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SOME COMPANIONS OF THE GRÜSS INEQUALITY IN INNER PRODUCT SPACES

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Abstract

Some companions of Grüss inequality in inner product spaces and applications for integrals are given.

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Contents

1. Introduction

The following inequality of Grüss type in real or complex linear spaces is known (see [1]).

Theorem 1.1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *and* $e\in H$, $\|e\|=1$. *If* ϕ,γ,Φ,Γ *are real or complex numbers and* x,y *are vectors in* H *such that the condition*

$$\operatorname{Re}\langle\Phi e-x,x-\phi e\rangle\geq 0 \text{ and } \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.1}$$

or, equivalently (see [3]),

$$\left\|x-\frac{\phi+\Phi}{2}e\right\|\leq\frac{1}{2}|\Phi-\phi| \text{ and } \left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma| \tag{1.2}$$

holds, then we have the inequality

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{1.3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Remark 1.1. *The case for* $\mathbb{K}=\mathbb{R}$ *for the above theorem has been published by the author in [2].*

The following particular instances for integrals and means are useful in applications.

Corollary 1.2. *Let* $f,g:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *be Lebesgue measurable and such that there exists the constants* $\phi,\gamma,\Phi,\Gamma\in\mathbb{K}$ *with the property*

$$\operatorname{Re}\left[(\Phi-f(x))\left(\overline{f(x)}-\overline{\phi}\right)\right]\geq 0,\qquad (1.4)$$
$$\operatorname{Re}\left[(\Gamma-g(x))\left(\overline{g(x)}-\overline{\gamma}\right)\right]\geq 0$$

for a.e. $x\in[a,b]$, *or, equivalently*

$$\left|f(x)-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}|\Phi-\phi|\ \text{ and }\ \left|g(x)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\qquad (1.5)$$

for a.e. $x\in[a,b]$.

Then we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\overline{g(x)}dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b\overline{g(x)}dx\right|\leq\frac{1}{4}|\Phi-\phi|\,|\Gamma-\gamma|.\qquad (1.6)$$

The constant $\frac{1}{4}$ *is best possible.*

The discrete case is incorporated in

Corollary 1.3. *Let* $\mathbf{x},\mathbf{y}\in\mathbb{K}^n$, *with* $\mathbf{x}=(x_1,\dots,x_n)$ *and* $\mathbf{y}=(y_1,\dots,y_n)$ *and* $\phi,\gamma,\Phi,\Gamma\in\mathbb{K}$ *be such that*

$$\operatorname{Re}\left[(\Phi-x_i)\left(\overline{x_i}-\overline{\phi}\right)\right]\geq 0\ \text{ and }\ \operatorname{Re}\left[(\Gamma-y_i)\left(\overline{y_i}-\overline{\gamma}\right)\right]\geq 0,\qquad (1.7)$$

for each $i \in \{1, \dots, n\}$ *, or, equivalently,*

$$\left| x_i - \frac{\phi + \Phi}{2} \right| \leq \frac{1}{2} |\Phi - \phi| \quad \text{and} \quad \left| y_i - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| , \tag{1.8}$$

for each $i \in \{1, \dots, n\}$ *.*

Then we have the inequality

$$\left| \frac{1}{n} \sum_{i=1}^{n} x_i \overline{y_i} - \frac{1}{n} \sum_{i=1}^{n} x_i \cdot \frac{1}{n} \sum_{i=1}^{n} \overline{y_i} \right| \leq \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma| . \tag{1.9}$$

The constant $\frac{1}{4}$ *is best possible in (1.9).*

For some recent results related to Grüss type inequalities in inner product spaces, see [3]. More applications of Theorem 1.1 for integral and discrete inequalities may be found in [4].

The main aim of this paper is to provide other inequalities of Grüss type in the general setting of inner product spaces over the real or complex number field $\mathbb{K}$. Applications for Lebesgue integrals are pointed out as well.

2. A Grüss Type Inequality

The following Grüss type inequality in inner product spaces holds.

