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SOME INEQUALITIES ASSOCIATED WITH A LINEAR OPERATOR DEFINED FOR A CLASS OF MULTIVALENT FUNCTIONS

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ABSTRACT. The authors derive several inequalities associated with differential subordinations between analytic functions and a linear operator defined for a certain family of p-valent functions, which is introduced here by means of this linear operator. Some special cases and consequences of the main results are also considered.

Key words and phrases: Analytic functions, Univalent and multivalent functions, Differential subordination, Schwarz function, Ruscheweyh derivatives, Hadamard product (or convolution), Linear operator, Convex functions, Starlike functions.

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1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Let $\mathcal{A}(p,n)$ denote the class of functions f *normalized* by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \qquad (p,n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.1}$$

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which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

In particular, we set

$$\mathcal{A}(p,1) =: \mathcal{A}_p \quad \text{and} \quad \mathcal{A}(1,1) =: \mathcal{A} = \mathcal{A}_1.$$

A function $f \in \mathcal{A}(p,n)$ is said to be in the class $\mathcal{A}(p,n;\alpha)$ if it satisfies the following inequality:

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \quad (z \in \mathbb{U}; \alpha > p). \tag{1.2}$$

We also denote by $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$, respectively, the usual subclasses of $\mathcal{A}$ consisting of functions which are *convex of order* α in $\mathbb{U}$ and *starlike of order* α in $\mathbb{U}$. Thus we have (see, for details, [3] and [9])

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\} \tag{1.3}$$

and

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{A} \quad \text{and} \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}. \tag{1.4}$$

In particular, we write

$$\mathcal{K}(0) =: \mathcal{K} \quad \text{and} \quad \mathcal{S}^*(0) =: \mathcal{S}^*.$$

For the above-defined class $\mathcal{A}(p,n;\alpha)$ of p-valent functions, Owa *et al.* [5] proved the following results.

Theorem A. (*Owa et al.* [5, p. 8, Theorem 1]). *If*

$$f(z) \in \mathcal{A}(p,n;\alpha) \quad \left(p < \alpha \leqq p + \frac{1}{2}n\right),$$

then

$$\Re\left(\frac{f(z)}{zf'(z)}\right) > \frac{2p+n}{(2\alpha+n)p} \quad (z \in \mathbb{U}). \tag{1.5}$$

Theorem B. (*Owa et al.* [5, p. 10, Theorem 2]). *If*

$$f(z) \in \mathcal{A}(p,n;\alpha) \quad \left(p < \alpha \leqq p + \frac{1}{2}n\right),$$

then

$$0 < \Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{(2\alpha+n)p}{2p+n} \quad (z \in \mathbb{U}). \tag{1.6}$$

In fact, as already observed by Owa *et al.* [5, p. 10], various *further* special cases of (for example) Theorem B when $p = n = 1$ were considered earlier by Nunokawa [4], Saitoh *et al.* [7], and Singh and Singh [8].

The main object of this paper is to present an extension of each of the inequalities (1.5) and (1.6) asserted by Theorem A and Theorem B, respectively, to hold true for a linear operator associated with a certain general class $\mathcal{A}(p,n;a,c,\alpha)$ of p-valent functions, which we introduce here.

For two functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+n}^{\infty} b_k\, z^k \qquad (p, n \in \mathbb{N}),$$

the Hadamard product (or convolution) $(f * g)(z)$ is defined, as usual, by

$$(f * g)(z) := z^p + \sum_{k=p+n}^{\infty} a_k\, b_k\, z^k =: (g * f)(z). \tag{1.7}$$

In terms of the Pochhammer symbol $(\lambda)_k$ or the *shifted* factorial, since

$$(1)_k = k! \quad (k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),$$

given by

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_k := \lambda(\lambda+1)\cdots(\lambda+k-1) \quad (k \in \mathbb{N}),$$

we now define the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) := z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k}\, z^{k+p} \tag{1.8}$$

$$\left(z \in \mathbb{U};\ a \in \mathbb{R};\ c \in \mathbb{R}\setminus \mathbb{Z}_0^-;\ \mathbb{Z}_0^- := \{0, -1, -2, \ldots\}\right).$$

