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GENERALIZED INEQUALITIES FOR INDEFINITE FORMS

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ABSTRACT. We establish abstract inequalities that give, as particular cases, many previously established Hölder-like inequalities. In addition to unifying the proofs of these inequalities, which, in most cases, tend to be technical and obscure, the proofs of our inequalities are quite simple and basic. Moreover, we show that sharper inequalities can be obtained by applying our results.

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1. INTRODUCTION

Let $n \geq 2$ be a fixed integer and let $a_i, b_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, be such that $a_1^2 - \sum_{i=2}^n a_i^2 \geq 0$ and $b_1^2 - \sum_{i=2}^n b_i^2 \geq 0$, where $\mathbb{R}$ is the set of real numbers. Then

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)^{\frac{1}{2}} \left(b_1^2 - \sum_{i=2}^n b_i^2\right)^{\frac{1}{2}} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{1.1}$$

This inequality was first considered by Aczél and Varga [2]. It was proved in detail by Aczél [1], who used it to present some applications of functional equations in non-Euclidean geometry. Inequality (1.1) was generalized by Popoviciu [8] as follows. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_i, b_i \geq 0$, $i = 1, 2, \dots, n$, with $a_1^p - \sum_{i=2}^n a_i^p \geq 0$ and $b_1^q - \sum_{i=2}^n b_i^q \geq 0$. Then

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i. \tag{1.2}$$

This is the "Hölder-like" generalization of (1.1). A simple proof of (1.2) may be found in [10]. Also, Chapter 5 in [6] contains generalizations of (1.2).

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For a fixed integer $n \geq 2$ and $p(\neq 0) \in \mathbb{R}$, the authors in [5] introduced the following definition:

$$\Phi_p(x) := \left(x_1^p - \sum_{i=2}^{n} x_i^p \right)^{\frac{1}{p}}, \; x \in R_p, \tag{1.3}$$

where

$$R_p = \left\{ x = (x_1, \dots, x_n) : x_i \geq (>)\, 0,\, x_1^p \geq (>) \sum_{i=2}^{n} x_i^p \right\} \text{ if } p > (<)\, 0. \tag{1.4}$$

There they presented inequalities for Φ_p from which they deduced, among other things, the inequalities (1.1) and (1.2).

Finally, in [9] the authors introduced the following definitions, which generalize (1.3) and (1.4). Let n be a positive integer, $n \geq 2$, and let M be a one-to-one real-valued function whose domain is a subset of $\mathbb{R}$. Then, for $\alpha \in \mathbb{R}$,

$$R_{\alpha,M} = \Bigg\{ x = (x_1, x_2, \dots, x_n) : x_1 > 0,\; \left(\frac{x_i}{x_1} \right) \in Domain\,(M) \text{ for } i = 2, \dots, n,$$
$$\text{and } \left[\alpha - \sum_{i=2}^{n} M\left(\frac{x_i}{x_1} \right) \right] \in Range\,(M) \Bigg\}$$

and, for $x \in R_{\alpha,M}$,

$$\Phi_{\alpha,M}\,(x) = x_1 M^{-1} \left[\alpha - \sum_{i=2}^{n} M\left(\frac{x_i}{x_1} \right) \right].$$

There the authors obtained generalizations of inequalities (1.1) and (1.2) and of the inequalities in [5].

It is our aim in this paper to establish inequalities (see Theorems 2.1 and 2.2) that give, as particular cases, all the inequalities mentioned above. In addition to unifying the proofs of these inequalities, which, in most cases, tend to be technical and obscure in the sense that it is not clear what really makes them work, the proofs of our inequalities are quite simple and basic. Moreover, we show that sharper inequalities can be obtained by applying our results.

2. GENERALIZED INEQUALITIES

Let $R_{\alpha,M}$ and $\Phi_{\alpha,M}$ be as defined above and let $m \geq 2$ be an integer.