Theorem 2.1. *Let* $x, y, e \in H$ *with* $\|e\| = 1$, *and the scalars* $a, A, b, B \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *such that* $\operatorname{Re}(\bar{a}A) > 0$ *and* $\operatorname{Re}(\bar{b}B) > 0$. *If*

$$\operatorname{Re}\langle Ae - x, x - ae\rangle \geq 0 \text{ and } \operatorname{Re}\langle Be - y, y - be\rangle \geq 0 \tag{2.1}$$

or, equivalently (see [3]),

$$\left\| x - \frac{a+A}{2}e \right\| \leq \frac{1}{2}|A - a| \text{ and } \left\| y - \frac{b+B}{2}e \right\| \leq \frac{1}{2}|B - b|, \tag{2.2}$$

then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \leq \frac{1}{4} \cdot \frac{|A - a| |B - b|}{\sqrt{\operatorname{Re}(\bar{a}A) \operatorname{Re}(\bar{b}B)}} |\langle x, e\rangle \langle e, y\rangle|. \tag{2.3}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it cannot be replaced by a smaller constant.*

Proof. Apply Schwartz's inequality in $(H; \langle \cdot, \cdot \rangle)$ for the vectors $x - \langle x, e\rangle e$ and $y - \langle y, e\rangle e$, to get (see also [1])

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \leq \left(\|x\|^2 - |\langle x, e\rangle|^2\right)\left(\|y\|^2 - |\langle y, e\rangle|^2\right). \tag{2.4}$$

Now, assume that $u, v \in H$, and $c, C \in \mathbb{K}$ with $\operatorname{Re}(\bar{c}C) > 0$ and $\operatorname{Re}\langle Cv - u, u - cv\rangle \geq 0$. This last inequality is equivalent to

$$\|u\|^2 + \operatorname{Re}(\bar{c}C)\|v\|^2 \leq \operatorname{Re}\left[C\overline{\langle u, v\rangle} + \bar{c}\langle u, v\rangle\right] = \operatorname{Re}\left[\left(\bar{C} + \bar{c}\right)\langle u, v\rangle\right], \tag{2.5}$$

since

$$\operatorname{Re}\left[C\overline{\langle u, v\rangle}\right] = \operatorname{Re}\left[\bar{C}\langle u, v\rangle\right].$$

Dividing this inequality by $[\operatorname{Re}(C\bar{c})]^{\frac{1}{2}} > 0$, we deduce

$$\frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}\|v\|^2 \leq \frac{\operatorname{Re}\left[\left(\bar{C} + \bar{c}\right)\langle u, v\rangle\right]}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}. \tag{2.6}$$

On the other hand, by the elementary inequality

$$\alpha p^2 + \frac{1}{\alpha}q^2 \geq 2pq, \quad \alpha > 0,\ p, q \geq 0,$$

we deduce

$$2\|u\|\|v\| \leq \frac{1}{[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}\|u\|^2 + [\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}\|v\|^2. \tag{2.7}$$

Making use of (2.6) and (2.7) and the fact that for any $z \in \mathbb{C}$, $\operatorname{Re}(z) \leq |z|$, we get

$$\|u\|\|v\| \leq \frac{\operatorname{Re}\left[\left(\bar{C} + \bar{c}\right)\langle u, v\rangle\right]}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}} \leq \frac{|C + c|}{2[\operatorname{Re}(\bar{c}C)]^{\frac{1}{2}}}|\langle u, v\rangle|.$$

Consequently

$$\|u\|^2\|v\|^2 - |\langle u,v\rangle|^2 \le \left[\frac{|C+c|^2}{4\left[\operatorname{Re}\left(\bar{c}C\right)\right]} - 1\right] |\langle u,v\rangle|^2 = \frac{1}{4}\cdot\frac{|C-c|^2}{\operatorname{Re}\left(\bar{c}C\right)} |\langle u,v\rangle|^2 . \tag{2.8}$$

Now, if we write (2.8) for the choices $u = x$, $v = e$ and $u = y$, $v = e$ respectively and use (2.4), we deduce the desired result (2.2). The sharpness of the constant will be proved in the case where H is a real inner product space. □

The following corollary which provides a simpler Grüss type inequality for real constants (and in particular, for real inner product spaces) holds.

