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SOME INEQUALITIES ASSOCIATED WITH A LINEAR OPERATOR DEFINED FOR A CLASS OF MULTIVALENT FUNCTIONS

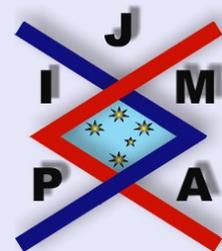
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Abstract

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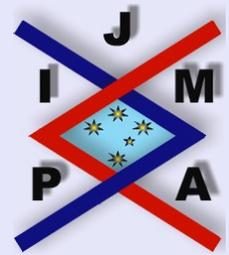


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Abstract

The authors derive several inequalities associated with differential subordinations between analytic functions and a linear operator defined for a certain family of p -valent functions, which is introduced here by means of this linear operator. Some special cases and consequences of the main results are also considered.

2000 Mathematics Subject Classification: Primary 30C45.

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1. Introduction, Definitions and Preliminaries

Let $\mathcal{A}(p, n)$ denote the class of functions f normalized by

$$(1.1) \quad f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\}),$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

In particular, we set

$$\mathcal{A}(p, 1) =: \mathcal{A}_p \quad \text{and} \quad \mathcal{A}(1, 1) =: \mathcal{A} = \mathcal{A}_1.$$

A function $f \in \mathcal{A}(p, n)$ is said to be in the class $\mathcal{A}(p, n; \alpha)$ if it satisfies the following inequality:

$$(1.2) \quad \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) < \alpha \quad (z \in \mathbb{U}; \alpha > p).$$

We also denote by $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$, respectively, the usual subclasses of \mathcal{A} consisting of functions which are *convex of order α* in \mathbb{U} and *starlike of order α* in \mathbb{U} . Thus we have (see, for details, [3] and [9])

$$(1.3) \quad \mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(1 + \frac{z f''(z)}{f'(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}$$



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and

$$(1.4) \quad \mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \text{ and } \Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\}.$$

In particular, we write

$$\mathcal{K}(0) =: \mathcal{K} \quad \text{and} \quad \mathcal{S}^*(0) =: \mathcal{S}^*.$$

For the above-defined class $\mathcal{A}(p, n; \alpha)$ of p -valent functions, Owa *et al.* [5] proved the following results.

Theorem A. (Owa *et al.* [5, p. 8, Theorem 1]). *If*

$$f(z) \in \mathcal{A}(p, n; \alpha) \quad \left(p < \alpha \leq p + \frac{1}{2}n \right),$$

then

$$(1.5) \quad \Re \left(\frac{f(z)}{zf'(z)} \right) > \frac{2p+n}{(2\alpha+n)p} \quad (z \in \mathbb{U}).$$

Theorem B. (Owa *et al.* [5, p. 10, Theorem 2]). *If*

$$f(z) \in \mathcal{A}(p, n; \alpha) \quad \left(p < \alpha \leq p + \frac{1}{2}n \right),$$

then

$$(1.6) \quad 0 < \Re \left(\frac{zf'(z)}{f(z)} \right) < \frac{(2\alpha+n)p}{2p+n} \quad (z \in \mathbb{U}).$$



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In fact, as already observed by Owa *et al.* [5, p. 10], various *further* special cases of (for example) Theorem B when $p = n = 1$ were considered earlier by Nunokawa [4], Saitoh *et al.* [7], and Singh and Singh [8].

The main object of this paper is to present an extension of each of the inequalities (1.5) and (1.6) asserted by Theorem A and Theorem B, respectively, to hold true for a linear operator associated with a certain general class $\mathcal{A}(p, n; a, c, \alpha)$ of p -valent functions, which we introduce here.