Corresponding to the function $\phi_p(a, c; z)$, Saitoh [6] introduced a linear operator $L_p(a, c)$ which is defined by means of the following Hadamard product (or convolution):

$$L_p(a, c)\, f(z) := \phi_p(a, c; z) * f(z) \quad (f \in \mathcal{A}_p) \tag{1.9}$$

or, equivalently, by

$$L_p(a, c)\, f(z) := z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k}\, a_{k+p}\, z^{k+p} \quad (z \in \mathbb{U}). \tag{1.10}$$

The definition (1.9) or (1.10) of the linear operator $L_p(a, c)$ is motivated essentially by the familiar Carlson-Shaffer operator

$$L(a, c) := L_1(a, c),$$

which has been used widely on such spaces of analytic and univalent functions in $\mathbb{U}$ as $\mathcal{K}(\alpha)$ and $\mathcal{S}^*(\alpha)$ defined by (1.3) and (1.4), respectively (see, for example, [9]). A linear operator $\mathcal{L}_p(a, c)$, analogous to $L_p(a, c)$ considered here, was investigated recently by Liu and Srivastava [2] on the space of *meromorphically* p-valent functions in $\mathbb{U}$. We remark in passing that a much more general convolution operator than the operator $L_p(a, c)$ considered here, involving the generalized hypergeometric function in the defining Hadamard product (or convolution), was introduced earlier by Dziok and Srivastava [1].

Making use of the linear operator $L_p(a, c)$ defined by (1.9) or (1.10), we say that a function $f \in \mathcal{A}(p, n)$ is in the aforementioned *general* class $\mathcal{A}(p, n; a, c, \alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{L_p(a+2, c)\, f(z)}{L_p(a+1, c)\, f(z)}\right) < \alpha \tag{1.11}$$

$$\left(z \in \mathbb{U};\ \alpha > 1;\ a \in \mathbb{R};\ c \in \mathbb{R}\setminus\mathbb{Z}_0^-\right).$$

The Ruscheweyh derivative of $f(z)$ of order $\delta + p - 1$ is defined by

$$D^{\delta+p-1} f(z) := \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (f \in \mathcal{A}(p, n);\ \delta \in \mathbb{R}\setminus(-\infty, -p]) \tag{1.12}$$

or, equivalently, by

$$D^{\delta+p-1}\,f(z):=z^p+\sum_{k=p+n}^{\infty}\binom{\delta+k-1}{k-p}a_k\,z^k \tag{1.13}$$

$$\left(f\in\mathcal{A}(p,n);\ \delta\in\mathbb{R}\backslash(-\infty,-p]\right).$$

In particular, if $\delta=l$ $(l+p\in\mathbb{N})$, we find from the definition (1.12) or (1.13) that

$$D^{l+p-1}\,f(z)=\frac{z^p}{(l+p-1)!}\,\frac{d^{l+p-1}}{dz^{l+p-1}}\left\{z^{l-1}\,f(z)\right\}, \tag{1.14}$$

$$\left(f\in\mathcal{A}(p,n);\ l+p\in\mathbb{N}\right).$$

Since

$$L_p(\delta+p,1)\,f(z)=D^{\delta+p-1}\,f(z), \tag{1.15}$$

$$\left(f\in\mathcal{A}(p,n);\ \delta\in\mathbb{R}\backslash(-\infty,-p]\right),$$

which can easily be verified by comparing the definitions (1.10) and (1.13), we may set

$$\mathcal{A}(p,n;\delta+p,1,\alpha)=:\mathcal{A}(p,n;\delta,\alpha). \tag{1.16}$$

Thus a function $f\in\mathcal{A}(p,n)$ is in the class $\mathcal{A}(p,n;\delta,\alpha)$ if it satisfies the following inequality:

$$\Re\left(\frac{D^{\delta+p+1}\,f(z)}{D^{\delta+p}\,f(z)}\right)<\alpha, \tag{1.17}$$

$$\left(z\in\mathbb{U};\ \alpha>1;\ \delta\in\mathbb{R}\backslash(-\infty,-p]\right).$$