Corollary 2.2. *With the assumptions of Theorem 2.1 and if $a, b, A, B \in \mathbb{R}$ are such that $A > a > 0$, $B > b > 0$ and*

$$\left\| x - \frac{a+A}{2}e \right\| \le \frac{1}{2}(A-a) \quad \text{and} \quad \left\| y - \frac{b+B}{2}e \right\| \le \frac{1}{2}(B-b), \tag{2.9}$$

then we have the inequality

$$|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \le \frac{1}{4}\cdot\frac{(A-a)(B-b)}{\sqrt{abAB}} |\langle x,e\rangle\langle e,y\rangle| . \tag{2.10}$$

The constant $\frac{1}{4}$ is best possible.

Proof. The prove the sharpness of the constant $\frac{1}{4}$ assume that the inequality (2.10) holds in real inner product spaces with $x = y$ and for a constant $k > 0$, i.e.,

$$\|x\|^2 - |\langle x, e\rangle|^2 \le k \cdot \frac{(A-a)^2}{aA} |\langle x, e\rangle|^2 \quad (A > a > 0), \tag{2.11}$$

provided $\left\|x - \frac{a+A}{2} e\right\| \le \frac{1}{2}(A - a)$, or equivalently, $\langle Ae - x, x - ae\rangle \ge 0$.

We choose $H = \mathbb{R}^2$, $x = (x_1, x_2) \in \mathbb{R}^2$, $e = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then we have

$$\|x\|^2 - |\langle x, e\rangle|^2 = x_1^2 + x_2^2 - \frac{(x_1 + x_2)^2}{2} = \frac{(x_1 - x_2)^2}{2},$$

$$|\langle x, e\rangle|^2 = \frac{(x_1 + x_2)^2}{2},$$

and by (2.11) we get

$$\frac{(x_1 - x_2)^2}{2} \le k \cdot \frac{(A-a)^2}{aA} \cdot \frac{(x_1 + x_2)^2}{2}. \tag{2.12}$$

Now, if we let $x_1 = \frac{a}{\sqrt{2}}$, $x_2 = \frac{A}{\sqrt{2}}$ $(A > a > 0)$, then obviously

$$\langle Ae - x, x - ae\rangle = \sum_{i=1}^{2} \left(\frac{A}{\sqrt{2}} - x_i\right)\left(x_i - \frac{a}{\sqrt{2}}\right) = 0,$$

which shows that the condition (2.9) is fulfilled, and by (2.12) we get

$$\frac{(A-a)^2}{4} \le k \cdot \frac{(A-a)^2}{aA} \cdot \frac{(a+A)^2}{4}$$

for any $A > a > 0$. This implies

$$(A + a)^2 k \geq aA \tag{2.13}$$

for any $A > a > 0$.

Finally, let $a = 1 - q$, $A = 1 + q$, $q \in (0, 1)$. Then from (2.13) we get $4k \geq 1 - q^2$ for any $q \in (0, 1)$ which produces $k \geq \frac{1}{4}$. □

Remark 2.1. *If $\langle x, e \rangle, \langle y, e \rangle$ are assumed not to be zero, then the inequality (2.3) is equivalent to*

$$\left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{|A - a| \, |B - b|}{\sqrt{\operatorname{Re}(\bar{a}A) \operatorname{Re}(\bar{b}B)}}, \tag{2.14}$$

while the inequality (2.10) is equivalent to

$$\left| \frac{\langle x, y \rangle}{\langle x, e \rangle \langle e, y \rangle} - 1 \right| \leq \frac{1}{4} \cdot \frac{(A - a)(B - b)}{\sqrt{abAB}}. \tag{2.15}$$

The constant $\frac{1}{4}$ is best possible in both inequalities.

3. Some Related Results

The following result holds.