For two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k \quad (p, n \in \mathbb{N}),$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(1.7) \quad (f * g)(z) := z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k =: (g * f)(z).$$

In terms of the Pochhammer symbol $(\lambda)_k$ or the *shifted* factorial, since

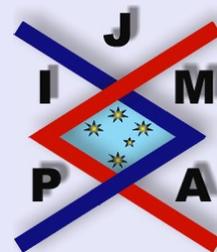
$$(1)_k = k! \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

given by

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_k := \lambda(\lambda + 1) \cdots (\lambda + k - 1) \quad (k \in \mathbb{N}),$$

we now define the function $\phi_p(a, c; z)$ by

$$(1.8) \quad \phi_p(a, c; z) := z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p}$$



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$$(z \in \mathbb{U}; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- := \{0, -1, -2, \dots\}).$$

Corresponding to the function $\phi_p(a, c; z)$, Saitoh [6] introduced a linear operator $L_p(a, c)$ which is defined by means of the following Hadamard product (or convolution):

$$(1.9) \quad L_p(a, c) f(z) := \phi_p(a, c; z) * f(z) \quad (f \in \mathcal{A}_p)$$

or, equivalently, by

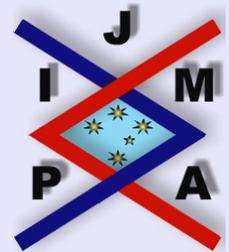
$$(1.10) \quad L_p(a, c) f(z) := z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+p} z^{k+p} \quad (z \in \mathbb{U}).$$

The definition (1.9) or (1.10) of the linear operator $L_p(a, c)$ is motivated essentially by the familiar Carlson-Shaffer operator

$$L(a, c) := L_1(a, c),$$

which has been used widely on such spaces of analytic and univalent functions in \mathbb{U} as $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$ defined by (1.3) and (1.4), respectively (see, for example, [9]). A linear operator $\mathcal{L}_p(a, c)$, analogous to $L_p(a, c)$ considered here, was investigated recently by Liu and Srivastava [2] on the space of *meromorphically* p -valent functions in \mathbb{U} . We remark in passing that a much more general convolution operator than the operator $L_p(a, c)$ considered here, involving the generalized hypergeometric function in the defining Hadamard product (or convolution), was introduced earlier by Dziok and Srivastava [1].

Making use of the linear operator $L_p(a, c)$ defined by (1.9) or (1.10), we say that a function $f \in \mathcal{A}(p, n)$ is in the aforementioned *general* class $\mathcal{A}(p, n; a, c, \alpha)$



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if it satisfies the following inequality:

$$(1.11) \quad \Re \left(\frac{L_p(a+2, c) f(z)}{L_p(a+1, c) f(z)} \right) < \alpha$$

$$(z \in \mathbb{U}; \alpha > 1; a \in \mathbb{R}; c \in \mathbb{R} \setminus \mathbb{Z}_0^-).$$

The Ruscheweyh derivative of $f(z)$ of order $\delta + p - 1$ is defined by

$$(1.12) \quad D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (f \in \mathcal{A}(p, n); \delta \in \mathbb{R} \setminus (-\infty, -p])$$

or, equivalently, by

$$(1.13) \quad D^{\delta+p-1} f(z) := z^p + \sum_{k=p+n}^{\infty} \binom{\delta+k-1}{k-p} a_k z^k$$

$$(f \in \mathcal{A}(p, n); \delta \in \mathbb{R} \setminus (-\infty, -p]).$$

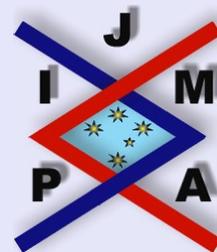
In particular, if $\delta = l$ ($l + p \in \mathbb{N}$), we find from the definition (1.12) or (1.13) that

$$(1.14) \quad D^{l+p-1} f(z) = \frac{z^p}{(l+p-1)!} \frac{d^{l+p-1}}{dz^{l+p-1}} \{z^{l-1} f(z)\},$$

$$(f \in \mathcal{A}(p, n); l + p \in \mathbb{N}).$$

Since

$$(1.15) \quad L_p(\delta + p, 1) f(z) = D^{\delta+p-1} f(z),$$



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$$(f \in \mathcal{A}(p, n); \delta \in \mathbb{R} \setminus (-\infty, -p]),$$

which can easily be verified by comparing the definitions (1.10) and (1.13), we may set

$$(1.16) \quad \mathcal{A}(p, n; \delta + p, 1, \alpha) =: \mathcal{A}(p, n; \delta, \alpha).$$

