

JAY M. JAHANGIRI AND K. FARAHMAND

Kent State University Burton,
Ohio 44021-9500, USA.

EMail: jay@geauga.kent.edu

University of Ulster,
Jordanstown, BT37 0QB,
United Kingdom.

EMail: k.farahmand@ulster.ac.uk



volume 4, issue 4, article 79,
2003.

*Received 15 July, 2002;
accepted 10 June, 2003.*

Communicated by: N.E. Cho

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

We determine conditions under which the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning.

2000 Mathematics Subject Classification: Primary 30C45; Secondary 26D05.

Key words: Partial Sums, Bounded Turning, Libera Integral Operator.

Contents

1	Introduction	3
2	Preliminary Lemmas	5
3	Proof of the Main Theorem	7
	References	



Partial Sums of Functions of Bounded Turning

Jay M. Jahangiri and
K. Farahmand

Title Page

Contents



Go Back

Close

Quit

Page 2 of 9

1. Introduction

Let \mathcal{A} denote the family of functions f which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and are normalized by

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{U}.$$

For $0 \leq \alpha < 1$, let $\mathcal{B}(\alpha)$ denote the class of functions f of the form (1.1) so that $\Re(f') > \alpha$ in \mathcal{U} . The functions in $\mathcal{B}(\alpha)$ are called functions of bounded turning (c.f. [3, Vol. II]). By the Nashiro-Warschowski Theorem (see e.g. [3, Vol. I]) the functions in $\mathcal{B}(\alpha)$ are univalent and also close-to-convex in \mathcal{U} .

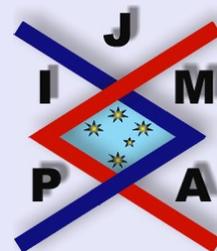
For f of the form (1.1), the Libera integral operator F is given by

$$F(z) = \frac{2}{z} \int_0^z f(\zeta) d\zeta = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k.$$

The n -th partial sums $F_n(z)$ of the Libera integral operator $F(z)$ are given by

$$F_n(z) = z + \sum_{k=2}^n \frac{2}{k+1} a_k z^k.$$

In [5] it was shown that if $f \in \mathcal{A}$ is starlike of order α , $\alpha = 0.294\dots$, then so is the Libera integral operator F . We also know that (see e.g. [1]), there are functions which are univalent or spiral-like in \mathcal{U} so that their Libera integral operators are not univalent or spiral-like in \mathcal{U} . Li and Owa [4] proved that if $f \in \mathcal{A}$ is univalent in \mathcal{U} , then $F_n(z)$ is starlike in $|z| < \frac{3}{8}$. The number $\frac{3}{8}$ is



Title Page

Contents



Go Back

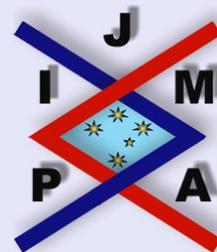
Close

Quit

Page 3 of 9

sharp. In this paper we make use of a result of Gasper [2] to provide a simple proof for the following theorem.

Theorem 1.1 (Main Theorem). *If $\frac{1}{4} \leq \alpha < 1$ and $f \in \mathcal{B}(\alpha)$, then $F_n \in \mathcal{B}\left(\frac{4\alpha-1}{3}\right)$.*



Partial Sums of Functions of Bounded Turning

Jay M. Jahangiri and
K. Farahmand

Title Page

Contents



Go Back

Close

Quit

Page 4 of 9

2. Preliminary Lemmas

To prove our Main Theorem, we shall need the following three lemmas. The first lemma is due to Gasper ([2, Theorem 1]) and the third lemma is a well-known and celebrated result (c.f. [3, Vol. I]) which can be derived from Herlotz's representation for positive real part functions.

Lemma 2.1. *Let θ be a real number and m and k be natural numbers. Then*

$$(2.1) \quad \frac{1}{3} + \sum_{k=1}^m \frac{\cos(k\theta)}{k+2} \geq 0.$$

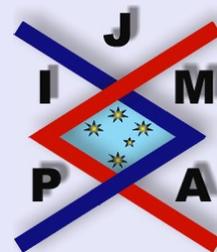
Lemma 2.2. *For $z \in \mathcal{U}$ we have*

$$\Re \left(\sum_{k=1}^m \frac{z^k}{k+2} \right) > -\frac{1}{3}.$$

Proof. For $0 \leq r < 1$ and for $0 \leq |\theta| \leq \pi$ write $z = re^{i\theta} = r(\cos(\theta) + i \sin(\theta))$. By DeMoivre's law and the minimum principle for harmonic functions, we have

$$(2.2) \quad \Re \left(\sum_{k=1}^m \frac{z^k}{k+2} \right) = \sum_{k=1}^m \frac{r^k \cos(k\theta)}{k+2} > \sum_{k=1}^m \frac{\cos(k\theta)}{k+2}.$$

