

# Journal of Inequalities in Pure and Applied Mathematics

## THE SPREAD OF THE SHAPE OPERATOR AS CONFORMAL INVARIANT

BOGDAN D. SUCEAVĂ

Department of Mathematics  
California State University, Fullerton  
CA 92834-6850, U.S.A.

*E*Mail: [bsuceava@fullerton.edu](mailto:bsuceava@fullerton.edu)

*U*RL: <http://math.fullerton.edu/bsuceava>

©2000 Victoria University  
ISSN (electronic): 1443-5756  
060-03



---

volume 4, issue 4, article 74,  
2003.

*Received 01 May, 2003;*  
*accepted 27 September, 2003.*

*Communicated by: S.S. Dragomir*

---

Abstract

Contents



Home Page

Go Back

Close

Quit

## Abstract

The notion the spread of a matrix was first introduced fifty years ago in algebra. In this article, we define the spread of the shape operator by applying the same idea to submanifolds of Riemannian manifolds. We prove that the spread of shape operator is a conformal invariant for any submanifold in a Riemannian manifold. Then, we prove that, for a compact submanifold of a Riemannian manifold, the spread of the shape operator is bounded above by a geometric quantity proportional to the Willmore-Chen functional. For a complete non-compact submanifold, we establish a relationship between the spread of the shape operator and the Willmore-Chen functional. In the last section, we obtain a necessary and sufficient condition for a surface of rotation to have finite integral of the spread of the shape operator.

*2000 Mathematics Subject Classification:* 53B25, 53B20, 53A30.

*Key words:* Principal curvatures, Shape operator, Extrinsic scalar curvature, Surfaces of rotation

## Contents

1	Introduction .....	3
2	Geometric Inequalities on Compact Submanifolds .....	7
3	The Noncompact Case .....	11
	References	



---

### The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 16

# 1. Introduction

In the classic matrix theory *spread of a matrix* has been defined by Mirsky in [7] and then mentioned in various references, as for example [6]. Let  $A \in M_n(\mathbb{C})$ ,  $n \geq 3$ , and let  $\lambda_1, \dots, \lambda_n$  be the characteristic roots of  $A$ . The *spread* of  $A$  is defined to be  $s(A) = \max_{i,j} |\lambda_i - \lambda_j|$ . Let us denote by  $\|A\|$  the Euclidean norm of the matrix  $A$ , i.e.:  $\|A\|^2 = \sum_{i,j=1}^{m,n} |a_{ij}|^2$ . We use also the classical notation  $E_2$  for the sum of all 2-square principal subdeterminants of  $A$ . If  $A \in M_n(\mathbb{C})$  then we have the following inequalities (see [6]):

$$(1.1) \quad s(A) \leq \left( 2\|A\|^2 - \frac{2}{n}|tr A|^2 \right)^{\frac{1}{2}},$$

$$(1.2) \quad s(A) \leq \sqrt{2}\|A\|.$$

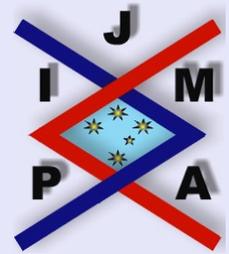
If  $A \in M_n(\mathbb{R})$ , then:

$$(1.3) \quad s(A) \leq \left[ 2 \left( 1 - \frac{1}{n} \right) (tr A)^2 - 4E_2(A) \right]^{\frac{1}{2}},$$

with equality if and only if  $n - 2$  of the characteristic roots of  $A$  are equal to the arithmetic mean of the remaining two.

Consider now an isometrically immersed submanifold  $M^n$  of dimension  $n \geq 2$  in a Riemannian manifold  $(\bar{M}^{n+s}, \bar{g})$ . Then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 3 of 16

$$\bar{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for every  $X, Y \in \Gamma(TM)$  and  $\xi \in \Gamma(\nu M)$ . Take a vector  $\eta \in \nu_p M$  and consider the linear mapping  $A_\eta : T_p M \rightarrow T_p M$ . Let us consider the eigenvalues  $\lambda_\eta^1, \dots, \lambda_\eta^n$  of  $A_\eta$ . We put

$$(1.4) \quad L_\eta(p) = \sup_{i=1, \dots, n} (\lambda_\eta^i) - \inf_{i=1, \dots, n} (\lambda_\eta^i).$$

$L_\eta$  is the spread of the shape operator in the direction  $\eta$ . We define *the spread of the shape operator at the point p* by

$$(1.5) \quad L(p) = \sup_{\eta \in \nu_p M} L_\eta(p).$$

Suppose  $M$  is a compact submanifold of  $\bar{M}$ .