Theorem 3.1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K}=\mathbb{C},\mathbb{R})$ *. If* $\gamma,\Gamma\in\mathbb{K}$, $e,x,y\in H$ *with* $\|e\|=1$ *and* $\lambda\in(0,1)$ *are such that*

$$\operatorname{Re}\langle\Gamma e-(\lambda x+(1-\lambda)y),(\lambda x+(1-\lambda)y)-\gamma e\rangle\geq 0, \tag{3.1}$$

or, equivalently,

$$\left\|\lambda x+(1-\lambda)y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma|, \tag{3.2}$$

then we have the inequality

$$\operatorname{Re}[\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle]\leq\frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}|\Gamma-\gamma|^2. \tag{3.3}$$

The constant $\frac{1}{16}$ *is the best possible constant in (3.3) in the sense that it cannot be replaced by a smaller one.*

Proof. We know that for any $z,u\in H$ one has

$$\operatorname{Re}\langle z,u\rangle\leq\frac{1}{4}\|z+u\|^2.$$

Then for any $a,b\in H$ and $\lambda\in(0,1)$ one has

$$\operatorname{Re}\langle a,b\rangle\leq\frac{1}{4\lambda(1-\lambda)}\|\lambda a+(1-\lambda)b\|^2. \tag{3.4}$$

Since

$$\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle = \langle x - \langle x,e\rangle e, y - \langle y,e\rangle e\rangle \qquad (\text{as } \|e\| = 1),$$

using (3.4), we have

$$\begin{aligned} \text{(3.5)}\quad \operatorname{Re}\left[\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right] &= \operatorname{Re}\left[\langle x - \langle x,e\rangle e, y - \langle y,e\rangle e\rangle\right] \\ &\le \frac{1}{4\lambda(1-\lambda)}\left\|\lambda(x - \langle x,e\rangle e) + (1-\lambda)(y - \langle y,e\rangle e)\right\|^2 \\ &= \frac{1}{4\lambda(1-\lambda)}\left\|\lambda x + (1-\lambda)y - \langle \lambda x + (1-\lambda)y, e\rangle e\right\|^2. \end{aligned}$$

Since, for $m,e \in H$ with $\|e\| = 1$, one has the equality

$$\text{(3.6)}\qquad \|m - \langle m,e\rangle e\|^2 = \|m\|^2 - |\langle m,e\rangle|^2,$$

then by (3.5) we deduce the inequality

$$\begin{aligned} \text{(3.7)}\quad \operatorname{Re}\left[\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\right] \\ \le \frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x + (1-\lambda)y\|^2 - |\langle \lambda x + (1-\lambda)y, e\rangle|^2\right]. \end{aligned}$$

Now, if we apply Grüss' inequality

$$0 \le \|a\|^2 - |\langle a,e\rangle|^2 \le \frac{1}{4}|\Gamma - \gamma|^2$$

provided

$$\operatorname{Re}\langle \Gamma e - a, a - \gamma e\rangle \geq 0,$$

for $a = \lambda x + (1-\lambda) y$, we have

$$\|\lambda x + (1-\lambda) y\|^2 - |\langle \lambda x + (1-\lambda) y, e\rangle|^2 \leq \frac{1}{4} |\Gamma - \gamma|^2 . \tag{3.8}$$

Utilising (3.7) and (3.8) we deduce the desired inequality (3.3). To prove the sharpness of the constant $\frac{1}{16}$, assume that (3.3) holds with a constant $C > 0$, provided (3.1) is valid, i.e.,

$$\operatorname{Re}[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle] \leq C \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2 . \tag{3.9}$$

If in (3.9) we choose $x = y$, provided (3.1) holds with $x = y$ and $\lambda \in (0,1)$, then

$$\|x\|^2 - |\langle x, e\rangle|^2 \leq C \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2 , \tag{3.10}$$

provided

$$\operatorname{Re}\langle \Gamma e - x, x - \gamma e\rangle \geq 0.$$

Since we know, in Grüss' inequality, the constant $\frac{1}{4}$ is best possible, then by (3.10), one has

$$\frac{1}{4} \leq \frac{C}{\lambda(1-\lambda)} \quad \text{for} \quad \lambda \in (0,1),$$

giving, for $\lambda = \frac{1}{2}$, $C \geq \frac{1}{16}$.

The theorem is completely proved. □

The following corollary is a natural consequence of the above result.