Now by Abel's lemma (c.f. Titchmarsh [6]) and condition (2.1) of Lemma 2.1 we conclude that the right hand side of (2.2) is greater than or equal to $-\frac{1}{3}$. \square



Partial Sums of Functions of
Bounded Turning

Jay M. Jahangiri and
K. Farahmand

Title Page

Contents



Go Back

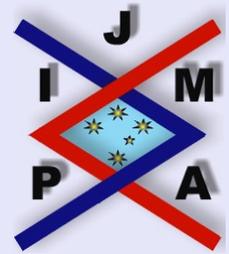
Close

Quit

Page 5 of 9

Lemma 2.3. Let $P(z)$ be analytic in \mathcal{U} , $P(0) = 1$, and $\Re(P(z)) > \frac{1}{2}$ in \mathcal{U} . For functions Q analytic in \mathcal{U} the convolution function $P * Q$ takes values in the convex hull of the image on \mathcal{U} under Q .

The operator “ $*$ ” stands for the Hadamard product or convolution of two power series $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ denoted by $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$.



Partial Sums of Functions of Bounded Turning

Jay M. Jahangiri and
K. Farahmand

Title Page

Contents



Go Back

Close

Quit

Page 6 of 9

3. Proof of the Main Theorem

Let f be of the form (1.1) and belong to $\mathcal{B}(\alpha)$ for $\frac{1}{4} \leq \alpha < 1$. Since $\Re(f'(z)) > \alpha$ we have

$$(3.1) \quad \Re \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1} \right) > \frac{1}{2}.$$

Applying the convolution properties of power series to $F'_n(z)$ we may write

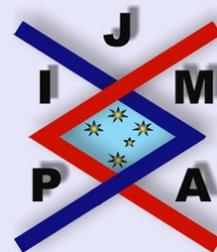
$$(3.2) \quad \begin{aligned} F'_n(z) &= 1 + \sum_{k=2}^n \frac{2k}{k+1} a_k z^{k-1} \\ &= \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1} \right) * \left(1 + (1-\alpha) \sum_{k=2}^n \frac{4}{k+1} z^{k-1} \right) \\ &= P(z) * Q(z). \end{aligned}$$

From Lemma 2.2 for $m = n - 1$ we obtain

$$(3.3) \quad \Re \left(\sum_{k=2}^n \frac{z^{k-1}}{k+1} \right) > -\frac{1}{3}.$$

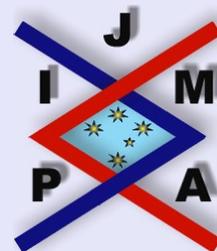
Applying a simple algebra to the above inequality (3.3) and $Q(z)$ in (3.2) yields

$$\Re(Q(z)) = \Re \left(1 + (1-\alpha) \sum_{k=2}^n \frac{4}{k+1} z^{k-1} \right) > \frac{4\alpha - 1}{3}.$$



On the other hand, the power series $P(z)$ in (3.2) in conjunction with the condition (3.1) yields $\Re(P(z)) > \frac{1}{2}$. Therefore, by Lemma 2.3, $\Re(F'_n(z)) > \frac{4\alpha-1}{3}$. This concludes the Main Theorem.

Remark 3.1. *The Main Theorem also holds for $\alpha < \frac{1}{4}$. We also note that $\mathcal{B}(\alpha)$ for $\alpha < 0$ is no longer a bounded turning family.*



Partial Sums of Functions of Bounded Turning

Jay M. Jahangiri and
K. Farahmand

Title Page

Contents



Go Back

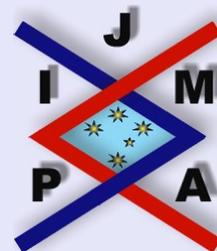
Close

Quit

Page 8 of 9

References

- [1] D.M. CAMPBELL AND V. SINGH, Valence properties of the solution of a differential equation, *Pacific J. Math.*, **84** (1979), 29–33.
- [2] G. GASPER, Nonnegative sums of cosines, ultraspherical and Jacobi polynomials, *J. Math. Anal. Appl.*, **26** (1969), 60–68.
- [3] A.W. GOODMAN, *Univalent Functions*, Vols. I & II, Mariner Pub. Co., Tampa, FL., 1983.
- [4] J.L. LI AND S. OWA, On partial sums of the Libera integral operator, *J. Math. Anal. Appl.*, **213** (1997), 444–454.
- [5] P.T. MOCANU, M.O. READE AND D. RIPEANU, The order of starlikeness of a Libera integral operator, *Mathematica (Cluj)*, **19** (1977), 67–73.
- [6] E.C. TITCHMARSH, *The Theory of Functions*, 2nd Ed., Oxford University Press, 1976.



Partial Sums of Functions of Bounded Turning

Jay M. Jahangiri and
K. Farahmand

Title Page

Contents



Go Back

Close

Quit

Page 9 of 9