Finally, for two functions f and g analytic in $\mathbb{U}$, we say that the function $f(z)$ is *subordinate* to $g(z)$ in $\mathbb{U}$, and write

$$f\prec g \quad \text{or} \quad f(z)\prec g(z) \quad (z\in\mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0)=0 \quad \text{and} \quad |w(z)|<1 \quad (z\in\mathbb{U}),$$

such that

$$f(z)=g\big(w(z)\big) \quad (z\in\mathbb{U}). \tag{1.18}$$

In particular, if the function g is *univalent* in $\mathbb{U}$, the above subordination is equivalent to

$$f(0)=g(0) \quad \text{and} \quad f(\mathbb{U})\subset g(\mathbb{U}).$$

In our present investigation of the above-defined general class $\mathcal{A}(p,n;a,c,\alpha)$, we shall require each of the following lemmas.

Lemma 1. (*cf. Miller and Mocanu* [3, p. 35, Theorem 2.3i (i)]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that $\Phi(u,v;z)$ is a complex-valued mapping:*

$$\Phi:\mathbb{C}^2\times\mathbb{U}\to\mathbb{C},$$

where

$$u=u_1+iu_2 \quad \textit{and} \quad v=v_1+iv_2.$$

Also let $\Phi(iu_2,v_1;z)\notin\Omega$ for all $z\in\mathbb{U}$ and for all real u_2 and v_1 such that

$$v_1\leqq-\frac{1}{2}n\left(1+u_2^2\right). \tag{1.19}$$

If

$$q(z)=1+c_n\,z^n+c_{n+1}\,z^{n+1}+\cdots$$

is analytic in $\mathbb{U}$ *and*

$$\Phi\left(q\left(z\right),zq'\left(z\right);z\right)\in\Omega\quad\left(z\in\mathbb{U}\right),$$

then

$$\Re\left\{q\left(z\right)\right\}>0\quad\left(z\in\mathbb{U}\right).$$

Lemma 2. (*cf. Miller and Mocanu* [3, p. 132, Theorem 3.4h]). *Let* $\psi\left(z\right)$ *be univalent in* $\mathbb{U}$ *and suppose that the functions* ϑ *and* φ *are analytic in a domain* $\mathbb{D}\supset\psi\left(\mathbb{U}\right)$ *with* $\varphi\left(\zeta\right)\neq0$ *when* $\zeta\in\psi\left(\mathbb{U}\right)$. *Define the functions* $Q\left(z\right)$ *and* $h\left(z\right)$ *by*

$$Q\left(z\right):=z\psi'\left(z\right)\varphi\Big(\psi\left(z\right)\Big)\quad\textit{and}\quad h\left(z\right):=\vartheta\Big(\psi\left(z\right)\Big)+Q\left(z\right),\tag{1.20}$$

and assume that

(i) $Q\left(z\right)$ *is starlike univalent in* $\mathbb{U}$

and

(ii) $\Re\left(\dfrac{zh'\left(z\right)}{Q\left(z\right)}\right)>0\quad\left(z\in\mathbb{U}\right).$

If

$$\vartheta\Big(q\left(z\right)\Big)+zq'\left(z\right)\varphi\Big(q\left(z\right)\Big)\prec h\left(z\right)\quad\left(z\in\mathbb{U}\right),\tag{1.21}$$

then

$$q\left(z\right)\prec\psi\left(z\right)\quad\left(z\in\mathbb{U}\right)$$

and $\psi\left(z\right)$ *is the best dominant.*

2. INEQUALITIES INVOLVING THE LINEAR OPERATOR $L_p(a,c)$

By appealing to Lemma 1 of the preceding section, we first prove Theorem 1 below.

Theorem 1. *Let the parameters* a *and* α *satisfy the following inequalities:*

$$a>-1\quad\textit{and}\quad 1<\alpha\leqq1+\frac{n}{2(a+1)}.\tag{2.1}$$

If $f(z)\in\mathcal{A}(p,n;a,c,\alpha)$, *then*

$$\Re\left(\frac{L_p(a,c)f(z)}{L_p(a+1,c)f(z)}\right)>\frac{2a+n}{2\alpha(a+1)-2+n}\quad(z\in\mathbb{U})\tag{2.2}$$

and

$$\Re\left(\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}\right)<\frac{2\alpha(a+1)-2+n}{2a+n}\quad(z\in\mathbb{U}).\tag{2.3}$$