Corollary 3.2. *Assume that* γ, Γ, e, x, y *and* λ *are as in Theorem 3.1. If*

$$\operatorname{Re}\langle \Gamma e - (\lambda x \pm (1-\lambda) y), (\lambda x \pm (1-\lambda) y) - \gamma e \rangle \geq 0, \tag{3.11}$$

or, equivalently,

$$\left\| \lambda x \pm (1-\lambda) y - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma|^2, \tag{3.12}$$

then we have the inequality

$$|\operatorname{Re}[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle]| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \tag{3.13}$$

The constant $\frac{1}{16}$ *is best possible in (3.13).*

Proof. Using Theorem 3.1 for $(-y)$ instead of y, we have that

$$\operatorname{Re}\langle \Gamma e - (\lambda x - (1-\lambda) y), (\lambda x - (1-\lambda) y) - \gamma e \rangle \geq 0,$$

which implies that

$$\operatorname{Re}[-\langle x, y\rangle + \langle x, e\rangle \langle e, y\rangle] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2$$

giving

$$\operatorname{Re}[\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle] \geq -\frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2. \tag{3.14}$$

Consequently, by (3.3) and (3.14) we deduce the desired inequality (3.13). □

Remark 3.1. *If $M, m \in \mathbb{R}$ with $M > m$ and, for $\lambda \in (0,1)$,*

$$\left\| \lambda x + (1-\lambda) y - \frac{M+m}{2} e \right\| \leq \frac{1}{2} (M-m) \tag{3.15}$$

then

$$\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle \leq \frac{1}{16} \cdot \frac{1}{\lambda (1-\lambda)} (M-m)^2 .$$

If (3.15) holds with "$\pm$" instead of "$+$", then

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{16} \cdot \frac{1}{\lambda (1-\lambda)} (M-m)^2 . \tag{3.16}$$

Remark 3.2. *If $\lambda = \frac{1}{2}$ in (3.1) or (3.2), then we obtain the result from [3], i.e.,*

$$\operatorname{Re} \left\langle \Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e \right\rangle \geq 0 \tag{3.17}$$

or, equivalently

$$\left\| \frac{x+y}{2} - \frac{\gamma + \Gamma}{2} e \right\| \leq \frac{1}{2} |\Gamma - \gamma| \tag{3.18}$$

implies

$$\operatorname{Re} [\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle] \leq \frac{1}{4} |\Gamma - \gamma|^2 . \tag{3.19}$$

The constant $\frac{1}{4}$ is best possible in (3.19).

For $\lambda = \frac{1}{2}$, Corollary 3.2 and Remark 3.1 will produce the corresponding results obtained in [3]. We omit the details.

4. Integral Inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of parts and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L^2(\Omega, \mathbb{K})$ the Hilbert space of all real or complex valued functions f defined on Ω and $2-$integrable on Ω, i.e.,

$$\int_{\Omega} |f(s)|^2 d\mu(s) < \infty.$$

The following proposition holds

Proposition 4.1. *If* $f, g, h \in L^2(\Omega, \mathbb{K})$ *and* $\varphi, \Phi, \gamma, \Gamma \in \mathbb{K}$*, are so that* $\operatorname{Re}(\Phi\overline{\varphi}) > 0, \operatorname{Re}(\Gamma\overline{\gamma}) > 0, \int_{\Omega} |h(s)|^2 d\mu(s) = 1$ *and*

$$\int_{\Omega} \operatorname{Re}\left[(\Phi h(s) - f(s))\left(\overline{f(s)} - \overline{\varphi}\overline{h(s)}\right)\right] d\mu(s) \geq 0 \tag{4.1}$$
$$\int_{\Omega} \operatorname{Re}\left[(\Gamma h(s) - g(s))\left(\overline{g(s)} - \overline{\gamma}\overline{h(s)}\right)\right] d\mu(s) \geq 0$$

or, equivalently

$$\left(\int_{\Omega} \left| f(s) - \frac{\Phi + \varphi}{2} h(s) \right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2} |\Phi - \varphi|, \tag{4.2}$$
$$\left(\int_{\Omega} \left| g(s) - \frac{\Gamma + \gamma}{2} h(s) \right|^2 d\mu(s)\right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|,$$

then we have the following Grüss type integral inequality

$$\left|\int_\Omega f(s)\overline{g(s)}d\mu(s) - \int_\Omega f(s)\overline{h(s)}d\mu(s)\int_\Omega h(s)\overline{g(s)}d\mu(s)\right| \\ \leq \frac{1}{4}\cdot\frac{|\Phi-\varphi||\Gamma-\gamma|}{\sqrt{\operatorname{Re}(\Phi\overline{\varphi})\operatorname{Re}(\Gamma\overline{\gamma})}}\left|\int_\Omega f(s)\overline{h(s)}d\mu(s)\int_\Omega h(s)\overline{g(s)}d\mu(s)\right|. \tag{4.3}$$

