}$$

and by (2.11) we get

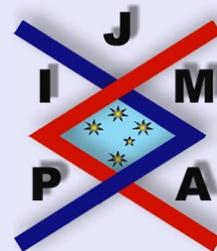
$$(2.12) \quad \frac{(x_1 - x_2)^2}{2} \leq k \cdot \frac{(A - a)^2}{aA} \cdot \frac{(x_1 + x_2)^2}{2}.$$

Now, if we let $x_1 = \frac{a}{\sqrt{2}}$, $x_2 = \frac{A}{\sqrt{2}}$ ($A > a > 0$), then obviously

$$\langle Ae - x, x - ae \rangle = \sum_{i=1}^2 \left(\frac{A}{\sqrt{2}} - x_i \right) \left(x_i - \frac{a}{\sqrt{2}} \right) = 0,$$

which shows that the condition (2.9) is fulfilled, and by (2.12) we get

$$\frac{(A - a)^2}{4} \leq k \cdot \frac{(A - a)^2}{aA} \cdot \frac{(a + A)^2}{4}$$



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for any $A > a > 0$. This implies

$$(2.13) \quad (A + a)^2 k \geq aA$$

for any $A > a > 0$.

Finally, let $a = 1 - q$, $A = 1 + q$, $q \in (0, 1)$. Then from (2.13) we get $4k \geq 1 - q^2$ for any $q \in (0, 1)$ which produces $k \geq \frac{1}{4}$. \square

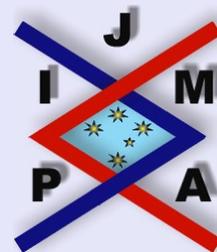
Remark 2.1. If $\langle x, e \rangle, \langle y, e \rangle$ are assumed not to be zero, then the inequality (2.3) is equivalent to

$$(2.14) \quad \left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{|A - a| |B - b|}{\sqrt{\operatorname{Re}(\bar{a}A) \operatorname{Re}(\bar{b}B)}},$$

while the inequality (2.10) is equivalent to

$$(2.15) \quad \left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}}.$$

The constant $\frac{1}{4}$ is best possible in both inequalities.



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3. Some Related Results

The following result holds.

Theorem 3.1. *Let $(H; \langle \cdot, \cdot \rangle)$ be an inner product space over \mathbb{K} ($\mathbb{K} = \mathbb{C}, \mathbb{R}$). If $\gamma, \Gamma \in \mathbb{K}$, $e, x, y \in H$ with $\|e\| = 1$ and $\lambda \in (0, 1)$ are such that*

$$(3.1) \quad \operatorname{Re} \langle \Gamma e - (\lambda x + (1 - \lambda) y), (\lambda x + (1 - \lambda) y) - \gamma e \rangle \geq 0,$$

or, equivalently,

$$(3.2) \quad \left\| \lambda x + (1 - \lambda) y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the inequality

$$(3.3) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2.$$

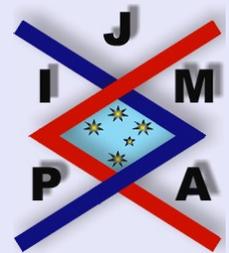
The constant $\frac{1}{16}$ is the best possible constant in (3.3) in the sense that it cannot be replaced by a smaller one.

Proof. We know that for any $z, u \in H$ one has

$$\operatorname{Re} \langle z, u \rangle \leq \frac{1}{4} \|z + u\|^2.$$

Then for any $a, b \in H$ and $\lambda \in (0, 1)$ one has

$$(3.4) \quad \operatorname{Re} \langle a, b \rangle \leq \frac{1}{4\lambda(1 - \lambda)} \|\lambda a + (1 - \lambda) b\|^2.$$



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Since

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle = \langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle \quad (\text{as } \|e\| = 1),$$

using (3.4), we have

$$\begin{aligned} (3.5) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] &= \operatorname{Re} [\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle] \\ &\leq \frac{1}{4\lambda(1-\lambda)} \|\lambda(x - \langle x, e \rangle e) + (1-\lambda)(y - \langle y, e \rangle e)\|^2 \\ &= \frac{1}{4\lambda(1-\lambda)} \|\lambda x + (1-\lambda)y - \langle \lambda x + (1-\lambda)y, e \rangle e\|^2. \end{aligned}$$