Thus a function $f \in \mathcal{A}(p, n)$ is in the class $\mathcal{A}(p, n; \delta, \alpha)$ if it satisfies the following inequality:

$$(1.17) \quad \Re \left(\frac{D^{\delta+p+1} f(z)}{D^{\delta+p} f(z)} \right) < \alpha,$$

$$(z \in \mathbb{U}; \alpha > 1; \delta \in \mathbb{R} \setminus (-\infty, -p]).$$

Finally, for two functions f and g analytic in \mathbb{U} , we say that the function $f(z)$ is *subordinate* to $g(z)$ in \mathbb{U} , and write

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in \mathbb{U} with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

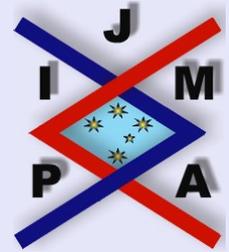
such that

$$(1.18) \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

In particular, if the function g is *univalent* in \mathbb{U} , the above subordination is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In our present investigation of the above-defined general class $\mathcal{A}(p, n; a, c, \alpha)$, we shall require each of the following lemmas.



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Lemma 1. (cf. Miller and Mocanu [3, p. 35, Theorem 2.3i (i)]). Let Ω be a set in the complex plane \mathbb{C} and suppose that $\Phi(u, v; z)$ is a complex-valued mapping:

$$\Phi : \mathbb{C}^2 \times \mathbb{U} \rightarrow \mathbb{C},$$

where

$$u = u_1 + iu_2 \quad \text{and} \quad v = v_1 + iv_2.$$

Also let $\Phi(iu_2, v_1; z) \notin \Omega$ for all $z \in \mathbb{U}$ and for all real u_2 and v_1 such that

$$(1.19) \quad v_1 \leq -\frac{1}{2}n(1 + u_2^2).$$

If

$$q(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots$$

is analytic in \mathbb{U} and

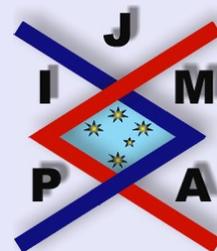
$$\Phi(q(z), zq'(z); z) \in \Omega \quad (z \in \mathbb{U}),$$

then

$$\Re \{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Lemma 2. (cf. Miller and Mocanu [3, p. 132, Theorem 3.4h]). Let $\psi(z)$ be univalent in \mathbb{U} and suppose that the functions ϑ and φ are analytic in a domain $\mathbb{D} \supset \psi(\mathbb{U})$ with $\varphi(\zeta) \neq 0$ when $\zeta \in \psi(\mathbb{U})$. Define the functions $Q(z)$ and $h(z)$ by

$$(1.20) \quad Q(z) := z\psi'(z)\varphi(\psi(z)) \quad \text{and} \quad h(z) := \vartheta(\psi(z)) + Q(z),$$



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and assume that

(i) $Q(z)$ is starlike univalent in \mathbb{U}

and

(ii) $\Re \left(\frac{zh'(z)}{Q(z)} \right) > 0 \quad (z \in \mathbb{U}).$

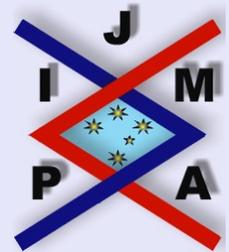
If

$$(1.21) \quad \vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec h(z) \quad (z \in \mathbb{U}),$$

then

$$q(z) \prec \psi(z) \quad (z \in \mathbb{U})$$

and $\psi(z)$ is the best dominant.



