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PARTIAL SUMS OF FUNCTIONS OF BOUNDED TURNING

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Abstract

We determine conditions under which the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning.

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Contents

1. Introduction

Let $\mathcal{A}$ denote the family of functions f which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and are normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{U}. \tag{1.1}$$

For $0 \leq \alpha < 1$, let $\mathcal{B}(\alpha)$ denote the class of functions f of the form (1.1) so that $\Re(f') > \alpha$ in $\mathcal{U}$. The functions in $\mathcal{B}(\alpha)$ are called functions of bounded turning (c.f. [3, Vol. II]). By the Nashiro-Warschowski Theorem (see e.g. [3, Vol. I]) the functions in $\mathcal{B}(\alpha)$ are univalent and also close-to-convex in $\mathcal{U}$.

For f of the form (1.1), the Libera integral operator F is given by

$$F(z) = \frac{2}{z} \int_0^z f(\zeta) d\zeta = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k.$$

The n-th partial sums $F_n(z)$ of the Libera integral operator $F(z)$ are given by

$$F_n(z) = z + \sum_{k=2}^{n} \frac{2}{k+1} a_k z^k.$$

In [5] it was shown that if $f \in \mathcal{A}$ is starlike of order α, $\alpha = 0.294...$, then so is the Libera integral operator F. We also know that (see e.g. [1]), there are functions which are univalent or spiral-like in $\mathcal{U}$ so that their Libera integral operators are not univalent or spiral-like in $\mathcal{U}$. Li and Owa [4] proved that if $f \in \mathcal{A}$ is univalent in $\mathcal{U}$, then $F_n(z)$ is starlike in $|z| < \frac{3}{8}$. The number $\frac{3}{8}$ is sharp. In this paper we make use of a result of Gasper [2] to provide a simple proof for the following theorem.

Theorem 1.1 (Main Theorem). *If* $\frac{1}{4} \le \alpha < 1$ *and* $f \in \mathcal{B}(\alpha)$, *then* $F_n \in \mathcal{B}\left(\frac{4\alpha-1}{3}\right)$.

2. Preliminary Lemmas

To prove our Main Theorem, we shall need the following three lemmas. The first lemma is due to Gasper ([2, Theorem 1]) and the third lemma is a well-known and celebrated result (c.f. [3, Vol. I]) which can be derived from Herglotz's representation for positive real part functions.

Lemma 2.1. *Let θ be a real number and m and k be natural numbers. Then*

$$\frac{1}{3}+\sum_{k=1}^{m}\frac{\cos(k\theta)}{k+2}\geq 0. \tag{2.1}$$

Lemma 2.2. *For $z\in\mathcal{U}$ we have*

$$\Re\left(\sum_{k=1}^{m}\frac{z^k}{k+2}\right)>-\frac{1}{3}.$$

Proof. For $0\leq r<1$ and for $0\leq|\theta|\leq\pi$ write $z=re^{i\theta}=r(\cos(\theta)+i\sin(\theta))$. By DeMoivre's law and the minimum principle for harmonic functions, we have

$$\Re\left(\sum_{k=1}^{m}\frac{z^k}{k+2}\right)=\sum_{k=1}^{m}\frac{r^k\cos(k\theta)}{k+2}>\sum_{k=1}^{m}\frac{\cos(k\theta)}{k+2}. \tag{2.2}$$

Now by Abel's lemma (c.f. Titchmarsh [6]) and condition (2.1) of Lemma 2.1 we conclude that the right hand side of (2.2) is greater than or equal to $\frac{-1}{3}$. □

Lemma 2.3. *Let $P(z)$ be analytic in $\mathcal{U}$, $P(0) = 1$, and $\Re(P(z)) > \frac{1}{2}$ in $\mathcal{U}$. For functions Q analytic in $\mathcal{U}$ the convolution function $P * Q$ takes values in the convex hull of the image on $\mathcal{U}$ under Q.*

The operator "$*$" stands for the Hadamard product or convolution of two power series $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ denoted by $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$.

3. Proof of the Main Theorem

Let f be of the form (1.1) and belong to $\mathcal{B}(\alpha)$ for $\frac{1}{4} \leq \alpha < 1$. Since $\Re(f'(z)) > \alpha$ we have

$$\Re\left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1}\right) > \frac{1}{2}. \tag{3.1}$$

Applying the convolution properties of power series to $F_n'(z)$ we may write

$$\begin{aligned} F_n'(z) &= 1 + \sum_{k=2}^{n} \frac{2k}{k+1} a_k z^{k-1} \\ &= \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1}\right) * \left(1 + (1-\alpha) \sum_{k=2}^{n} \frac{4}{k+1} z^{k-1}\right) \\ &= P(z) * Q(z). \end{aligned} \tag{3.2}$$

From Lemma 2.2 for $m = n-1$ we obtain

$$\Re\left(\sum_{k=2}^{n} \frac{z^{k-1}}{k+1}\right) > -\frac{1}{3}. \tag{3.3}$$

Applying a simple algebra to the above inequality (3.3) and $Q(z)$ in (3.2) yields

$$\Re(Q(z)) = \Re\left(1 + (1-\alpha) \sum_{k=2}^{n} \frac{4}{k+1} z^{k-1}\right) > \frac{4\alpha - 1}{3}.$$

On the other hand, the power series $P(z)$ in (3.2) in conjunction with the condition (3.1) yields $\Re(P(z)) > \frac{1}{2}$. Therefore, by Lemma 2.3, $\Re(F_n'(z)) > \frac{4\alpha-1}{3}$. This concludes the Main Theorem.

Remark 3.1. *The Main Theorem also holds for $\alpha < \frac{1}{4}$. We also note that $\mathcal{B}(\alpha)$ for $\alpha < 0$ is no longer a bounded turning family.*

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