Proof. Define the function $q(z)$ by

$$(1-\beta)q(z)+\beta=\frac{L_p(a,c)f(z)}{L_p(a+1,c)f(z)}\quad(z\in\mathbb{U}),\tag{2.4}$$

where

$$\beta:=\frac{2a+n}{2\alpha(a+1)-2+n}.\tag{2.5}$$

Then, clearly, $q(z)$ is analytic in $\mathbb{U}$ and

$$q(z)=1+c_n\,z^n+c_{n+1}\,z^{n+1}+\cdots\quad(z\in\mathbb{U}).$$

By a simple computation, we observe from (2.4) that

$$\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} = \frac{z\big(L_p(a,c)f(z)\big)'}{L_p(a,c)f(z)} - \frac{z\big(L_p(a+1,c)f(z)\big)'}{L_p(a+1,c)f(z)}. \tag{2.6}$$

Making use of the familiar identity:

$$z\big(L_p(a,c)f(z)\big)' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z), \tag{2.7}$$

we find from (2.6) that

$$\frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta} = 1 + a\,\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - (a+1)\,\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)},$$

which, in view of (2.4), yields

$$\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} = \frac{1}{a+1} + \frac{1}{a+1}\left(\frac{a}{(1-\beta)q(z)+\beta} - \frac{(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta}\right)$$

or, equivalently,

$$\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} = \frac{1}{a+1}\left(1 + \frac{a-(1-\beta)zq'(z)}{(1-\beta)q(z)+\beta}\right). \tag{2.8}$$

If we define $\Phi(u,v;z)$ by

$$\Phi(u,v;z) := \frac{1}{a+1}\left(1 + \frac{a-(1-\beta)v}{(1-\beta)u+\beta}\right), \tag{2.9}$$

then, by the hypothesis of Theorem 1 that $f \in \mathcal{A}(p,n;a,c,\alpha)$, we have

$$\Re\left\{\Phi\left(q(z), zq'(z); z\right)\right\} = \Re\left(\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)}\right) < \alpha \quad (z \in \mathbb{U};\ \alpha > 1).$$

We will now show that

$$\Re\left\{\Phi\left(iu_2, v_1; z\right)\right\} \geqq \alpha$$

for all $z \in \mathbb{U}$ and for all real u_2 and v_1 constrained by the inequality (1.19). Indeed we find from (2.9) that

$$\begin{aligned}\Re\left\{\Phi\left(iu_2, v_1; z\right)\right\} &= \frac{1}{a+1}\left[1 + \Re\left(\frac{a-(1-\beta)v_1}{(1-\beta)iu_2+\beta}\right)\right] \\ &= \frac{1}{a+1}\left[1 + \Re\left(\frac{[a-(1-\beta)v_1][\beta-(1-\beta)iu_2]}{(1-\beta)^2u_2^2+\beta^2}\right)\right] \\ &= \frac{1}{a+1}\left(1 + \frac{[a-(1-\beta)v_1]\beta}{(1-\beta)^2u_2^2+\beta^2}\right),\end{aligned}$$

so that, by using (1.19), we have

$$\Re\left\{\Phi\left(iu_2, v_1; z\right)\right\} \geqq \frac{1}{a+1}\left(1 + \frac{\beta[a+\frac{1}{2}n(1-\beta)(1+u_2^2)]}{(1-\beta)^2u_2^2+\beta^2}\right) \quad (z \in \mathbb{U}). \tag{2.10}$$

From the inequalities in (2.1), we get

$$\frac{n}{2}\beta^2 \geqq \left(a + \frac{1}{2}n(1-\beta)\right)(1-\beta),$$

and hence the function

$$\frac{a+\frac{1}{2}n(1-\beta)(1+x^2)}{(1-\beta)^2x^2+\beta^2}$$

is an increasing function for $x \geqq 0$. Thus we find from (2.10) that

$$\Re\left\{\Phi\left(iu_2, v_1; z\right)\right\} \geqq \frac{1}{a+1}\left(1+\frac{a+\frac{1}{2}n(1-\beta)}{\beta}\right) = \alpha \quad (z \in \mathbb{U}).$$

The *first* assertion (2.2) of Theorem 1 follows by applying Lemma 1.