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2. Inequalities Involving the Linear Operator $L_p(a, c)$

By appealing to Lemma 1 of the preceding section, we first prove Theorem 1 below.

Theorem 1. *Let the parameters a and α satisfy the following inequalities:*

$$(2.1) \quad a > -1 \quad \text{and} \quad 1 < \alpha \leq 1 + \frac{n}{2(a+1)}.$$

If $f(z) \in \mathcal{A}(p, n; a, c, \alpha)$, then

$$(2.2) \quad \Re \left(\frac{L_p(a, c)f(z)}{L_p(a+1, c)f(z)} \right) > \frac{2a+n}{2\alpha(a+1)-2+n} \quad (z \in \mathbb{U})$$

and

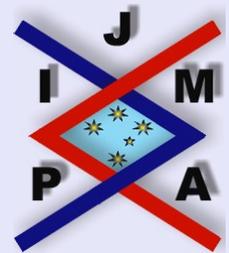
$$(2.3) \quad \Re \left(\frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} \right) < \frac{2\alpha(a+1)-2+n}{2a+n} \quad (z \in \mathbb{U}).$$

Proof. Define the function $q(z)$ by

$$(2.4) \quad (1-\beta)q(z) + \beta = \frac{L_p(a, c)f(z)}{L_p(a+1, c)f(z)} \quad (z \in \mathbb{U}),$$

where

$$(2.5) \quad \beta := \frac{2a+n}{2\alpha(a+1)-2+n}.$$



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Then, clearly, $q(z)$ is analytic in \mathbb{U} and

$$q(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \dots \quad (z \in \mathbb{U}).$$

By a simple computation, we observe from (2.4) that

$$(2.6) \quad \frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} = \frac{z(L_p(a,c)f(z))'}{L_p(a,c)f(z)} - \frac{z(L_p(a+1,c)f(z))'}{L_p(a+1,c)f(z)}.$$

Making use of the familiar identity:

$$(2.7) \quad z(L_p(a,c)f(z))' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z),$$

we find from (2.6) that

$$\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} = 1 + a \frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - (a+1) \frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)},$$

which, in view of (2.4), yields

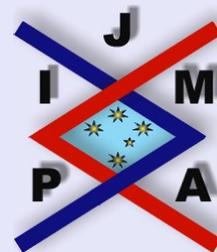
$$\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} = \frac{1}{a+1} + \frac{1}{a+1} \left(\frac{a}{(1-\beta)q(z)+\beta} - \frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} \right)$$

or, equivalently,

$$(2.8) \quad \frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} = \frac{1}{a+1} \left(1 + \frac{a - (1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} \right).$$

If we define $\Phi(u, v; z)$ by

$$(2.9) \quad \Phi(u, v; z) := \frac{1}{a+1} \left(1 + \frac{a - (1-\beta)v}{(1-\beta)u + \beta} \right),$$



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then, by the hypothesis of Theorem 1 that $f \in \mathcal{A}(p, n; a, c, \alpha)$, we have

$$\Re \{ \Phi (q(z), zq'(z); z) \} = \Re \left(\frac{L_p(a+2, c)f(z)}{L_p(a+1, c)f(z)} \right) < \alpha \quad (z \in \mathbb{U}; \alpha > 1).$$

We will now show that

$$\Re \{ \Phi (iu_2, v_1; z) \} \geq \alpha$$

for all $z \in \mathbb{U}$ and for all real u_2 and v_1 constrained by the inequality (1.19). Indeed we find from (2.9) that

$$\begin{aligned} \Re \{ \Phi (iu_2, v_1; z) \} &= \frac{1}{a+1} \left[1 + \Re \left(\frac{a - (1-\beta)v_1}{(1-\beta)iu_2 + \beta} \right) \right] \\ &= \frac{1}{a+1} \left[1 + \Re \left(\frac{[a - (1-\beta)v_1][\beta - (1-\beta)iu_2]}{(1-\beta)^2u_2^2 + \beta^2} \right) \right] \\ &= \frac{1}{a+1} \left(1 + \frac{[a - (1-\beta)v_1]\beta}{(1-\beta)^2u_2^2 + \beta^2} \right), \end{aligned}$$