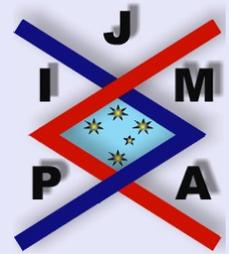
Let us remark that when  $M^2$  is a surface we have

$$L_\nu^2(p) = (\lambda_\nu^1(p) - \lambda_\nu^2(p))^2 = 4(|H(p)|^2 - K(p)),$$

where  $\nu$  is the normal vector at  $p$ ,  $H$  is the mean curvature, and  $K$  is the Gaussian curvature. In [1] it is proved that for a surface  $M^2$  in  $\mathbb{E}^{2+s}$  the geometric quantity  $(|H|^2 - K)dV$  is a conformal invariant. As a corollary, one obtains for an orientable surface in  $\mathbb{E}^{2+s}$  that  $L_\nu^2 dV$  is a conformal invariant.

Let  $\xi_{n+1}, \dots, \xi_{n+s}$  be an orthonormal frame in the normal fibre bundle  $\nu M$ . Let us recall the definition of *the extrinsic scalar curvature* from [2]:

$$ext = \frac{2}{n(n-1)} \sum_{r=1}^s \sum_{i < j} \lambda_{n+r}^i \lambda_{n+r}^j.$$



## The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 4 of 16

In [2] it is proved that for a submanifold  $M^n$  of a Riemannian manifold  $(\bar{M}, \bar{g})$ , the geometric quantity  $(|H|^2 - ext)g$  is invariant under any conformal change of metric. If  $M$  is compact (see also [2]), this result implies that for  $M$ , a  $n$ -dimensional compact submanifold of a Riemannian manifold  $(\bar{M}, \bar{g})$ , the geometric quantity  $\int (|H|^2 - ext)^{\frac{n}{2}} dV$  is a conformal invariant.

Let us prove the following fact.

**Proposition 1.1.** *Let  $M^n$  be a submanifold of the Riemannian manifold  $(\bar{M}, \bar{g})$ . Then the spread of the shape operator is a conformal invariant.*

*Proof.* The context and the idea of the proof are similar to the one given in [3, pp. 204-205]. Let us consider  $\rho$  a nowhere vanishing positive function on  $\bar{M}$ . We have the conformal change of metric in the ambient space  $\bar{M}$  given by

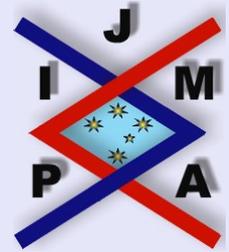
$$\bar{g}^* = \rho^2 \bar{g}.$$

Let us denote by  $h$  and  $h^*$  the second fundamental forms of  $M$  in  $(\bar{M}, \bar{g})$  and  $(\bar{M}, \bar{g}^*)$ , respectively. Then we have (see [3]):

$$g(A_\xi^* X, Y) = g(A_\xi X, Y) + g(X, Y) \bar{g}(U, \xi),$$

where  $U$  is the vector field defined by  $U = (d\rho)^\#$ . Let  $e_1, \dots, e_n$  be the principal normal directions of  $A_\xi$  with respect to  $g$ . Then  $\rho^{-1}e_1, \dots, \rho^{-1}e_n$ , form an orthonormal frame of  $M$  with respect to  $g^*$ , and they are the principal directions of  $A_\xi^*$ . Therefore

$$L^*(p) = \sup_{\xi^* \in \nu_p M; \|\xi^*\|_* = 1} L_{\xi^*}^*$$




---

### The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

---

Title Page

Contents



Go Back

Close

Quit

Page 5 of 16

$$\begin{aligned}
&= \sup_{\xi^* \in \nu_p M; \|\xi^*\|_* = 1} \left( \sup_{i=1, \dots, n} (\lambda_\xi^i)^* - \inf_{j=1, \dots, n} (\lambda_\xi^j)^* \right) \\
&= \sup_{\xi \in \nu_p M; \|\xi\| = 1} \left[ \sup_{i=1, \dots, n} (\lambda_\xi^i + \bar{g}(U, \xi)) - \inf_{j=1, \dots, n} (\lambda_\xi^j + \bar{g}(U, \xi)) \right] \\
&= \sup_{\xi \in \nu_p M; \|\xi\| = 1} \left[ \sup_{i=1, \dots, n} (\lambda_\xi^i) - \inf_{j=1, \dots, n} (\lambda_\xi^j) \right] = L(p).
\end{aligned}$$

This proves the proposition.  $\square$

When  $M$  is a surface, both  $L$  and  $L^2 dV$  are conformal invariants.