The constant $\frac{1}{4}$ *is best possible.*

The proof follows by Theorem 3.1 on choosing $H = L^2(\Omega, \mathbb{K})$ with the inner product

$$\langle f, g\rangle := \int_\Omega f(s)\overline{g(s)}d\mu(s).$$

We omit the details.

Remark 4.1. *It is obvious that a sufficient condition for* (4.1) *to hold is*

$$\operatorname{Re}\left[(\Phi h(s) - f(s))\left(\overline{f(s)} - \overline{\varphi}\overline{h(s)}\right)\right] \geq 0,$$

and

$$\operatorname{Re}\left[(\Gamma h(s) - g(s))\left(\overline{g(s)} - \overline{\gamma}\overline{h(s)}\right)\right] \geq 0,$$

for $\mu-$*a.e.* $s \in \Omega$, *or equivalently,*

$$\left|f(s) - \frac{\Phi+\varphi}{2}h(s)\right| \leq \frac{1}{2}|\Phi-\varphi||h(s)| \quad \text{and}$$
$$\left|g(s) - \frac{\Gamma+\gamma}{2}h(s)\right| \leq \frac{1}{2}|\Gamma-\gamma||h(s)|,$$

for $\mu-$*a.e.* $s \in \Omega$.

The following result may be stated as well.

Corollary 4.2. *If* $z, Z, t, T \in \mathbb{K}$, *with* $\operatorname{Re}(\bar{z}Z), \operatorname{Re}(\bar{t}T) > 0$, $\mu(\Omega) < \infty$ *and* $f, g \in L^2(\Omega, \mathbb{K})$ *are such that:*

$$\operatorname{Re}\left[(Z - f(s))\left(\overline{f(s)} - \bar{z}\right)\right] \geq 0, \tag{4.4}$$
$$\operatorname{Re}\left[(T - g(s))\left(\overline{g(s)} - \bar{t}\right)\right] \geq 0 \text{ for a.e. } s \in \Omega$$

or, equivalently

$$\left|f(s) - \frac{z+Z}{2}\right| \leq \frac{1}{2}|Z - z|, \tag{4.5}$$
$$\left|g(s) - \frac{t+T}{2}\right| \leq \frac{1}{2}|T - t| \text{ for a.e. } s \in \Omega;$$

then we have the inequality

$$\left|\frac{1}{\mu(\Omega)}\int_\Omega f(s)\overline{g(s)}d\mu(s) - \frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s) \cdot \frac{1}{\mu(\Omega)}\int_\Omega \overline{g(s)}d\mu(s)\right| \tag{4.6}$$
$$\leq \frac{1}{4} \cdot \frac{|Z - z|\,|T - t|}{\sqrt{\operatorname{Re}(\bar{z}Z)\operatorname{Re}(\bar{t}T)}} \times \left|\frac{1}{\mu(\Omega)}\int_\Omega f(s)\,d\mu(s) \cdot \frac{1}{\mu(\Omega)}\int_\Omega \overline{g(s)}d\mu(s)\right|.$$

Remark 4.2. *The case of real functions incorporates the following interesting inequality*

$$\left| \frac{\mu(\Omega) \int_\Omega f(s) g(s) d\mu(s)}{\int_\Omega f(s) d\mu(s) \int_\Omega g(s) d\mu(s)} - 1 \right| \leq \frac{1}{4} \cdot \frac{(Z-z)(T-t)}{\sqrt{ztZT}} \tag{4.7}$$

provided $\mu(\Omega) < \infty$,

$$z \leq f(s) \leq Z, t \leq g(s) \leq T$$

for $\mu - a.e.\ s \in \Omega$, *where* z, t, Z, T *are real numbers and the integrals at the denominator are not zero. Here the constant* $\frac{1}{4}$ *is best possible in the sense mentioned above.*

Using Theorem 3.1 we may state the following result as well.