Next, we define the function $\psi(z)$ by

$$\psi(z) := \frac{L_p(a,c)f(z)}{L_p(a+1,c)f(z)} \quad (z \in \mathbb{U}),$$

where β is given by (2.5). Then, in view of the already proven assertion (2.2) of Theorem 1, we have

$$\Re\left\{\psi(z)\right\} > \beta > 0 \quad (z \in \mathbb{U}) \tag{2.11}$$

so that

$$\Re\left(\frac{1}{\psi(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{2.12}$$

Since (2.12) holds true, we have

$$\Re\left\{\psi(z)\right\}\Re\left(\frac{1}{\psi(z)}\right) \leqq |\psi(z)| \cdot \frac{1}{|\psi(z)|} = 1,$$

or

$$\Re\left(\frac{1}{\psi(z)}\right) \leqq \frac{1}{\Re\left\{\psi(z)\right\}} \quad (z \in \mathbb{U}),$$

which, in view of (2.11), yields

$$0 < \Re\left(\frac{1}{\psi(z)}\right) < \frac{1}{\beta} \qquad (z \in \mathbb{U})$$

which is the *second* assertion (2.3) of Theorem 1. □

The following result is a special case of Theorem 1 obtained by taking

$$a = \delta + p \quad \text{and} \quad c = 1.$$

Corollary 1. *If*

$$f(z) \in \mathcal{A}(p,n;\delta,\alpha) \quad \left(\delta+p>1;\ 1 \leqq \alpha < 1+\frac{n}{2(\delta+p+1)}\right),$$

then

$$\Re\left(\frac{D^{\delta+p-1}f(z)}{D^{\delta+p}f(z)}\right) > \frac{2\delta+2p+n}{2\alpha(\delta+p+1)-2+n} \quad (z \in \mathbb{U}),$$

and

$$\Re\left(\frac{D^{\delta+p}f(z)}{D^{\delta+p-1}f(z)}\right) < \frac{2\alpha(\delta+p+1)-2+n}{2\delta+2p+n} \quad (z \in \mathbb{U}).$$

3. FURTHER RESULTS INVOLVING DIFFERENTIAL SUBORDINATION BETWEEN ANALYTIC FUNCTIONS

We begin by proving the following result.

Lemma 3. *Let the functions $q(z)$ and $\psi(z)$ be analytic in $\mathbb{U}$ and suppose that*

$$\psi(z) \neq 0 \quad (z \in \mathbb{U})$$

is also univalent in $\mathbb{U}$ and that $z\psi'(z)/\psi(z)$ is starlike univalent in $\mathbb{U}$. If

$$\Re\left(\frac{\alpha}{\beta}\frac{1}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0, \tag{3.1}$$

$$(z \in \mathbb{U};\ \alpha, \beta \in \mathbb{C};\ \beta \neq 0)$$

and

$$\frac{\alpha}{q(z)} - \beta\,\frac{zq'(z)}{q(z)} \prec \frac{\alpha}{\psi(z)} - \beta\,\frac{z\psi'(z)}{\psi(z)}, \tag{3.2}$$

$$(z \in \mathbb{U};\ \alpha, \beta \in \mathbb{C};\ \beta \neq 0),$$

then

$$q(z) \prec \psi(z) \quad (z \in \mathbb{U})$$

and $q(z)$ is the best dominant.

Proof. By setting

$$\vartheta(\zeta) = \frac{\alpha}{\zeta} \quad \text{and} \qquad \varphi(\zeta) = -\frac{\beta}{\zeta},$$

it is easily observed that both $\vartheta(\zeta)$ and $\varphi(\zeta)$ are analytic in $\mathbb{C}\backslash\{0\}$ and that

$$\varphi(\zeta) \neq 0 \quad (\zeta \in \mathbb{C}\backslash\,\{0\}).$$

Also, by letting

$$Q(z) = z\psi'(z)\varphi\big(\psi(z)\big) = -\beta\,\frac{z\psi'(z)}{\psi(z)} \tag{3.3}$$

and

$$h(z) = \vartheta\big(\psi(z)\big) + Q(z) = \frac{\alpha}{\psi(z)} - \beta\,\frac{z\psi'(z)}{\psi(z)}, \tag{3.4}$$

we find that $Q(z)$ is starlike univalent in $\mathbb{U}$ and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\alpha}{\beta}\,\frac{1}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0,$$

$$(z \in \mathbb{U};\ \alpha, \beta \in \mathbb{C};\ \beta \neq 0),$$

by the hypothesis (3.1) of Lemma 3. Thus, by applying Lemma 2, our proof of Lemma 3 is completed. □

We now prove the following result involving differential subordination between analytic functions.