so that, by using (1.19), we have

$$(2.10) \quad \Re \{ \Phi (iu_2, v_1; z) \} \geq \frac{1}{a+1} \left(1 + \frac{\beta[a + \frac{1}{2}n(1-\beta)(1+u_2^2)]}{(1-\beta)^2u_2^2 + \beta^2} \right) \quad (z \in \mathbb{U}).$$

From the inequalities in (2.1), we get

$$\frac{n}{2}\beta^2 \geq \left(a + \frac{1}{2}n(1-\beta) \right) (1-\beta),$$



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and hence the function

$$\frac{a + \frac{1}{2}n(1 - \beta)(1 + x^2)}{(1 - \beta)^2x^2 + \beta^2}$$

is an increasing function for $x \geq 0$. Thus we find from (2.10) that

$$\Re \{ \Phi(iu_2, v_1; z) \} \geq \frac{1}{a + 1} \left(1 + \frac{a + \frac{1}{2}n(1 - \beta)}{\beta} \right) = \alpha \quad (z \in \mathbb{U}).$$

The *first* assertion (2.2) of Theorem 1 follows by applying Lemma 1.

Next, we define the function $\psi(z)$ by

$$\psi(z) := \frac{L_p(a, c)f(z)}{L_p(a + 1, c)f(z)} \quad (z \in \mathbb{U}),$$

where β is given by (2.5). Then, in view of the already proven assertion (2.2) of Theorem 1, we have

$$(2.11) \quad \Re \{ \psi(z) \} > \beta > 0 \quad (z \in \mathbb{U})$$

so that

$$(2.12) \quad \Re \left(\frac{1}{\psi(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

Since (2.12) holds true, we have

$$\Re \{ \psi(z) \} \Re \left(\frac{1}{\psi(z)} \right) \leq |\psi(z)| \cdot \frac{1}{|\psi(z)|} = 1,$$



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or

$$\Re \left(\frac{1}{\psi(z)} \right) \leq \frac{1}{\Re \{ \psi(z) \}} \quad (z \in \mathbb{U}),$$

which, in view of (2.11), yields

$$0 < \Re \left(\frac{1}{\psi(z)} \right) < \frac{1}{\beta} \quad (z \in \mathbb{U})$$

which is the *second* assertion (2.3) of Theorem 1. □

The following result is a special case of Theorem 1 obtained by taking

$$a = \delta + p \quad \text{and} \quad c = 1.$$

Corollary 1. *If*

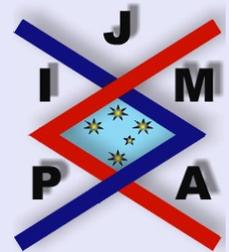
$$f(z) \in \mathcal{A}(p, n; \delta, \alpha) \quad \left(\delta + p > 1; 1 \leq \alpha < 1 + \frac{n}{2(\delta + p + 1)} \right),$$

then

$$\Re \left(\frac{D^{\delta+p-1} f(z)}{D^{\delta+p} f(z)} \right) > \frac{2\delta + 2p + n}{2\alpha(\delta + p + 1) - 2 + n} \quad (z \in \mathbb{U}),$$

and

$$\Re \left(\frac{D^{\delta+p} f(z)}{D^{\delta+p-1} f(z)} \right) < \frac{2\alpha(\delta + p + 1) - 2 + n}{2\delta + 2p + n} \quad (z \in \mathbb{U}).$$



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3. Further Results Involving Differential Subordination Between Analytic Functions

We begin by proving the following result.