The *shape discriminant* of the submanifold  $M$  in  $\bar{M}$  w.r.t. a normal direction  $\eta$  was discussed in [9]. Let  $A_\eta$  be the shape operator associated with an arbitrary normal vector  $\eta$  at  $p$ . The shape discriminant of  $\eta$  is defined by

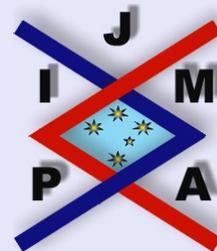
$$(1.6) \quad D_\eta = 2\|A_\eta\|^2 - \frac{2}{n}(\text{trace } A_\eta)^2,$$

where  $\|A_\eta\|^2 = (\lambda_\eta^1)^2 + \dots + (\lambda_\eta^n)^2$ , at every point  $p \in M \subset \bar{M}$ .

The following pointwise double inequality was proved in [9]:

$$(1.7) \quad D_\eta \bigg/ \binom{n}{2} \leq L_\eta^2 \leq D_\eta,$$

We will use this inequality later on. The proof of this fact is algebraically related to the proof of Chen's fundamental inequality with classical curvature invariants (see [4]). The alternate proof of this result is presented in [10].



## The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 6 of 16

## 2. Geometric Inequalities on Compact Submanifolds

In this section, we study the relationship between the spread of the shape operator's spectrum and the conformal invariant from [2]. The main result is Proposition 2.1. For its proof we need a few preliminary steps.

**Proposition 2.1.** *Let  $M^n$  be a compact submanifold of a Riemannian manifold  $\bar{M}^{n+s}$ . Then the following inequality holds:*

$$(2.1) \quad \left( \int_M LdV \right)^2 (vol(M))^{\frac{n}{2n-2}} \leq 2n(n-1) \left( \int_M (|H|^2 - ext)^{\frac{n}{2}} dV \right)^{\frac{2}{n}}.$$

*The equality holds if and only if either  $n = 2$  or  $M$  is a totally umbilical submanifold of dimension  $n \geq 3$ .*

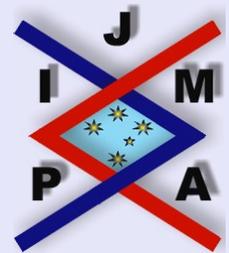
Before presenting the proof, let us see what this inequality means. For any conformal diffeomorphism  $\phi$  of the ambient space  $\bar{M}$ , the quantity

$$\left( \int_{\phi(M)} LdV_\phi \right)^2 (vol(\phi(M)))^{\frac{n}{2n-2}}$$

is bounded above by the conformal invariant geometric quantity expressed in (2.1).

First, let us prove the following.

**Lemma 2.2.** *Let  $M^n \subset \bar{M}^{n+s}$  be a compact submanifold and  $p$  an arbitrary point in  $M$ . Consider an orthonormal normal frame  $\xi_1, \dots, \xi_s$  at  $p$  and let  $D_\alpha$*



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 7 of 16

be the shape discriminant corresponding to  $\xi_\alpha$ , where  $\alpha = 1, \dots, s$ . Then we have

$$(2.2) \quad \frac{1}{2n(n-1)} \sum_{\alpha=1}^s D_\alpha = |H|^2 - ext.$$

*Proof.* Since

$$H = \frac{1}{n} \sum_{\alpha=1}^s \left( \sum_{i=1}^n \lambda_\alpha^i \right) \xi_\alpha,$$

$$ext = \frac{2}{n(n-1)} \sum_{\alpha=1}^s \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j,$$

we have

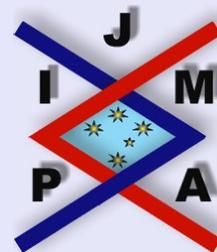
$$(2.3) \quad |H|^2 - ext = \frac{1}{n^2} \sum_{\alpha=1}^s \sum_{i=1}^n (\lambda_\alpha^i)^2 - \frac{2}{n^2(n-1)} \sum_{\alpha=1}^s \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j.$$

A direct computation yields

$$(2.4) \quad D_\alpha = \frac{2(n-1)}{n} \sum_{i=1}^n (\lambda_\alpha^i)^2 - \frac{4}{n} \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j.$$

Summing from  $\alpha = 1$  to  $\alpha = s$  in (2.4) and comparing the result with (2.3) one may get (2.2).  $\square$

From the cited result in [2] and the previous lemma, we have:



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 8 of 16

**Corollary 2.3.** *If  $M$  is a compact submanifold in the ambient space  $\bar{M}$ , then*

$$\int_M \left( \sum_{\alpha=1}^s D_\alpha \right)^{\frac{n}{2}} dV$$

*is a conformal invariant.*

Let us remark that for  $n = 2$  this is a well-known fact.

