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A NEW UPPER BOUND OF THE LOGARITHMIC MEAN

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ABSTRACT. Let a and b be positive numbers with $a \neq b$. The inequalities about the logarithmic-mean

$$L(a,b) < H_p(a,b) < M_q(a,b)$$

are obtained, where $p \geq \frac{1}{2}$ and $q \geq \frac{2}{3}p$. We would point out that $p = \frac{1}{2}$ and $q = \frac{1}{3}$ are the best constants such that above inequalities hold.

Key words and phrases: Logarithmic mean; Power mean; Heron mean; Best constant.

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1. Introduction and Main Results

The aim of this paper is to establish a new upper bound for the logarithmic mean.

Let a and b be positive numbers with $a \neq b$, $p > 0, q > 0$. The logarithmic mean is defined as

$$L(a,b) = \frac{b-a}{\log b - \log a},$$

The power mean is defined by

$$M_q(a,b) = \left(\frac{a^q + b^q}{2}\right)^{\frac{1}{q}},$$

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and the Heron mean is defined as

$$H_p(a,b) = \left(\frac{a^p + (ab)^{p/2} + b^p}{3}\right)^{\frac{1}{p}}.$$

There are many important results concerning $L(a,b)$, $M_p(a,b)$ and $H_q(a,b)$. The well known Lin Tong-Po inequality (see [1]) is stated as

$$L(a,b) < M_{\frac{1}{3}}(a,b). \tag{1.1}$$

In [2], Yang Z.H. obtained the inequalities

$$L(a,b) < M_{\frac{1}{2}}(a,b) < H_1(a,b). \tag{1.2}$$

In [1], Kuang J. C. summarized and stated the interpolation inequalities

$$L(a,b) < M_{\frac{1}{3}}(a,b) < M_{\frac{1}{2}}(a,b) < H_1(a,b) < M_{\frac{2}{3}}(a,b). \tag{1.3}$$

In this paper, we further improve the upper bound of the logarithmic mean and obtain the following theorem:

Theorem 1.1. *Let $p \geq \frac{1}{2}$, $q \geq \frac{2}{3}p$, and a, b be positive numbers with $a \neq b$. We then have*

$$L(a,b) < H_p(a,b) < M_q(a,b). \tag{1.4}$$

Furthermore, $p = \frac{1}{2}, q = \frac{2}{3}$ are the best constants for (1.4).

2. Proof of Theorem 1.1

In this section, there are two goals: the first is to state and prove some fundamental lemmas. The second is to prove our main result by virtue of these lemmas.

Lemma 2.1. *(*[3], [4]*). Suppose a and b are fixed positive numbers with $a \neq b$. For $p > 0$, then $H_p(a,b)$ and $M_p(a,b)$ are strictly monotone increasing functions with respect to p.*

Lemma 2.2. *Let $x > 1$. Then*

$$\frac{x-1}{\log x} < \left(\frac{x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1}{3}\right)^2. \tag{2.1}$$

Proof. Taking $t = x^{\frac{1}{4}}$, where $x > 1$, it is easy to see that inequality (2.1) is equivalent to

$$\frac{t^4 - 1}{4\log t} < \frac{1}{9}(t^2 + t + 1)^2. \tag{2.2}$$

Define the function

$$f(t) = \frac{4}{9}\log t - \frac{t^4 - 1}{(t^2 + t + 1)^2}. \tag{2.3}$$

Calculating the derivative for $f(t)$, we get

$$\begin{aligned} f'(t) &= \frac{4}{9t} - \frac{4t^3(t^2+t+1) - 2(t^4-1)(2t+1)}{(t^2+t+1)^3} \\ &= \frac{2(t-1)^4(2t^2+5t+2)}{9t(t^2+t+1)^3}. \end{aligned}$$

Since $t = x^{\frac{1}{4}} > 1$, we find that $f'(t) > 0$. Obviously, $f'(1) = 0$. So $f(t) > 0$ for $t > 1$. i.e. (2.1) holds. □

Lemma 2.3. *Let $x > 1$, then the following inequality holds*

$$\left(\frac{x^{\frac{1}{2}} + x^{\frac{1}{4}} + 1}{3}\right)^2 < \left(\frac{x^{\frac{1}{3}} + 1}{2}\right)^3. \tag{2.4}$$