Since, for $m, e \in H$ with $\|e\| = 1$, one has the equality

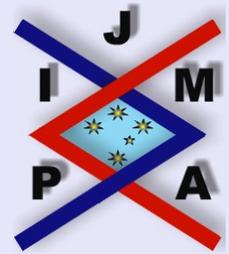
$$(3.6) \quad \|m - \langle m, e \rangle e\|^2 = \|m\|^2 - |\langle m, e \rangle|^2,$$

then by (3.5) we deduce the inequality

$$(3.7) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4\lambda(1-\lambda)} [\|\lambda x + (1-\lambda)y\|^2 - |\langle \lambda x + (1-\lambda)y, e \rangle|^2].$$

Now, if we apply Grüss' inequality

$$0 \leq \|a\|^2 - |\langle a, e \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2$$



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provided

$$\operatorname{Re} \langle \Gamma e - a, a - \gamma e \rangle \geq 0,$$

for $a = \lambda x + (1 - \lambda) y$, we have

$$(3.8) \quad \|\lambda x + (1 - \lambda) y\|^2 - |\langle \lambda x + (1 - \lambda) y, e \rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

Utilising (3.7) and (3.8) we deduce the desired inequality (3.3). To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.3) holds with a constant $C > 0$, provided (3.1) is valid, i.e.,

$$(3.9) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq C \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2.$$

If in (3.9) we choose $x = y$, provided (3.1) holds with $x = y$ and $\lambda \in (0, 1)$, then

$$(3.10) \quad \|x\|^2 - |\langle x, e \rangle|^2 \leq C \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2,$$

provided

$$\operatorname{Re} \langle \Gamma e - x, x - \gamma e \rangle \geq 0.$$

Since we know, in Grüss' inequality, the constant $\frac{1}{4}$ is best possible, then by (3.10), one has

$$\frac{1}{4} \leq \frac{C}{\lambda(1 - \lambda)} \quad \text{for } \lambda \in (0, 1),$$

giving, for $\lambda = \frac{1}{2}$, $C \geq \frac{1}{16}$.

The theorem is completely proved. \square



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The following corollary is a natural consequence of the above result.

Corollary 3.2. Assume that γ, Γ, e, x, y and λ are as in Theorem 3.1. If

$$(3.11) \quad \operatorname{Re} \langle \Gamma e - (\lambda x \pm (1 - \lambda) y), (\lambda x \pm (1 - \lambda) y) - \gamma e \rangle \geq 0,$$

or, equivalently,

$$(3.12) \quad \left\| \lambda x \pm (1 - \lambda) y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|^2,$$

then we have the inequality

$$(3.13) \quad |\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle]| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2.$$

The constant $\frac{1}{16}$ is best possible in (3.13).

Proof. Using Theorem 3.1 for $(-y)$ instead of y , we have that

$$\operatorname{Re} \langle \Gamma e - (\lambda x - (1 - \lambda) y), (\lambda x - (1 - \lambda) y) - \gamma e \rangle \geq 0,$$

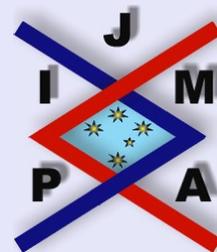
which implies that

$$\operatorname{Re} [-\langle x, y \rangle + \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2$$

giving

$$(3.14) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \geq -\frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |\Gamma - \gamma|^2.$$

Consequently, by (3.3) and (3.14) we deduce the desired inequality (3.13). \square



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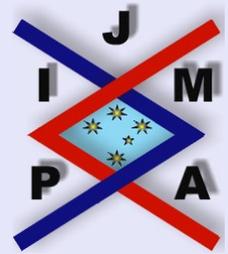


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Remark 3.1. If $M, m \in \mathbb{R}$ with $M > m$ and, for $\lambda \in (0, 1)$,

$$(3.15) \quad \left\| \lambda x + (1 - \lambda) y - \frac{M + m}{2} e \right\| \leq \frac{1}{2} (M - m)$$

then

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} (M - m)^2.$$

If (3.15) holds with “ \pm ” instead of “+”, then

$$(3.16) \quad |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} (M - m)^2.$$

Remark 3.2. If $\lambda = \frac{1}{2}$ in (3.1) or (3.2), then we obtain the result from [3], i.e.,

$$(3.17) \quad \operatorname{Re} \left\langle \Gamma e - \frac{x + y}{2}, \frac{x + y}{2} - \gamma e \right\rangle \geq 0$$

or, equivalently

$$(3.18) \quad \left\| \frac{x + y}{2} - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|$$

implies

$$(3.19) \quad \operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2.$$

The constant $\frac{1}{4}$ is best possible in (3.19).