Theorem 2. *Let the function* $\psi(z) \neq 0$ $(z \in \mathbb{U})$ *be analytic and univalent in* $\mathbb{U}$ *and suppose that* $z\psi'(z)/\psi(z)$ *is starlike univalent in* $\mathbb{U}$ *and*

$$\Re\left(\frac{a}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0 \tag{3.5}$$
$$(z \in \mathbb{U};\ a \in \mathbb{C}\setminus\{-1\}).$$

If $f \in \mathcal{A}_p$ *satisfies the following subordination:*

$$\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} \prec \frac{1}{a+1}\left(1 + \frac{a - z\psi'(z)}{\psi(z)}\right) \quad (z \in \mathbb{U}), \tag{3.6}$$

then

$$\frac{L_p(a,c)f(z)}{L_p(a+1,c)f(z)} \prec \psi(z) \quad (z \in \mathbb{U}) \tag{3.7}$$

and $\psi(z)$ *is the best dominant.*

Proof. Let the function $q(z)$ be defined by

$$q(z) := \frac{L_p(a,c)f(z)}{L_p(a+1,c)f(z)} \quad (z \in \mathbb{U};\ f \in \mathcal{A}_p),$$

so that, by a straightforward computation, we have

$$\frac{zq'(z)}{q(z)} = \frac{z\big(L_p(a,c)f(z)\big)'}{L_p(a,c)f(z)} - \frac{z\big(L_p(a+1,c)f(z)\big)'}{L_p(a+1,c)f(z)}, \tag{3.8}$$

which follows also from (2.6) in the special case when $\beta = 0$.

Making use of the familiar identity (2.7) once again (or *directly* from (2.8) with $\beta = 0$), we find that

$$\begin{aligned}\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} &= a\,\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - (a+1)\,\frac{L_p(a+2,c)f(z)}{L_p(a+1,c)f(z)} + 1\\ &= \frac{1}{a+1}\left(1 + \frac{a}{q(z)} - \frac{zq'(z)}{q(z)}\right),\end{aligned}$$

which, in light of the hypothesis (3.6) of Theorem 2, yields the following subordination:

$$\frac{a}{q(z)} - \frac{zq'(z)}{q(z)} \prec \frac{a}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}).$$

The assertion (3.7) of Theorem 2 now follows from Lemma 3. □

Remark 1. *If the function* $\psi(z)$ *is such that*

$$\Re\{\psi(z)\} > 0 \quad (z \in \mathbb{U})$$

and if $z\psi'(z)/\psi(z)$ *is starlike in* $\mathbb{U}$*, then the condition* (3.5) *is satisfied for* $a > 0$*.*

In its special case when

$$a = \delta + p \quad \text{and} \quad c = 1,$$

Theorem 2 yields the following result.

Corollary 2. *Let the function* $\psi(z) \neq 0$ $(z \in \mathbb{U})$ *be analytic and univalent in* $\mathbb{U}$ *and suppose that* $z\psi'(z)/\psi(z)$ *is starlike univalent in* $\mathbb{U}$ *and*

$$\Re\left(\frac{\delta+p}{\psi(z)} + \left[1 + \frac{z\psi''(z)}{\psi'(z)} - \frac{z\psi'(z)}{\psi(z)}\right]\right) > 0 \quad (z \in \mathbb{U};\ \delta \in \mathbb{R}\setminus(-\infty, p]).$$

If $f \in \mathcal{A}$ satisfies the following subordination:

$$\frac{D^{\delta+p+1}f(z)}{D^{\delta+p}f(z)} \prec \frac{1}{\delta+p+1}\left(1+\frac{\delta+p-z\psi'(z)}{\psi(z)}\right) \quad (z \in \mathbb{U}),$$

then

$$\frac{D^{\delta+p-1}f(z)}{D^{\delta+p}f(z)} \prec \psi(z) \quad (z \in \mathbb{U}).$$

Lastly, by using a similar technique as above, we can prove Theorem 3 below.