**Lemma 2.4.** *Let  $M$  be a submanifold in the arbitrary ambient space  $\bar{M}$ . With the previous notations we have*

$$4(|H|^2 - ext) \leq \sum_{\alpha}^s L_{\alpha}^2(p) \leq 2n(n-1)(|H|^2 - ext)$$

*at each point  $p \in M$ . The equalities holds if and only if  $p$  is an umbilical point.*

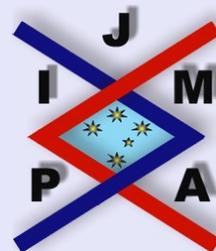
*Proof.* This is a direct consequence of Lemma 2.2 and (1.7). □

*Proof.* We may prove now Proposition 2.1. Let  $p$  be an arbitrary point of  $M$  and let  $\eta_0$  be a normal direction such that  $L(p) = L_{\eta_0}(p)$ . Consider the completion of  $\eta_0$  up to a orthonormal normal base  $\eta_0 = \eta_1, \dots, \eta_s$ . Then we have

$$(2.5) \quad L^2(p) = L_{\eta_0}^2(p) \leq \sum_{\alpha=1}^s L_{\alpha}^2(p) \leq 2n(n-1)(|H|^2 - ext).$$

By applying Hölder's inequality, one has:

$$\left( \int_M L dV \right)^2 \leq \left( \int_M L^2 dV \right) (vol(M)).$$



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 9 of 16

Applying Hölder's inequality one more time yields

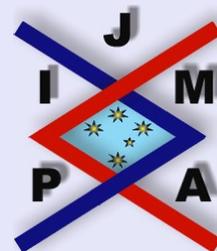
$$\int_M (|H|^2 - ext) dV \leq \left( \int_M (|H|^2 - ext)^{\frac{n}{2}} dV \right)^{\frac{2}{n}} (vol(M))^{\frac{n-2}{n}}.$$

Therefore, by using the inequality established in Lemma 2.4, we have

$$\begin{aligned} \left( \int_M L dV \right)^2 &\leq \left( \int_M L^2 dV \right) (vol(M)) \\ &\leq 2n(n-1) vol(M) \int_M (|H|^2 - ext) dV \\ &\leq 2n(n-1) (vol(M))^{\frac{2n-2}{n}} \left( \int_M (|H|^2 - ext)^{\frac{n}{2}} dV \right)^{\frac{2}{n}}. \end{aligned}$$

Let us discuss when the equality case may occur. We have seen that we get an identity if  $n = 2$ .

Now, let us assume  $n \geq 3$ . The first inequality in (2.5) is equality at  $p$  if there exist  $s - 1$  umbilical directions (i.e.  $L_\alpha(p) = 0$  for  $s = 2, \dots, n$ ). The second inequality in (2.5) is equality if and only if  $p$  is an umbilical point (see [9]). Finally, the two Hölder inequalities are indeed equalities if and only if there exist real numbers  $\theta$  and  $\mu$  satisfying  $L(p) = \theta$  and  $|H|^2 - ext = \mu$  at every  $p \in M$ . The first equality conditions impose pointwise  $L(p) = 0$ , which yields  $\theta = \mu = 0$ . This means that  $M$  is totally umbilical.  $\square$



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 10 of 16

### 3. The Noncompact Case

Let  $M$  be an  $n$ -dimensional noncompact submanifold of an  $(n+d)$ -dimensional Riemannian manifold  $(\bar{M}, g)$ .