Lemma 3. *Let the functions $q(z)$ and $\psi(z)$ be analytic in \mathbb{U} and suppose that*

$$\psi(z) \neq 0 \quad (z \in \mathbb{U})$$

is also univalent in \mathbb{U} and that $z\psi'(z)/\psi(z)$ is starlike univalent in \mathbb{U} . If

$$(3.1) \quad \Re \left(\frac{\alpha}{\beta} \frac{1}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right] \right) > 0,$$

$$(z \in \mathbb{U}; \alpha, \beta \in \mathbb{C}; \beta \neq 0)$$

and

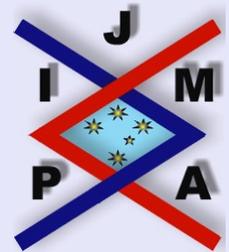
$$(3.2) \quad \frac{\alpha}{q(z)} - \beta \frac{zq'(z)}{q(z)} \prec \frac{\alpha}{\psi(z)} - \beta \frac{z\psi'(z)}{\psi(z)},$$

$$(z \in \mathbb{U}; \alpha, \beta \in \mathbb{C}; \beta \neq 0),$$

then

$$q(z) \prec \psi(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.



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Proof. By setting

$$\vartheta(\zeta) = \frac{\alpha}{\zeta} \quad \text{and} \quad \varphi(\zeta) = -\frac{\beta}{\zeta},$$

it is easily observed that both $\vartheta(\zeta)$ and $\varphi(\zeta)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\varphi(\zeta) \neq 0 \quad (\zeta \in \mathbb{C} \setminus \{0\}).$$

Also, by letting

$$(3.3) \quad Q(z) = z\psi'(z)\varphi(\psi(z)) = -\beta \frac{z\psi'(z)}{\psi(z)}$$

and

$$(3.4) \quad h(z) = \vartheta(\psi(z)) + Q(z) = \frac{\alpha}{\psi(z)} - \beta \frac{z\psi'(z)}{\psi(z)},$$

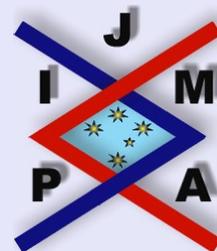
we find that $Q(z)$ is starlike univalent in \mathbb{U} and that

$$\Re \left(\frac{zh'(z)}{Q(z)} \right) = \Re \left(\frac{\alpha}{\beta} \frac{1}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right] \right) > 0,$$

$$(z \in \mathbb{U}; \alpha, \beta \in \mathbb{C}; \beta \neq 0),$$

by the hypothesis (3.1) of Lemma 3. Thus, by applying Lemma 2, our proof of Lemma 3 is completed. \square

We now prove the following result involving differential subordination between analytic functions.



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Theorem 2. Let the function $\psi(z) \neq 0$ ($z \in \mathbb{U}$) be analytic and univalent in \mathbb{U} and suppose that $z\psi'(z)/\psi(z)$ is starlike univalent in \mathbb{U} and

$$(3.5) \quad \Re \left(\frac{a}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right] \right) > 0$$

$$(z \in \mathbb{U}; a \in \mathbb{C} \setminus \{-1\}).$$

If $f \in \mathcal{A}_p$ satisfies the following subordination:

$$(3.6) \quad \frac{L_p(a+2, c)f(z)}{L_p(a+1, c)f(z)} \prec \frac{1}{a+1} \left(1 + \frac{a - z\psi'(z)}{\psi(z)} \right) \quad (z \in \mathbb{U}),$$

then

$$(3.7) \quad \frac{L_p(a, c)f(z)}{L_p(a+1, c)f(z)} \prec \psi(z) \quad (z \in \mathbb{U})$$

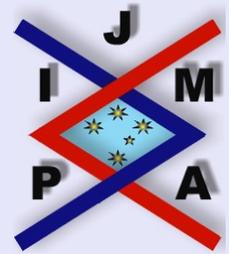
and $\psi(z)$ is the best dominant.