For $\lambda = \frac{1}{2}$, Corollary 3.2 and Remark 3.1 will produce the corresponding results obtained in [3]. We omit the details.

4. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , Σ a σ -algebra of parts and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L^2(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and 2-integrable on Ω , i.e.,

$$\int_{\Omega} |f(s)|^2 d\mu(s) < \infty.$$

The following proposition holds

Proposition 4.1. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$, are so that $\operatorname{Re}(\Phi\bar{\varphi}) > 0$, $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$ and*

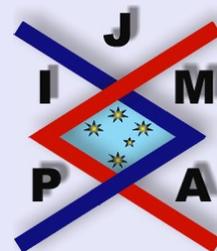
$$(4.1) \quad \int_{\Omega} \operatorname{Re} \left[(\Phi h(s) - f(s)) \left(\overline{f(s)} - \overline{\varphi h(s)} \right) \right] d\mu(s) \geq 0$$

$$\int_{\Omega} \operatorname{Re} \left[(\Gamma h(s) - g(s)) \left(\overline{g(s)} - \overline{\gamma h(s)} \right) \right] d\mu(s) \geq 0$$

or, equivalently

$$(4.2) \quad \left(\int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|,$$

$$\left(\int_{\Omega} \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|,$$



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then we have the following Grüss type integral inequality

$$(4.3) \quad \left| \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right| \\ \leq \frac{1}{4} \cdot \frac{|\Phi - \varphi| |\Gamma - \gamma|}{\sqrt{\operatorname{Re}(\Phi \bar{\varphi}) \operatorname{Re}(\Gamma \bar{\gamma})}} \left| \int_{\Omega} f(s) \overline{h(s)} d\mu(s) \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right|.$$

The constant $\frac{1}{4}$ is best possible.

The proof follows by Theorem 3.1 on choosing $H = L^2(\Omega, \mathbb{K})$ with the inner product

$$\langle f, g \rangle := \int_{\Omega} f(s) \overline{g(s)} d\mu(s).$$

We omit the details.

Remark 4.1. It is obvious that a sufficient condition for (4.1) to hold is

$$\operatorname{Re} \left[(\Phi h(s) - f(s)) \left(\overline{f(s)} - \overline{\varphi h(s)} \right) \right] \geq 0,$$

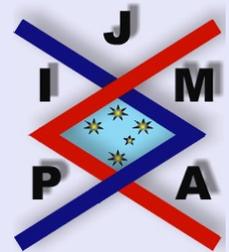
and

$$\operatorname{Re} \left[(\Gamma h(s) - g(s)) \left(\overline{g(s)} - \overline{\gamma h(s)} \right) \right] \geq 0,$$

for μ -a.e. $s \in \Omega$, or equivalently,

$$\left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right| \leq \frac{1}{2} |\Phi - \varphi| |h(s)| \quad \text{and} \\ \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)|,$$

for μ -a.e. $s \in \Omega$.



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The following result may be stated as well.

Corollary 4.2. *If $z, Z, t, T \in \mathbb{K}$, with $\operatorname{Re}(\bar{z}Z), \operatorname{Re}(\bar{t}T) > 0, \mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that:*

$$(4.4) \quad \operatorname{Re} \left[(Z - f(s)) \left(\overline{f(s)} - \bar{z} \right) \right] \geq 0,$$

$$\operatorname{Re} \left[(T - g(s)) \left(\overline{g(s)} - \bar{t} \right) \right] \geq 0 \text{ for a.e. } s \in \Omega$$

or, equivalently

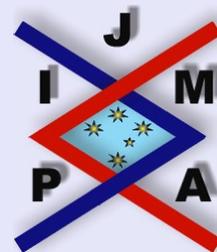
$$(4.5) \quad \left| f(s) - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z|,$$

$$\left| g(s) - \frac{t + T}{2} \right| \leq \frac{1}{2} |T - t| \text{ for a.e. } s \in \Omega;$$

then we have the inequality

$$(4.6) \quad \left| \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) \overline{g(s)} d\mu(s) - \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right|$$

$$\leq \frac{1}{4} \cdot \frac{|Z - z| |T - t|}{\sqrt{\operatorname{Re}(\bar{z}Z) \operatorname{Re}(\bar{t}T)}} \times \left| \frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right|.$$



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Remark 4.2. *The case of real functions incorporates the following interesting inequality*

$$(4.7) \quad \left| \frac{\mu(\Omega) \int_{\Omega} f(s)g(s) d\mu(s)}{\int_{\Omega} f(s) d\mu(s) \int_{\Omega} g(s) d\mu(s)} - 1 \right| \leq \frac{1}{4} \cdot \frac{(Z-z)(T-t)}{\sqrt{ztZT}}$$

provided $\mu(\Omega) < \infty$,

$$z \leq f(s) \leq Z, t \leq g(s) \leq T$$

for μ -a.e. $s \in \Omega$, where z, t, Z, T are real numbers and the integrals at the denominator are not zero. Here the constant $\frac{1}{4}$ is best possible in the sense mentioned above.