Proof. Taking $t = x^{\frac{1}{12}}$, where $x > 1$, it is easy to see that inequality (2.4) is equivalent to

$$9(t^4 + 1)^3 > 8(t^6 + t^3 + 1)^2. \tag{2.5}$$

Define a function $g(t)$ as

$$g(t) = 9(t^4 + 1)^3 - 8(t^6 + t^3 + 1)^2.$$

Factorizing $g(t)$, we obtain

$$\begin{aligned} g(t) &= (t-1)^4(1 + 4t + 10t^2 + 4t^3 - 2t^4 + 4t^5 + 10t^6 + 4t^7 + t^8) \\ &= (t-1)^4((t^4-1)^2 + 4t + 10t^2 + 4t^3 + 4t^5 + 10t^6 + 4t^7). \end{aligned}$$

The proof is completed. □

Proof of Theorem 1.1. We first prove, for $p = \frac{1}{2}, q = \frac{1}{3}$, that (1.4) is true. In fact, since $a > 0, b > 0$ and $a \neq b$, there is no harm in supposing $b > a$. If we take $x = \frac{b}{a}$, using Lemma 2.2 and Lemma 2.3, we have

$$L(a,b) < H_{\frac{1}{2}}(a,b) < M_{\frac{1}{3}}(a,b). \tag{2.6}$$

For $q \geq \frac{2}{3}p$, there is the known result ([1])

$$H_p(a,b) < M_q(a,b), \quad (a \neq b). \tag{2.7}$$

Using Lemma 2.1, combining (2.6) and (2.7), we can conclude that

$$L(a,b) < H_{\frac{1}{2}}(a,b) < H_p(a,b) < M_q(a,b), \quad \left(p \geq \frac{1}{2},\, q \geq \frac{2}{3}p\right).$$

Next, we prove that $p = \frac{1}{2}$ and $q = \frac{1}{3}$ are the best constants for (1.4). Suppose we know that the following inequalities

$$L(x,1) < H_p(x,1) < M_q(x,1), \tag{2.8}$$

hold for any $x > 1$. There is no harm in supposing $1 < x \leq 2$. (In fact, if $n < x \leq n+1$, we can take $t = x - n$, where n is a positive integer.) Taking $t = x - 1$, applying Taylor's Theorem to the functions $L(x,1), H_p(x,1)$ and $M_q(x,1)$, we have

$$L(x,1) = L(t+1,1) = 1 + \frac{1}{2}t - \frac{1}{12}t^2 + \cdots, \tag{2.9}$$

$$H_p(x,1) = H_p(t+1,1) = 1 + \frac{1}{2}t + \frac{2p-3}{24}t^2 + \cdots, \tag{2.10}$$

$$M_q(x,1) = M_q(t+1,1) = 1 + \frac{1}{2}t + \frac{q-1}{8}t^2 + \cdots, \tag{2.11}$$

With simple manipulations (2.9), (2.10) and (2.11), together with (2.8), yield

$$-\frac{1}{12} \leq \frac{2p-3}{24} \leq \frac{q-1}{8}. \tag{2.12}$$

From (2.12), it immediately follows that

$$p \geq \frac{1}{2}, \text{ and } q \geq \frac{2}{3}p.$$

We then have, by virtue of Lemma 2.1, that $p = \frac{1}{2}$ and $q = \frac{1}{3}$ are the best constants for (1.4). □

Remark 2.4. It is easy to see that the best lower bound of the logarithmic mean is $H_0(a,b) = \sqrt{ab}$, namely $H_0 = G$, the geometric mean. In addition, using Lemma 2.1, combining (1.4), (2.7), (2.8) and the related results in [1], we derive the following graceful inequalities

$$\sqrt{ab} < L(a,b) < H_{\frac{1}{2}}(a,b) < M_{\frac{1}{3}}(a,b) < M_\alpha(a,b) < H_\beta(a,b) < M_\gamma(a,b),$$

where $\frac{1}{3} < \alpha < \frac{\log 2}{\log 3}\beta$, $\gamma \geq \frac{2}{3}\beta$, $\beta > \frac{\log 3}{3\log 2}$.

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