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NEIGHBOURHOODS AND PARTIAL SUMS OF STARLIKE FUNCTIONS BASED ON RUSCHEWEYH DERIVATIVES

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ABSTRACT. In this paper a new class $S_p^\lambda(\alpha,\beta)$ of starlike functions is introduced. A subclass $TS_p^\lambda(\alpha,\beta)$ of $S_p^\lambda(\alpha,\beta)$ with negative coefficients is also considered. These classes are based on Ruscheweyh derivatives. Certain neighbourhood results are obtained. Partial sums $f_n(z)$ of functions $f(z)$ in these classes are considered and sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f'_n(z)$ are determined.

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1. INTRODUCTION

Let S denote the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Also denote by T, the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \tag{1.2}$$

which are univalent and normalized in U.

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For $f \in S$, and of the form (1.1) and $g(z) \in S$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or convolution) $f * g$ of f and g by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k. \tag{1.3}$$

For $-1 \leq \alpha < 1$ and $\beta \geq 0$, we let $S_p^{\lambda}(\alpha, \beta)$ be the subclass of S consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{z\left(D^{\lambda} f(z)'\right)}{D^{\lambda} f(z)} - \alpha\right\} > \beta \left|\frac{z\left(D^{\lambda} f(z)'\right)}{D^{\lambda} f(z)} - 1\right|, \tag{1.4}$$

where D^{λ} is the Ruscheweyh derivative [6] defined by

$$D^{\lambda} f(z) = f(z) * \frac{1}{(1-z)^{\lambda+1}} = z + \sum_{k=2}^{\infty} B_k(\lambda) a_k z^k$$

and

$$B_k(\lambda) = \frac{(\lambda+1)_{k-1}}{(k-1)!} = \frac{(\lambda+1)(\lambda+1)\cdots(\lambda+k-1)}{(k-1)!}, \quad \lambda \geq 0. \tag{1.5}$$

We also let $TS_p^{\lambda}(\alpha, \beta) = S_p^{\lambda}(\alpha, \beta) \cap T$. It can be seen that, by specializing on the parameters α, β, λ the class $TS_p^{\lambda}(\alpha, \beta)$ reduces to the classes introduced and studied by various authors [1, 9, 11, 12].

The main aim of this work is to study coefficient bounds and extreme points of the general class $TS_p^{\lambda}(\alpha, \beta)$. Furthermore, we obtain certain neighbourhoods results for functions in $TS_p^{\lambda}(\alpha, \beta)$. Partial sums $f_n(z)$ of functions $f(z)$ in the class $S_p^{\lambda}(\alpha, \beta)$ are considered.

2. THE CLASSES $S_p^{\lambda}(\alpha, \beta)$ AND $TS_p^{\lambda}(\alpha, \beta)$

In this section we obtain a necessary and sufficient condition and extreme points for functions $f(z)$ in the class $TS_p^{\lambda}(\alpha, \beta)$.

Theorem 2.1. *A sufficient condition for a function $f(z)$ of the form (1.1) to be in $S_p^{\lambda}(\alpha, \beta)$ is that*

$$\sum_{k=2}^{\infty} \frac{[(1+\beta)k - (\alpha+\beta)]}{1-\alpha} B_k(\lambda) |a_k| \leq 1, \tag{2.1}$$

$-1 \leq \alpha < 1, \beta \geq 0, \lambda \geq 0$ *and* $B_k(\lambda)$ *is as defined in (1.5).*

Proof. It suffices to show that

$$\beta \left|\frac{z\left(D^{\lambda} f(z)\right)'}{D^{\lambda} f(z)} - 1\right| - \operatorname{Re}\left\{\frac{z\left(D^{\lambda} f(z)\right)'}{D^{\lambda} f(z)} - 1\right\} \leq 1 - \alpha.$$

We have

$$\begin{aligned}
\beta \left|\frac{z\left(D^{\lambda} f(z)\right)'}{D^{\lambda} f(z)} - 1\right| - \operatorname{Re}\left\{\frac{z\left(D^{\lambda} f(z)\right)'}{D^{\lambda} f(z)} - 1\right\} &\leq (1+\beta)\left|\frac{z\left(D^{\lambda} f(z)\right)'}{D^{\lambda} f(z)} - 1\right| \\
&\leq \frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)B_k(\lambda)|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k||z|^{k-1}} \\
&\leq \frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)B_k(\lambda)|a_k|}{1-\sum_{k=2}^{\infty}B_k(\lambda)|a_k|}.
\end{aligned}$$

This last expression is bounded above by $1-\alpha$ if

$$\sum_{k=2}^{\infty}\left[(1+\beta)k-(\alpha+\beta)\right]B_k(\lambda)\left|a_k\right|\leq 1-\alpha,$$

and the proof is complete. □

Now we prove that the above condition is also necessary for $f\in T$.