Proof. Let the function $q(z)$ be defined by

$$q(z) := \frac{L_p(a, c)f(z)}{L_p(a+1, c)f(z)} \quad (z \in \mathbb{U}; f \in \mathcal{A}_p),$$

so that, by a straightforward computation, we have

$$(3.8) \quad \frac{zq'(z)}{q(z)} = \frac{z(L_p(a, c)f(z))'}{L_p(a, c)f(z)} - \frac{z(L_p(a+1, c)f(z))'}{L_p(a+1, c)f(z)},$$



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which follows also from (2.6) in the special case when $\beta = 0$.

Making use of the familiar identity (2.7) once again (or *directly* from (2.8) with $\beta = 0$), we find that

$$\begin{aligned} \frac{L_p(a+2, c)f(z)}{L_p(a+1, c)f(z)} &= a \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} - (a+1) \frac{L_p(a+2, c)f(z)}{L_p(a+1, c)f(z)} + 1 \\ &= \frac{1}{a+1} \left(1 + \frac{a}{q(z)} - \frac{zq'(z)}{q(z)} \right), \end{aligned}$$

which, in light of the hypothesis (3.6) of Theorem 2, yields the following subordination:

$$\frac{a}{q(z)} - \frac{zq'(z)}{q(z)} \prec \frac{a}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}).$$

The assertion (3.7) of Theorem 2 now follows from Lemma 3. □

Remark 1. *If the function $\psi(z)$ is such that*

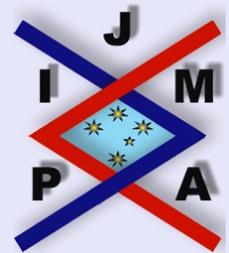
$$\Re \{ \psi(z) \} > 0 \quad (z \in \mathbb{U})$$

and if $z\psi'(z)/\psi(z)$ is starlike in \mathbb{U} , then the condition (3.5) is satisfied for $a > 0$.

In its special case when

$$a = \delta + p \quad \text{and} \quad c = 1,$$

Theorem 2 yields the following result.



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Corollary 2. Let the function $\psi(z) \neq 0$ ($z \in \mathbb{U}$) be analytic and univalent in \mathbb{U} and suppose that $z\psi'(z)/\psi(z)$ is starlike univalent in \mathbb{U} and

$$\Re \left(\frac{\delta + p}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)} \right] \right) > 0 \quad (z \in \mathbb{U}; \delta \in \mathbb{R} \setminus (-\infty, p]).$$

If $f \in \mathcal{A}$ satisfies the following subordination:

$$\frac{D^{\delta+p+1}f(z)}{D^{\delta+p}f(z)} \prec \frac{1}{\delta + p + 1} \left(1 + \frac{\delta + p - z\psi'(z)}{\psi(z)} \right) \quad (z \in \mathbb{U}),$$

then

$$\frac{D^{\delta+p-1}f(z)}{D^{\delta+p}f(z)} \prec \psi(z) \quad (z \in \mathbb{U}).$$

Lastly, by using a similar technique as above, we can prove Theorem 3 below.

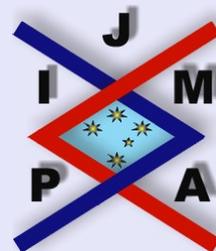
Theorem 3. If $f \in \mathcal{A}(p, n)$ and

$$(3.9) \quad 1 + \frac{zf''(z)}{f'(z)} \prec p \frac{1 + Bz^n}{1 + Az^n} - \frac{n(A - B)z^n}{(1 + Az^n)(1 + Bz^n)},$$

$$(z \in \mathbb{U}; -1 \leq B < A \leq 1),$$

then

$$(3.10) \quad \frac{pf(z)}{zf'(z)} \prec \frac{1 + Az^n}{1 + Bz^n} \quad (z \in \mathbb{U}).$$