Theorem 3. *If $f \in \mathcal{A}(p,n)$ and*

(3.9) $$1+\frac{zf''(z)}{f'(z)} \prec p\,\frac{1+Bz^n}{1+Az^n} - \frac{n(A-B)z^n}{(1+Az^n)(1+Bz^n)},$$

$$(z \in \mathbb{U};\ -1 \leqq B < A \leqq 1),$$

then

(3.10) $$\frac{pf(z)}{zf'(z)} \prec \frac{1+Az^n}{1+Bz^n} \quad (z \in \mathbb{U}).$$

Proof. Let the function $q(z)$ be defined by

(3.11) $$q(z) := \frac{pf(z)}{zf'(z)} \quad (z \in \mathbb{U};\ f \in \mathcal{A}(p,n)),$$

so that

(3.12) $$1+\frac{zf''(z)}{f'(z)} = \frac{p}{q(z)} - \frac{zq'(z)}{q(z)}.$$

If the function $\psi(z)$ is defined by

$$\psi(z) := \frac{1+Az^n}{1+Bz^n} \quad (-1 \leqq B < A \leqq 1;\ z \in \mathbb{U}),$$

then, in view of (3.9) and (3.12), we get

$$\frac{p}{q(z)} - \frac{zq'(z)}{q(z)} \prec \frac{p}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)} \quad (z \in \mathbb{U}).$$

The result (Theorem 3) now follows from Lemma 3 (with $\alpha = p$ and $\beta = 1$). □

The following result is a simple consequence of Theorem 3.

Corollary 3. *If $f \in \mathcal{A}$ satisfies the following subordination:*

$$1+\frac{zf''(z)}{f'(z)} \prec \frac{1-4z+z^2}{1-z^2} \quad (z \in \mathbb{U}),$$

then

(3.13) $$\Re\left(\frac{f(z)}{zf'(z)}\right) > 0 \quad (z \in \mathbb{U})$$

or, equivalently, f is starlike in $\mathbb{U}$ (that is, $f \in \mathcal{S}^*$).

Remark 2. *The foregoing analysis can be applied* ***mutatis mutandis*** *in order to rederive Theorem* A *of Owa et al.* [5]. *Indeed, if*

$$f(z) \in \mathcal{A}(p,n;\alpha) \quad \left(p < \alpha \leqq p + \frac{1}{2}n\right), \tag{3.14}$$

then we can first show that

$$1 + \frac{zf''(z)}{f'(z)} \prec \psi(z) \quad (z \in \mathbb{U}),$$

where

$$\psi(z) := p\,\frac{1+Bz^n}{1+Az^n} - \frac{n(A-B)z^n}{(1+Az^n)(1+Bz^n)} = \frac{p(1+Bz^n)^2 - n(A+1)z^n}{(1+Az^n)(1-z^n)}$$

$$\left(A = 1-2\beta;\ B = -1;\ \beta = \frac{2p+n}{2\alpha+n}\right).$$

By letting

$$u(\theta) := \Re\{\psi(z)\} \quad \left(z = e^{i\theta/n} \in \partial\mathbb{U};\ 0 \leqq \theta \leqq 2n\pi\right),$$

it is easily seen for

$$u(\theta) = \frac{(1-A)\left[2p + n(1+A) - 2p\cos\theta\right]}{2(1+A^2+2A\cos\theta)} \quad (0 \leqq \theta \leqq 2n\pi)$$

that

$$u(\theta) \geqq u(\pi) = \frac{(1-A)\left[2p + n(1+A) + 2p\right]}{2(1-A)^2} = \alpha \quad (0 \leqq \theta \leqq 2n\pi), \tag{3.15}$$

which shows that $q(\mathbb{U})$ *contains the half-plane* $\Re(w) \leqq \alpha$, *where* $q(z)$ *is given, as before, by* (3.11). *Thus, under the hypothesis* (3.14), *we have the subordination* (3.9) *and hence* (*by Theorem* 3) *also the subordination* (3.10), *which leads us to the assertion* (1.5) *of Theorem* A.

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