Theorem 2.2. *A necessary and sufficient condition for f of the form (1.2) namely $f(z)=z-\sum_{k=2}^{\infty}b_kz^k$, $a_k\geq 0$, $z\in U$ to be in $TS_p^{\lambda}(\alpha,\beta)$, $-1\leq\alpha<1$, $\beta\geq 0$, $\lambda\geq 0$ is that*

$$\sum_{k=2}^{\infty}\left[(1+\beta)k-(\alpha+\beta)\right]B_k(\lambda)a_k\leq 1-\alpha. \tag{2.2}$$

Proof. In view of Theorem 2.1, we need only to prove the necessity. If $f\in TS_p^{\lambda}(\alpha,\beta)$ and z is real then

$$\frac{1-\sum_{k=2}^{\infty}ka_kB_k(\lambda)z^{k-1}}{1-\sum_{k=2}^{\infty}a_kB_k(\lambda)z^{k-1}}-\alpha\geq\frac{1-\sum_{k=2}^{\infty}(k-1)a_kB_k(\lambda)z^{k-1}}{1-\sum_{k=2}^{\infty}a_kB_k(\lambda)z^{k-1}}.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{k=2}^{\infty}\left[(1+\beta)k-(\alpha+\beta)\right]B_k(\lambda)a_k\leq 1-\alpha.$$

□

Theorem 2.3. *The extreme points of $TS_p^{\lambda}(\alpha,\beta)$, $-1\leq\alpha<1$, $\beta\geq 0$ are the functions given by*

$$f_1(z)=1 \ \text{ and } \ f_k(z)=z-\frac{1-\alpha}{\left[(1+\beta)k-(\alpha+\beta)\right]B_k(\lambda)}z^k, \tag{2.3}$$

$k=2,3,\ldots$ *where $\lambda>-1$ and $B_k(\lambda)$ is as defined in (1.5).*

Corollary 2.4. *A function $f\in TS_p^{\lambda}(\alpha,\beta)$ if and only if f may be expressed as $\sum_{k=1}^{\infty}\mu_kf_k(z)$ where $\mu_k\geq 0$, $\sum_{k=1}^{\infty}\mu_k=1$ and $f_1,f_2,\ldots$ are as defined in (2.3).*

3. NEIGHBOURHOOD RESULTS

The concept of neighbourhoods of analytic functions was first introduced by Goodman [4] and then generalized by Ruscheweyh [5]. In this section we study neighbourhoods of functions in the family $TS_p^{\lambda}(\alpha,\beta)$.

Definition 3.1. For $f\in S$ of the form (1.1) and $\delta\geq 0$, we define $\eta-\delta$- neighbourhood of f by

$$M_{\delta}^{\eta}(f)=\left\{g\in S:g(z)=z+\sum_{k=2}^{\infty}b_kz^k \ \text{ and } \ \sum_{k=2}^{\infty}k^{\eta+1}\left|a_k-b_k\right|\leq\delta\right\},$$

where η is a fixed positive integer.

We may write $M_{\delta}^{\eta}(f)=N_{\delta}(f)$ and $M_{\delta}^{1}(f)=M_{\delta}(f)$ [5]. We also notice that $M_{\delta}(f)$ was defined and studied by Silverman [7] and also by others [2, 3].

We need the following two lemmas to study the $\eta-\delta$- neighbourhood of functions in $TS_p^{\lambda}(\alpha,\beta)$.

Lemma 3.1. *Let $m\geq 0$ and $-1\leq\gamma<1$. If $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$ satisfies $\sum_{k=2}^{\infty}k^{\mu+1}\left|b^k\right|\leq\frac{1-\gamma}{1+\beta}$ then $g\in S_p^{\mu}(\gamma,\beta)$. The result is sharp.*

Proof. In view of the first part of Theorem 2.1, it is sufficient to show that

$$\frac{k(1+\beta)-(\gamma+\beta)}{1-\gamma}B_k(\mu)=\frac{k^{\mu+1}}{(1-\gamma)}(1+\beta).$$