**Proposition 3.1.** *Let  $M^n \subset \bar{M}^{n+d}$  be a complete noncompact submanifold and  $\eta_1, \dots, \eta_d$  an orthonormal basis of the normal bundle. Suppose that  $\sum \lambda_\alpha^i \lambda_\alpha^j \geq 0$  and  $L_\alpha \in L^2(M)$ . Then*

$$\int_M (|H|^2 - ext) dV < \infty.$$

*Proof.* We use the inequality (1.7). It is sufficient to prove locally the inequality:

$$|H|^2 - ext \leq \sum_{i=1}^d D_i$$

This is true since, elementary, the following inequality holds:

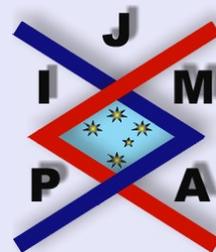
$$(\lambda_\alpha^1)^2 + \dots + (\lambda_\alpha^d)^2 - \frac{2n}{n-1} \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j \leq 2[(\lambda_\alpha^1)^2 + \dots + (\lambda_\alpha^d)^2] - \frac{2}{n} \left\{ \sum_{i=1}^d (\lambda_\alpha^i) \right\}^2.$$

This is equivalent to

$$n(n-1) \sum_{i=1}^d (\lambda_\alpha^i)^2 - 2n^2 \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j \leq 2(n-1)^2 \sum_{i=1}^d (\lambda_\alpha^i)^2 - 4(n-1) \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j$$

or

$$(n^2 - 3n + 2) \left\{ \sum_{i=1}^d (\lambda_\alpha^i)^2 \right\} + 2(n^2 - 2n + 2) \sum_{i < j} \lambda_\alpha^i \lambda_\alpha^j \geq 0,$$



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 11 of 16

which holds by using the hypothesis and that  $n \geq 2$ .

The inequality is the  $\alpha$ -component of the invariant inequality we are going to prove. By adding up  $d$  such inequalities and by considering the improper integral on  $M$  of the appropriate functions, the conclusion follows. This is due to

$$\int_M (|H|^2 - ext)dV \leq \int_M \sum_{i=1}^d D_i dV \leq \binom{n}{2} \sum_{i=1}^d \int_M L_i^2 dV$$

by the first inequality in 1.7. □

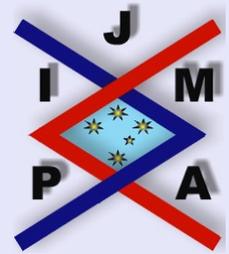
In the next proposition we establish a relation between  $\int_M [L(p)]^2 dV$  and the Willmore-Chen integral  $\int_M (|H|^2 - ext)dV$ , studied in [2].

**Proposition 3.2.** *Let  $M^n \subset \bar{M}^{n+d}$  be a complete noncompact orientable submanifold. If  $L(p) \in L^2(M)$ , then  $\int_M (|H|^2 - ext)dV < \infty$ .*

*Proof.* By direct computation, we have:

$$\begin{aligned} (3.1) \quad \int_M (|H|^2 - ext)dV &= \frac{1}{n^2(n-1)} \int_M \sum_{\alpha=1}^d \sum_{i < j} (\lambda_\alpha^1 - \lambda_\alpha^j)^2 dV \\ &\leq \frac{1}{n^2(n-1)} \int_M \sum_{\alpha=1}^d \sum_{i < j} L^2(p) dV \\ &= \frac{d}{2n} \int_M L^2(p) dV. \end{aligned}$$

□



The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 12 of 16

Let us discuss now two examples. First, let us consider *the catenoid* defined by

$$f_c(u, v) = \left( c \cos u \cosh \frac{v}{c}, c \sin u \cosh \frac{v}{c}, v \right).$$

Using the classical formulas for example from [8] one finds:

$$\lambda_1 = -\lambda_2 = \frac{1}{c} \cosh^{-2} \frac{v}{c}.$$

Therefore, we have

$$\int_{-\infty}^{\infty} L(p) dv = \int_{-\infty}^{\infty} \frac{2}{c} \cosh^{-2} \frac{v}{c} dv = 4 \int_{-\infty}^{\infty} \frac{e^t dt}{e^{2t} + 1} = 4\pi < \infty.$$

Let us consider *the pseudosphere* whose profile functions are given by (see, for example [5]):

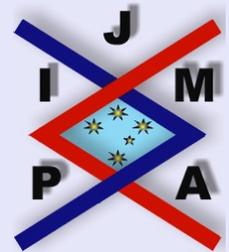
$$c_1(v) = ae^{-v/a}$$

$$c_2(v) = \int_0^v \sqrt{1 - e^{-2t/a}} dt$$

for  $0 \leq v < \infty$ . For simplicity, let us consider just the “upper” part of the pseudosphere. We have

$$\lambda_1 = \frac{e^{v/a}}{a} \sqrt{1 - e^{-2v/a}},$$

$$\lambda_2 = - \left( ae^{v/a} \sqrt{1 - e^{-2v/a}} \right)^{-1}.$$



## The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

Page 13 of 16

Remark that:

$$\int_M LdV = \int_0^\infty \frac{e^{t/a}}{a\sqrt{1-e^{-2t/a}}} dt = \frac{1}{2} \int_1^\infty \frac{dy}{\sqrt{y-1}} = \infty.$$

A natural question is to find a characterization for surfaces of rotation that have finite integral of the spread of shape operator.