Theorem 2.1. *Let $M_1, M_2, \dots, M_m$ be one-to-one real-valued functions defined in $\mathbb{R}$ and let M be a real-valued function defined on $Domain\,(M_1) \times \cdots \times Domain\,(M_m)$ and satisfying, for all $(t_1, \dots, t_m)$,*

$$M\,(t_1, t_2, \dots, t_m) \leq (\geq) \sum_{k=1}^{m} \sigma_k M_k\,(t_k)\,, \tag{2.1}$$

where $\sigma_1, \sigma_2, \dots \sigma_m$ are fixed real numbers. Then

$$M\left[\frac{\Phi_{\alpha_1,M_1}\,(x_1)}{x_{11}}, \dots, \frac{\Phi_{\alpha_m,M_m}\,(x_m)}{x_{m1}} \right] \leq (\geq) \sum_{k=1}^{m} \sigma_k \alpha_k - \sum_{i=2}^{n} M\left(\frac{x_{1i}}{x_{11}}, \dots, \frac{x_{mi}}{x_{m1}} \right) \tag{2.2}$$

for all $\alpha_k \in \mathbb{R}$ satisfying $R_{\alpha_k,M_k} \neq \emptyset$ and all $x_k \in R_{\alpha_k,M_k}$, $k = 1, \dots, m$.

Proof. Using (2.1) and the definition of $\Phi_{\alpha_k,M_k}(x_k)$, we obtain

$$\begin{aligned} M\left[\frac{\Phi_{\alpha_1,M_1}(x_1)}{x_{11}},\dots,\frac{\Phi_{\alpha_m,M_m}(x_m)}{x_{m1}}\right] &\le (\ge) \sum_{k=1}^{m}\sigma_k M_k\left(\frac{\Phi_{\alpha_k,M_k}(x_k)}{x_{k1}}\right) \\ &= \sum_{k=1}^{m}\sigma_k\left[\alpha_k-\sum_{i=2}^{n}M_k\left(\frac{x_{ki}}{x_{k1}}\right)\right] \\ &= \sum_{k=1}^{m}\sigma_k\alpha_k-\sum_{i=2}^{n}\sum_{k=1}^{m}\sigma_k M_k\left(\frac{x_{ki}}{x_{k1}}\right) \\ &\le (\ge) \sum_{k=1}^{m}\sigma_k\alpha_k-\sum_{i=2}^{n}M\left(\frac{x_{1i}}{x_{11}},\dots\frac{x_{mi}}{x_{m1}}\right). \end{aligned}$$

This ends the proof. □

Theorem 2.1, besides giving a unified and much simpler proof, is more general than many previously established inequalities. Indeed, as is shown below in the remarks following Corollary 3.2, these inequalities can be obtained as consequences of Theorem 2.1 with appropriate choices for the M_k's and with $M(t_1,\dots,t_m) := \prod_{k=1}^{m} t_k$.

Moreover, since inequality (2.2) is sharper whenever M is larger (smaller), we can obtain sharper inequalities each time we keep the same M_k's while modifying M so that the surface $t_{m+1} = M(t_1,\dots t_m)$ in $\mathbb{R}^{m+1}$ is distinct from and is between the two surfaces $t_{m+1} = P(t_1,\dots,t_m) := \prod_{k=1}^{m} t_k$ and $t_{m+1} = S(t_1,\dots,t_m) := \sum_{k=1}^{m}\sigma_k M_k(t_k)$. In other words, each time we chose $M \ne P, S$ such that, for every $(t_1,\dots,t_m) \in \mathit{Domain}(M_1)\times\cdots\times \mathit{Domain}(M_m)$,

$$\prod_{k=1}^{m} t_k \le (\ge)\ M(t_1,\dots t_m) \le (\ge) \sum_{k=1}^{m}\sigma_k M_k(t_k). \tag{2.3}$$

The closer M gets to S, the sharper the inequality is. Clearly, the optimum M is $M(t_1,\dots,t_m) := \sum_{k=1}^{m}\sigma_k M_k(t_k)$, in which case equality is attained in (2.2). But the idea is to choose an M that satisfies (2.3) while being simple enough to yield a "nice inequality". This, of course is most useful when the M_k's are not that simple. Nevertheless, any choice of M satisfying (2.3) will give a new inequality, strange as it may look.

To further clarify the above remarks, we establish the following consequence of Theorem 2.1, in which it is apparent that previous inequalities are particular cases and that Theorem 2.1 does indeed lead to sharper inequalities:

Theorem 2.2. *Let $M_1, M_2, \dots, M_m$ be one-to-one real-valued functions defined in $\mathbb{R}$ and satisfying, for all $(t_1,\dots,t_m) \in \mathit{Domain}(M_1)\times\cdots\times \mathit{Domain}(M_m)$,*

$$\prod_{k=1}^{m} t_k \le (\ge) \sum_{k=1}^{m}\sigma_k M_k(t_k), \tag{2.4}$$

where $\sigma_1,\sigma_2,\dots,\sigma_m$ are fixed real numbers. Let μ be any real-valued function defined on $\mathit{Domain}(M_1)\times\cdots\times \mathit{Domain}(M_m)$ and satisfying, for every $(t_1,\dots,t_m)$,

$$0 \le \mu(t_1,\dots,t_m) \le 1.$$

Then

$$\prod_{k=1}^{m}\Phi_{\alpha_k,M_k}(x_k)\le(\ge)\left(\left(\sum_{k=1}^{m}\sigma_k\alpha_k\right)\prod_{k=1}^{m}x_{k1}-\sum_{i=2}^{n}\prod_{k=1}^{m}x_{ki}\right.$$
$$\left.-\sum_{i=2}^{n}\left(1-\mu\left(\frac{x_{1i}}{x_{11}},\dots,\frac{x_{mi}}{x_{m1}}\right)\right)\left(\sum_{k=1}^{m}\sigma_kM_k\left(\frac{x_{ki}}{x_{k1}}\right)-\prod_{k=1}^{m}\frac{x_{ki}}{x_{k1}}\right)\prod_{k=1}^{m}x_{k1}\right) \tag{2.5}$$

for all $\alpha_k\in\mathbb{R}$ satisfying $R_{\alpha_k,M_k}\neq\emptyset$ and all $x_k\in R_{\alpha_k,M_k}$, $k=1,\dots,m$.

Proof. For simplicity of notation, let

$$\alpha:=\sum_{k=1}^{m}\sigma_k\alpha_k,\ P(\Phi):=\prod_{k=1}^{m}\frac{\Phi_{\alpha_k,M_k}(x_k)}{x_{k1}},\ P_i(x):=\prod_{k=1}^{m}\frac{x_{ki}}{x_{k1}},$$
$$S(\Phi):=\sum_{k=1}^{m}\sigma_kM_k\left(\frac{\Phi_{\alpha_k,M_k}(x_k)}{x_{k1}}\right),\ S_i(x):=\sum_{k=1}^{m}\sigma_kM_k\left(\frac{x_{ki}}{x_{k1}}\right),$$
$$\mu(\Phi):=\mu\left(\frac{\Phi_{\alpha_k,M_k}(x_k)}{x_{k1}},\dots,\frac{\Phi_{\alpha_k,M_k}(x_k)}{x_{k1}}\right),\ \mu_i(x):=\mu\left(\frac{x_{1i}}{x_{11}},\dots,\frac{x_{mi}}{x_{m1}}\right).$$

Let

$$M(t_1,\dots,t_m):=\prod_{k=1}^{m}t_k+(1-\mu(t_1,\dots,t_m))\left(\sum_{k=1}^{m}\sigma_kM_k(t_k)-\prod_{k=1}^{m}t_k\right).$$

Then M satisfies the inequalities in (2.3). Therefore we may apply Theorem 2.1 to obtain

$$P(\Phi)+(1-\mu(\Phi))(S(\Phi)-P(\Phi))\le(\ge)\ \alpha-\sum_{i=2}^{n}\left(P_i(x)+(1-\mu_i(x))(S_i(x)-P_i(x))\right).$$

Rearranging the terms, we get

$$P(\Phi)\le(\ge)\left(\alpha-\sum_{i=2}^{n}P_i(x)\right)-\left(\sum_{i=2}^{n}(1-\mu_i(x))(S_i(x)-P_i(x))\right)-(1-\mu(\Phi))(S(\Phi)-P(\Phi)).$$

Since the inequalities in (2.3) hold, we may drop the last term to obtain

$$P(\Phi)\le(\ge)\ \left(\alpha-\sum_{i=2}^{n}P_i(x)\right)-\left(\sum_{i=2}^{n}(1-\mu_i(x))(S_i(x)-P_i(x))\right).$$

Multiplying both sides by $\prod_{k=1}^{m}x_{k1}$, which is positive, we obtain the result (2.5). This ends the proof. □

3. APPLICATIONS

Let $p_1,p_2,\dots,p_m\neq0$ be real numbers satisfying $\frac{1}{p_1}+\frac{1}{p_2}+\cdots+\frac{1}{p_m}=1$. It is well known that

$$\prod_{k=1}^{m}t_k\le\sum_{k=1}^{m}\frac{1}{p_k}t_k^{p_k}, \tag{3.1}$$

for every $(t_1,\dots,t_m)\in\mathbb{R}_+^m:=(0,\infty)^m$, if and only if all p_i's are positive. Inequality (3.1) is known as Hölder's inequality.