But

$$\begin{aligned}\frac{k(1+\beta)-(\gamma+\beta)}{1-\gamma}B_k(\mu)&=\frac{(k(1+\beta)-(\gamma+\beta))(\mu+1)\cdots(\mu+k-1)}{(1-\gamma)(k-1)!}\\&\leq\frac{k(1+\beta)(\mu+1)(\mu+2)\cdots(\mu+k-1)}{(1-\gamma)(k-1)!}.\end{aligned}$$

Therefore we need to prove that

$$H(k,\mu)=\frac{(\mu+1)(\mu+2)\cdots(\mu+k-1)}{k^{\mu}(k-1)!}\leq 1.$$

Since$H(k,\mu)=[(\mu+1)/2^{\mu}]\leq 1$, we need only to show that $H(k,\mu)$ is a decreasing function of k. But $H(k+1,\mu)\leq H(k,\mu)$ is equivalent to $(1+\mu/k)\leq(1+1/k)^{\mu}$. The result follows because the last inequality holds for all $k\geq 2$. □

Lemma 3.2. *Let* $f(z)=z-\sum_{k=2}^{\infty}a_kz^k\in T$, $-1\leq\alpha<1$, $\beta\geq 0$ *and* $\varepsilon\geq 0$. *If* $\frac{f(z)+\varepsilon z}{1+\varepsilon}\in TS_p^{\lambda}(\alpha,\beta)$ *then*

$$\sum_{k=2}^{\infty}k^{\mu+1}a_k\leq\frac{2^{\eta+1}(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)},$$

where either $\eta=0$ *and* $\lambda\geq 0$ *or* $\eta=1$ *and* $1\leq\lambda\leq 2$. *The result is sharp with the extremal function*

$$f(z)=z-\frac{(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)}z^2,\quad z\in U.$$

Proof. Letting $g(z)=\frac{f(z)+\varepsilon z}{1+\varepsilon}$ we have $g(z)=z-\sum_{k=2}^{\infty}\frac{a_k}{1+\varepsilon}z^k$, $z\in U$.

In view of Corollary 2.4 $g(z)$, may be written as $g(z)=\sum_{k=1}^{\infty}\mu_kg_k(z)$, where $\mu_k\geq 0,\sum_{k=1}^{\infty}\mu_k=1$,

$$g_1(z)=z\ \text{ and }\ g_k(z)=z-\frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)}z^k,\qquad k=2,3,\ldots.$$

Therefore we obtain

$$\begin{aligned}g(z)&=\mu_1z+\sum_{k=2}^{\infty}\mu_k\left(z-\frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)}z^k\right)\\&=z-\sum_{k=2}^{\infty}\mu_k\left(\frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)}\right)z^k.\end{aligned}$$

Since $\mu_k\geq 0$ and $\sum_{k=1}^{\infty}\mu_k\leq 1$, it follows that

$$\sum_{k=2}^{\infty}k^{\eta+1}a_k\leq\sup_{k\geq 2}k^{\eta+1}\left(\frac{(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)}\right).$$

The result will follow if we can show that $A(k,\eta,\alpha,\varepsilon,\lambda)=\frac{k^{\eta+1}(1-\alpha)(1+\varepsilon)}{(k-\alpha+\beta)B_k(\lambda)}$ is a decreasing function of k. In view of $B_{k+1}(\lambda)=\frac{\lambda+k}{k}B_k(\lambda)$ the inequality

$$A(k+1,\eta,\alpha,\varepsilon,\lambda)\leq A(k,\eta,\alpha,\varepsilon,\lambda)$$

is equivalent to

$$(k+1)^{\eta+1}(k-\alpha+\beta) \leq k^{\eta}(k+1-\alpha+\beta)(\lambda+k).$$

This yields

$$\lambda(k-\alpha+\beta)+\lambda+\alpha-\beta \geq 0 \tag{3.1}$$

whenever $\eta = 0$ and $\lambda \geq 0$ and

$$k[(k+1)(\lambda-1)+(2-\lambda)(\alpha-\beta)]+\alpha-\beta \geq 0, \tag{3.2}$$

whenever $\eta = 1$ and $1 \leq \lambda \leq 2$. Since (3.1) and (3.2) holds for all $k \geq 2$, the proof is complete. □

Theorem 3.3. *Suppose either $\eta = 0$ and $\lambda \geq 0$ or $\eta = 1$ and $1 \leq \lambda \leq 2$.*
Let $-1 \leq \alpha < 1$, and

$$-1 \leq \gamma < \frac{(2-\alpha+\beta)(1+\lambda)-2^{\eta+1}(1-\alpha)(1+\varepsilon)(1+\beta)}{(2-\alpha+\beta)(1+\lambda)(1+\beta)}.$$