Consider surfaces of revolution whose profile curves are described as  $c(s) = (y(s), s)$  (see, for example, [8]). Then we have the following.

**Proposition 3.3.** *Let  $M$  be a surface of rotation in Euclidean 3-space defined by*

$$f(s, t) = (y(s) \cos t, y(s) \sin t, s).$$

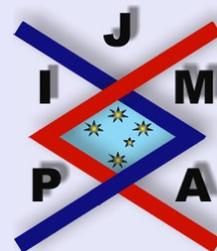
*Then the integral of the spread of the shape operator on  $M$  is finite if and only if there exists an integrable  $C^\infty(\mathbb{R})$  function  $f > 0$  which satisfies the following second order differential equation:*

$$-yy'' = 1 + (y')^2 \pm f(s)y(1 + (y')^2)^{\frac{3}{2}}.$$

*Proof.* For the proof, we use the classical formulas from [5, p. 228]. We have for  $\lambda_1 = k_{meridian}$ , and respectively for  $\lambda_2 = k_{parallel}$  :

$$\lambda_1 = \frac{-y''}{[1 + (y')^2]^{\frac{3}{2}}},$$

$$\lambda_2 = \frac{1}{y[1 + (y')^2]^{\frac{1}{2}}}.$$



## The Spread of the Shape Operator as Conformal Invariant

Bogdan D. Suceavă

Title Page

Contents



Go Back

Close

Quit

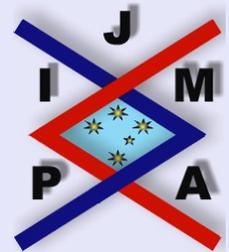
Page 14 of 16

Then, the condition that the integral is finite means that there exists an integrable function  $f > 0$  such that

$$\int_R |\lambda_1 - \lambda_2| ds = \int_R f(s) ds.$$

If we assume that  $f \in C^\infty$ , then the equality between the function under the integral holds everywhere and a straightforward computation yields the claimed equality.  $\square$

For example, for the catenoid  $f(s) = 0$ .



---

**The Spread of the Shape Operator as Conformal Invariant**

Bogdan D. Suceavă

---

Title Page

Contents



Go Back

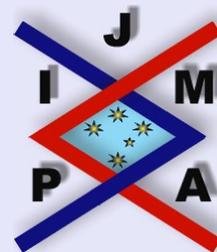
Close

Quit

Page 15 of 16

## References

- [1] B.Y. CHEN, An invariant of conformal mappings, *Proc. Amer. Math. Soc.*, **40** (1973), 563–564.
- [2] B.Y. CHEN, Some conformal invariants of submanifolds and their applications, *Boll. Un. Mat. Ital.*, (4) **10** (1974) 380–385.
- [3] B.Y. CHEN, *Total Mean Curvature and Submanifolds of Finite Type*, World Scientific, 1984.
- [4] B.Y. CHEN, Mean curvature and shape operator of isometric immersions in real-space-forms, *Glasgow Math.J.*, **38** (1996) 87–97.
- [5] A. GRAY, *Modern Differential Geometry of Curves and Surfaces with Matematica*, CRC Press, Second edition, 1998.
- [6] M. MARCUS AND H. MINC, *A survey of matrix theory and matrix inequalities*, Prindle, Weber & Schmidt, 1969.
- [7] L. MIRSKY, The spread of a matrix, *Mathematika*, **3**(1956), 127–130.
- [8] M. SPIVAK, *A Comprehensive Introduction to Differential Geometry*, Volume three, Publish or Perish, Inc., Second edition, 1979.
- [9] B. SUCEAVĂ, Some theorems on austere submanifolds, *Balkan. J. Geom. Appl.*, **2** (1997), 109–115.
- [10] B. SUCEAVĂ, Remarks on B.Y. Chen's inequality involving classical invariants, *Anal. Ști. Univ. Al. I. Cuza Iași, Matematică*, **XLV** (1999), 405–412.



---

The Spread of the Shape  
Operator as Conformal Invariant

Bogdan D. Suceavă

---

Title Page

Contents



Go Back

Close

Quit

Page 16 of 16