Also, one has the following reverse inequality to (3.1):

$$\prod_{k=1}^{m} t_k \geq \sum_{k=1}^{m} \frac{1}{p_k} t_k^{p_k}, \tag{3.2}$$

for every $(t_1, \ldots, t_m) \in \mathbb{R}_+^m$, if and only if all p_i's are negative except for exactly one of them, [9] and [11].

Setting $M_k(t) := t^{p_k}$, $\sigma_k = \frac{1}{p_k}$, and $\alpha_k = 1$, $k = 1, \ldots, m$, in Theorem 2.2, we obtain immediately the following corollary:

Corollary 3.1. *Let $p_1, p_2, \ldots, p_m \neq 0$ be real numbers satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$ and let μ be any real-valued function defined on $\mathbb{R}_+^m$ and satisfying, for every $(t_1, \ldots, t_m)$,*

$$0 \leq \mu(t_1, \ldots, t_m) \leq 1. \tag{3.3}$$

If all p_i's are positive (all p_i's are negative except for exactly one of them), then

$$\begin{aligned} \prod_{k=1}^{m} \left(x_{k1}^{p_k} - \sum_{i=2}^{n} x_{ki}^{p_k} \right)^{\frac{1}{p_k}} \leq (\geq) \left(\prod_{k=1}^{m} x_{k1} - \sum_{i=2}^{n} \prod_{k=1}^{m} x_{ki} \right) \\ - \sum_{i=2}^{n} \left(1 - \mu\left(\frac{x_{1i}}{x_{11}}, \ldots, \frac{x_{mi}}{x_{m1}} \right) \right) \left(\sum_{k=1}^{m} \frac{1}{p_k} \left(\frac{x_{ki}}{x_{k1}} \right)^{p_k} - \prod_{k=1}^{m} \frac{x_{ki}}{x_{k1}} \right) \prod_{k=1}^{m} x_{k1} \end{aligned} \tag{3.4}$$

for all $x_k \in R_{1,t^{p_k}}$, $k = 1, \ldots, m$.

Dropping the last term in (3.4), we obtain Corollary 1 of [5]:

Corollary 3.2. *Let $p_1, p_2, \ldots, p_m \neq 0$ be real numbers satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} = 1$. The inequality*

$$\prod_{k=1}^{m} \Phi_{p_k}(x_k) \leq (\geq) \prod_{k=1}^{m} x_{k1} - \sum_{i=2}^{n} \prod_{k=1}^{m} x_{ki} \tag{3.5}$$

holds for all $x_k \in R_{p_k}$, $k = 1, \ldots, m$, if and only if all p_k's are positive (all p_k's are negative except for exactly one of them).

Note that inequality (3.4) is sharper than inequality (3.5). Choosing $\mu \equiv 1$, (3.4) gives (3.5). But any other choice of μ, satisfying (3.3), will give a sharper inequality. Of course, one may choose $\mu \equiv 0$ to obtain the sharpest inequality from (3.4). But, by keeping μ in (3.4), we give ourselves the freedom of choosing μ in such a way as to make the last term in (3.4) as simple as possible. This is a trade we have to make between the sharpness of inequality (3.4) and its simplicity.

Finally, we note that inequalities (1.1) and (1.2) are particular cases of inequality (3.5) and, consequently, of inequality (3.4).

We conclude by noting that from Páles's paper [7] and from Losonczi's papers [3] and [4] it follows that inequalities (3.1) and (3.2), written in the form

$$\prod_{k=1}^{m} t_k - 1 \leq (\geq) \sum_{k=1}^{m} \frac{t_k^{p_k} - 1}{p_k}, \ (t_1, t_2, \ldots, t_m) \in \mathbb{R}^m,$$

are equivalent to

$$M_{n,1}\left(\prod_{k=1}^{m} x_k \right) \leq (\geq) \prod_{k=1}^{m} M_{n,p_k}(x_k), \ n \in \mathbb{N}, \ x_k \in \mathbb{R}_+^n, \ k = 1, 2, \ldots, m, \tag{3.6}$$

where $x_k := (x_{k1}, x_{k2}, \ldots, x_{kn})$, $k = 1, 2, \ldots, m$, and

$$M_{n,p}(x) := M_{n,p}(x_1, x_2, \ldots, x_n) := \begin{cases} \left(\sum_{j=1}^{n} \frac{x_j^p}{n}\right)^{\frac{1}{p}} & \text{if} \quad p \neq 0, \\ \sqrt[n]{x_1 x_2 \cdots x_n} & \text{if} \quad p = 0. \end{cases}$$

Inequality (3.6) was completely settled by Páles, [7, corollary on p. 464].

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