Let $f \in T$ and for all real numbers $0 \leq \varepsilon < \delta$, assume $\frac{f(z)+\varepsilon z}{1+\varepsilon} \in TS_p^{\lambda}(\alpha,\beta)$.
Then the η-δ - neighbourhood of f, namely $M_{\delta}^{\eta}(f) \subset S_p^{\eta}(\gamma,\beta)$ where

$$\delta = \frac{(1-\gamma)(2-\alpha+\beta)(1+\lambda)-2^{\eta+1}(1-\alpha)(1+\varepsilon)(1+\beta)}{(2-\alpha+\beta)(1+\lambda)(1+\beta)}.$$

The result is sharp, with the extremal function $f(z) = \frac{(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)} z^2$.

Proof. For a function f of the form (1.2), let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be in $M_{\delta}^{\eta}(f)$. In view of Lemma 3.2, we have

$$\begin{aligned}\sum_{k=2}^{\infty} k^{\eta+1}|b_k| &= \sum_{k=2}^{\infty} k^{\eta+1}|a_k-b_k-a_k| \\ &\leq \delta + \frac{2^{\eta+1}(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)}.\end{aligned}$$

Applying Lemma 3.1, it follows that $g \in S_p^{\eta}(\gamma,\beta)$ if $\delta + \frac{2^{\eta+1}(1-\alpha)(1+\varepsilon)}{(2-\alpha+\beta)(1+\lambda)} \leq \frac{1-\gamma}{1+\beta}$. That is, if

$$\delta \leq \frac{(1-\gamma)(2-\alpha+\beta)(1+\lambda)-2^{\eta+1}(1-\alpha)(1+\varepsilon)(1+\beta)}{(2-\alpha+\beta)(1+\lambda)(1+\beta)}.$$

This completes the proof. □

Remark 3.4. By taking $\beta = 0$ and letting $\lambda = 0$, $\lambda = 1$ and $\eta = 0 = \varepsilon$, we note that Theorems 1,2,4 in [8] follow immediately from Theorem 3.3.

4. PARTIAL SUMS

Following the earlier works by Silverman [8] and Silvia [10] on partial sums of analytic functions. We consider in this section partial sums of functions in the class $S_p^{\lambda}(\alpha,\beta)$ and obtain sharp lower bounds for the ratios of real part of $f(z)$ to $f_n(z)$ and $f'(z)$ to $f_n'(z)$.

Theorem 4.1. *Let $f(z) \in S_p^{\lambda}(\alpha,\beta)$ be given by (1.1) and define the partial sums $f_1(z)$ and $f_n(z)$, by*

$$f_1(z) = z; \quad \text{and} \quad f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad (n \in \mathbb{N}/\{1\}) \tag{4.1}$$

Suppose also that

$$\sum_{k=2}^{\infty} c_k |a_k| \leq 1, \tag{4.2}$$

where $\left(c_k := \frac{[(1+\beta)k-(\alpha+\beta)]B_k(\lambda)}{1-\alpha}\right)$. *Then* $f \in S_p^{\lambda}(\alpha, \beta)$. *Furthermore,*

$$\operatorname{Re}\left\{\frac{f(z)}{f_n(z)}\right\} > 1 - \frac{1}{c_{n+1}} z \in U, \quad n \in \mathbb{N} \tag{4.3}$$

and

$$\operatorname{Re}\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{n+1}}{1+c_{n+1}}. \tag{4.4}$$

Proof. It is easily seen that $z \in S_p^{\lambda}(\alpha, \beta)$. Thus from Theorem 3.3 and by hypothesis (4.2), we have

$$N_1(z) \subset S_p^{\lambda}(\alpha, \beta), \tag{4.5}$$

which shows that $f \in S_p^{\lambda}(\alpha, \beta)$ as asserted by Theorem 4.1.

Next, for the coefficients c_k given by (4.2) it is not difficult to verify that

$$c_{k+1} > c_k > 1. \tag{4.6}$$

Therefore we have

$$\sum_{k=2}^{n} |a_k| + c_{n+1} \sum_{k=n+1}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} c_k |a_k| \leq 1 \tag{4.7}$$

by using the hypothesis (4.2).