Proposition 4.3. *If* $f, g, h \in L^2(\Omega, \mathbb{K})$ *and* $\gamma, \Gamma \in \mathbb{K}$ *are such that* $\int_\Omega |h(s)|^2 d\mu(s) = 1$ *and*

$$\int_\Omega \Big\{ \operatorname{Re}\left[\Gamma h(s) - (\lambda f(s) + (1-\lambda) g(s))\right] \times \left[\lambda \overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma}\bar{h}(s)\right] \Big\} d\mu(s) \geq 0 \tag{4.8}$$

or, equivalently,

$$\left(\int_\Omega \left| \lambda f(s) + (1-\lambda) g(s) - \frac{\gamma + \Gamma}{2} h(s) \right|^2 d\mu(s) \right)^{\frac{1}{2}} \leq \frac{1}{2} |\Gamma - \gamma|, \tag{4.9}$$

then we have the inequality

$$I := \int_{\Omega} \mathrm{Re}\left[f(s)\overline{g(s)}\right] d\mu(s) - \mathrm{Re}\left[\int_{\Omega} f(s)\overline{h(s)}d\mu(s) \cdot \int_{\Omega} h(s)\overline{g(s)}d\mu(s)\right] \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2 . \tag{4.10}$$

The constant $\frac{1}{16}$ *is best possible.*

If (4.8) and (4.9) hold with " $\pm$ " instead of " $+$ " (see Corollary 3.2), then

$$|I| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |\Gamma - \gamma|^2 . \tag{4.11}$$

Remark 4.3. *It is obvious that a sufficient condition for (4.8) to hold is*

$$\mathrm{Re}\left\{ [\Gamma h(s) - (\lambda f(s) + (1-\lambda) g(s))] \times \left[\lambda\overline{f(s)} + (1-\lambda)\overline{g(s)} - \bar{\gamma}\bar{h}(s)\right]\right\} \geq 0 \tag{4.12}$$

for a.e. $s \in \Omega$, *or equivalently*

$$\left|\lambda f(s) + (1-\lambda) g(s) - \frac{\gamma + \Gamma}{2} h(s)\right| \leq \frac{1}{2} |\Gamma - \gamma| |h(s)| \tag{4.13}$$

for a.e. $s \in \Omega$.

Finally, the following corollary holds.

Corollary 4.4. *If $Z, z \in \mathbb{K}$, $\mu(\Omega) < \infty$ and $f, g \in L^2(\Omega, \mathbb{K})$ are such that*

$$\operatorname{Re}\left[\left(Z - (\lambda f(s) + (1-\lambda) g(s))\right) \times \left(\lambda \overline{f(s)} + (1-\lambda)\overline{g(s)} - \overline{z}\right)\right] \geq 0 \tag{4.14}$$

for a.e. $s \in \Omega$, or, equivalently

$$\left| \lambda f(s) + (1-\lambda) g(s) - \frac{z+Z}{2} \right| \leq \frac{1}{2} |Z - z|, \tag{4.15}$$

for a.e. $s \in \Omega$, then we have the inequality

$$\begin{aligned} J &:= \frac{1}{\mu(\Omega)} \int_\Omega \operatorname{Re}\left[f(s) \overline{g(s)} \right] d\mu(s) \\ &\quad - \operatorname{Re}\left[\frac{1}{\mu(\Omega)} \int_\Omega f(s) \, d\mu(s) \cdot \frac{1}{\mu(\Omega)} \int_\Omega \overline{g(s)} d\mu(s) \right] \\ &\leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2 . \end{aligned}$$

If (4.14) and (4.15) hold with " $\pm$ " instead of " $+$ ", then

$$|J| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} |Z - z|^2 . \tag{4.16}$$

Remark 4.4. *It is obvious that if one chooses the discrete measure above, then all the inequalities in this section may be written for sequences of real or complex numbers. We omit the details.*

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