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Proof. Let the function $q(z)$ be defined by

$$(3.11) \quad q(z) := \frac{pf(z)}{zf'(z)} \quad (z \in \mathbb{U}; f \in \mathcal{A}(p, n)),$$

so that

$$(3.12) \quad 1 + \frac{zf''(z)}{f'(z)} = \frac{p}{q(z)} - \frac{zq'(z)}{q(z)}.$$

If the function $\psi(z)$ is defined by

$$\psi(z) := \frac{1 + Az^n}{1 + Bz^n} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}),$$

then, in view of (3.9) and (3.12), we get

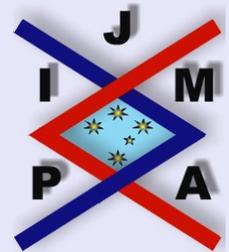
$$\frac{p}{q(z)} - \frac{zq'(z)}{q(z)} \prec \frac{p}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}).$$

The result (Theorem 3) now follows from Lemma 3 (with $\alpha = p$ and $\beta = 1$). □

The following result is a simple consequence of Theorem 3.

Corollary 3. *If $f \in \mathcal{A}$ satisfies the following subordination:*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1 - 4z + z^2}{1 - z^2} \quad (z \in \mathbb{U}),$$



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then

$$(3.13) \quad \Re \left(\frac{f(z)}{zf'(z)} \right) > 0 \quad (z \in \mathbb{U})$$

or, equivalently, f is starlike in \mathbb{U} (that is, $f \in \mathcal{S}^*$).

Remark 2. The foregoing analysis can be applied *mutatis mutandis* in order to rederive Theorem A of Owa et al. [5]. Indeed, if

$$(3.14) \quad f(z) \in \mathcal{A}(p, n; \alpha) \quad \left(p < \alpha \leq p + \frac{1}{2}n \right),$$

then we can first show that

$$1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \quad (z \in \mathbb{U}),$$

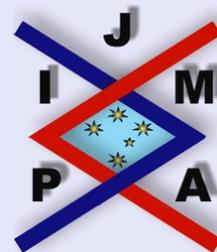
where

$$\psi(z) := p \frac{1 + Bz^n}{1 + Az^n} - \frac{n(A - B)z^n}{(1 + Az^n)(1 + Bz^n)} = \frac{p(1 + Bz^n)^2 - n(A + 1)z^n}{(1 + Az^n)(1 - z^n)}$$

$$\left(A = 1 - 2\beta; B = -1; \beta = \frac{2p + n}{2\alpha + n} \right).$$

By letting

$$u(\theta) := \Re \{ \psi(z) \} \quad (z = e^{i\theta/n} \in \partial\mathbb{U}; 0 \leq \theta \leq 2n\pi),$$



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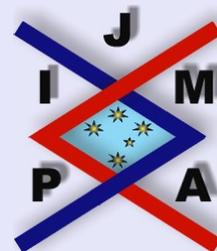
it is easily seen for

$$u(\theta) = \frac{(1 - A) [2p + n(1 + A) - 2p \cos \theta]}{2(1 + A^2 + 2A \cos \theta)} \quad (0 \leq \theta \leq 2n\pi)$$

that

$$(3.15) \quad u(\theta) \geq u(\pi) = \frac{(1 - A) [2p + n(1 + A) + 2p]}{2(1 - A)^2} = \alpha \quad (0 \leq \theta \leq 2n\pi),$$

which shows that $q(\mathbb{U})$ contains the half-plane $\Re(w) \leq \alpha$, where $q(z)$ is given, as before, by (3.11). Thus, under the hypothesis (3.14), we have the subordination (3.9) and hence (by Theorem 3) also the subordination (3.10), which leads us to the assertion (1.5) of Theorem A.



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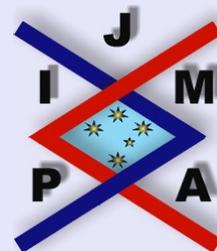
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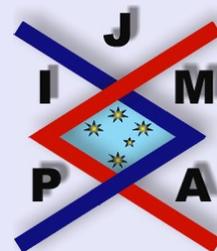
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