By setting

$$\begin{aligned} g_1(z) &= c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1 - \frac{1}{c_{n+1}}\right)\right\} \\ &= 1 + \frac{c_{n+1}\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{n} a_k z^{k-1}} \end{aligned} \tag{4.8}$$

and applying (4.7), we find that

$$\begin{aligned} \left|\frac{g_1(z)-1}{g_1(z)+1}\right| &\leq \frac{c_{n+1}\sum_{k=n+1}^{\infty} |a_k|}{2 - 2\sum_{k=2}^{n} |a_k| - c_{n+1}\sum_{k=n+1}^{\infty} |a_k|} \\ &\leq 1, \quad z \in U, \end{aligned} \tag{4.9}$$

which readily yields the assertion (4.3) of Theorem 4.1. In order to see that

$$f(z) = z + \frac{z^{n+1}}{c_{n+1}} \tag{4.10}$$

gives sharp result, we observe that for $z = re^{i\pi/n}$ that $\frac{f(z)}{f_n(z)} = 1 + \frac{z^n}{c_{n+1}} \to 1 - \frac{1}{c_{n+1}}$ as $z \to 1^-$.

Similarly, if we take

$$\begin{aligned} g_2(z) &= (1 + c_{n+1})\left\{\frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1+c_{n+1}}\right\} \\ &= 1 - \frac{(1+c_{n+1})\sum_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum_{k=2}^{\infty} a_k z^{k-1}} \end{aligned} \tag{4.11}$$

and making use of (4.7), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1+c_{n+1})\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{n}|a_k|-(1+c_{n+1})\sum_{k=n+1}^{\infty}|a_k|} \leq 1, \quad z \in U, \tag{4.12}$$

which leads us immediately to the assertion (4.4) of Theorem 4.1.

The bound in (4.4) is sharp for each $n \in \mathbb{N}$ with the extremal function $f(z)$ given by (4.10). The proof of Theorem 4.1. is thus complete. □

Theorem 4.2. *If $f(z)$ of the form (1.1) satisfies the condition (2.1). Then*

$$\operatorname{Re}\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq 1-\frac{n+1}{c_{n+1}}. \tag{4.13}$$

Proof. By setting

$$\begin{aligned} g(z) &= c_{n+1}\left\{\frac{f'(z)}{f_n'(z)}-\left(1-\frac{n+1}{c_{n+1}}\right)\right\} \\ &= \frac{1+\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty}ka_kz^{k-1}+\sum_{k=2}^{\infty}ka_kz^{k-1}}{1+\sum_{k=2}^{n}ka_kz^{k-1}} \\ &= 1+\frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty}ka_kz^{k-1}}{1+\sum_{k=2}^{n}ka_kz^{k-1}}, \end{aligned} \tag{4.14}$$

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty}k|a_k|}{2-2\sum_{k=2}^{n}k|a_k|-\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty}k|a_k|}.$$

Now $\left|\frac{g(z)-1}{g(z)+1}\right| \leq 1$ if

$$\sum_{k=2}^{n}k|a_k|+\frac{c_{n+1}}{n+1}\sum_{k=n+1}^{\infty}k|a_k| \leq 1 \tag{4.15}$$

since the left hand side of (4.15) is bounded above by $\sum_{k=2}^{n}c_k|a_k|$ if

$$\sum_{k=2}^{n}(c_k-k)|a_k|+\sum_{k=n+1}^{\infty}c_k-\frac{c_{n+1}}{n+1}k|a_k| \geq 0, \tag{4.16}$$

and the proof is complete. The result is sharp for the extremal function $f(z)=z+\frac{z^{n+1}}{c_{n+1}}$. □

Theorem 4.3. *If $f(z)$ of the form (1.1) satisfies the condition (2.1) then*

$$\operatorname{Re}\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{n+1+c_{n+1}}.$$

Proof. By setting

$$\begin{aligned} g(z) &= [(n+1)+c_{n+1}]\left\{\frac{f_n'(z)}{f'(z)}-\frac{c_{n+1}}{n+1+c_{n+1}}\right\} \\ &= 1-\frac{\left(1+\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty}ka_kz^{k-1}}{1+\sum_{k=2}^{n}ka_kz^{k-1}} \end{aligned}$$

and making use of (4.16), we can deduce that

$$\left|\frac{g(z)-1}{g(z)+1}\right| \leq \frac{\left(1+\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty}k|a_k|}{2-2\sum_{k=2}^{n}k|a_k|-\left(1+\frac{c_{n+1}}{n+1}\right)\sum_{k=n+1}^{\infty}k|a_k|} \leq 1,$$

which leads us immediately to the assertion of the Theorem 4.3. □

Remark 4.4. We note that $\beta = 1$, and choosing $\lambda = 0$, $\lambda = 1$ these results coincide with the results obtained in [13].

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