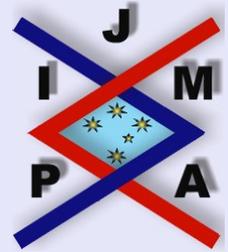
Using Theorem 3.1 we may state the following result as well.

Proposition 4.3. *If $f, g, h \in L^2(\Omega, \mathbb{K})$ and $\gamma, \Gamma \in \mathbb{K}$ are such that $\int_{\Omega} |h(s)|^2 d\mu(s) = 1$ and*

$$(4.8) \quad \int_{\Omega} \left\{ \operatorname{Re} [\Gamma h(s) - (\lambda f(s) + (1-\lambda)g(s))] \right. \\ \left. \times \left[\lambda \overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma} \bar{h}(s) \right] \right\} d\mu(s) \geq 0$$

or, equivalently,

$$(4.9) \quad \left(\int_{\Omega} \left| \lambda f(s) + (1-\lambda)g(s) - \frac{\gamma + \Gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|,$$



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then we have the inequality

$$(4.10) \quad I := \int_{\Omega} \operatorname{Re} \left[f(s) \overline{g(s)} \right] d\mu(s) \\ - \operatorname{Re} \left[\int_{\Omega} f(s) \overline{h(s)} d\mu(s) \cdot \int_{\Omega} h(s) \overline{g(s)} d\mu(s) \right] \\ \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2.$$

The constant $\frac{1}{16}$ is best possible.

If (4.8) and (4.9) hold with “ \pm ” instead of “ $+$ ” (see Corollary 3.2), then

$$(4.11) \quad |I| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2.$$

Remark 4.3. It is obvious that a sufficient condition for (4.8) to hold is

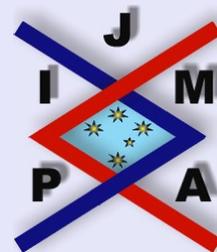
$$(4.12) \quad \operatorname{Re} \left\{ [\Gamma h(s) - (\lambda f(s) + (1-\lambda)g(s))] \right. \\ \left. \times [\lambda \overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma} \bar{h}(s)] \right\} \geq 0$$

for a.e. $s \in \Omega$, or equivalently

$$(4.13) \quad \left| \lambda f(s) + (1-\lambda)g(s) - \frac{\gamma + \Gamma}{2} h(s) \right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)|$$

for a.e. $s \in \Omega$.

Finally, the following corollary holds.



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Corollary 4.4. *If $Z, z \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that*

$$(4.14) \quad \operatorname{Re} \left[(Z - (\lambda f(s) + (1 - \lambda)g(s))) \right. \\ \left. \times \left(\overline{\lambda f(s) + (1 - \lambda)g(s)} - \bar{z} \right) \right] \geq 0$$

for a.e. $s \in \Omega$, or, equivalently

$$(4.15) \quad \left| \lambda f(s) + (1 - \lambda)g(s) - \frac{z + Z}{2} \right| \leq \frac{1}{2} |Z - z|,$$

for a.e. $s \in \Omega$, then we have the inequality

$$J := \frac{1}{\mu(\Omega)} \int_{\Omega} \operatorname{Re} \left[f(s) \overline{g(s)} \right] d\mu(s) \\ - \operatorname{Re} \left[\frac{1}{\mu(\Omega)} \int_{\Omega} f(s) d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} \overline{g(s)} d\mu(s) \right] \\ \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |Z - z|^2.$$

If (4.14) and (4.15) hold with “ \pm ” instead of “ $+$ ”, then

$$(4.16) \quad |J| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1 - \lambda)} |Z - z|^2.$$

Remark 4.4. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*



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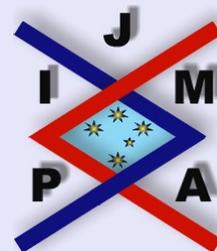
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