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INEQUALITIES FOR THE TRANSFORMATION OPERATORS AND APPLICATIONS

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ABSTRACT. Inequalities for the transformation operator kernel $A(x, y)$ in terms of F-function are given, and vice versa. These inequalities are applied to inverse scattering on the half-line. Characterization of the scattering data corresponding to the usual scattering class $L_{1,1}$ of the potentials, to the class of compactly supported potentials, and to the class of square integrable potentials is given. Invertibility of each of the steps in the inversion procedure is proved. The novel points in this paper include: a) inequalities for the transformation operators in terms of the function F, constructed from the scattering data, b) a considerably shorter way to study the inverse scattering problem on the half-axis and to get necessary and sufficient conditions on the scattering data for the potential to belong to some class of potentials, for example, to the class $L_{1,1}$, to its subclass $L^a_{1,1}$ of potentials vanishing for $x > a$, and for the class of potentials belonging to $L^2(\mathbb{R}_+)$.

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1. INTRODUCTION

Consider the half-line scattering problem data:

$$\mathcal{S} = \{S(k), k_j, s_j, 1 \leq j \leq J\}, \tag{1.1}$$

where $S(k) = \frac{f(-k)}{f(k)}$ is the S-matrix, $f(k)$ is the Jost function, $f(ik_j) = 0$, $\dot{f}(ik_j) := \frac{df(ik_j)}{dk} \neq 0$, $k_j > 0$, $s_j > 0$, J is a positive integer, it is equal to the number of negative eigenvalues of the Dirichlet operator $\ell u := -u'' + q(x)u$ on the half-line. The potential q is real-valued throughout, $q \in L_{1,1} := \left\{q : \int_0^\infty x|q|dx < \infty\right\}$. In [4] the class $L_{1,1} := \left\{q : \int_0^\infty (1+x)|q|dx < \infty\right\}$ was defined in the way, which is convenient for the usage in the problems on the whole line. The definition of $L_{1,1}$ in this paper allows for a larger class of potentials on the half-line: these

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potentials may have singularities at $x = 0$ which are not integrable. For $q \in L_{1,1}$ the scattering data $\mathcal{S}$ have the following properties:

A) $k_j, s_j > 0,\ S(-k) = \overline{S(k)} = S^{-1}(k),\ k \in \mathbb{R},\ S(\infty) = 1,$
B) $\kappa := indS(k) := \frac{1}{2\pi}\int_{-\infty}^{\infty} d\log S(k)$ is a nonpositive integer,
C) $F \in L^p$, $p = 1$ and $p = \infty$, $xF' \in L^1$, $L^p := L^p(0,\infty)$.

Here

$$F(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty}[1 - S(k)]e^{ikx}dk + \sum_{j=1}^{J} s_j e^{-k_j x}, \tag{1.2}$$

and

$$\kappa = -2J \quad \text{if } f(0) \neq 0, \quad \kappa = -2J - 1 \text{ if } f(0) = 0.$$

The Marchenko inversion method is described in the following manner:

$$\mathcal{S} \Rightarrow F(x) \Rightarrow A(x,y) \Rightarrow q(x), \tag{1.3}$$

where the step $\mathcal{S} \Rightarrow F(x)$ is done by formula (1.2), the step $F(x) \Rightarrow A(x,y)$ is done by solving the Marchenko equation:

$$(I + \mathbf{F}_x)A := A(x,y) + \int_x^{\infty} A(x,t)F(t+y)\,dt = -F(x+y), \quad y \geq x \geq 0, \tag{1.4}$$

and the step $A(x,y) \Rightarrow q(x)$ is done by the formula:

$$q(x) = -2\dot{A}(x,x) := -2\frac{dA(x,x)}{dx}. \tag{1.5}$$

Our aim is to study the estimates for A and F, which give a simple way of finding necessary and sufficient conditions for the data (1.1) to correspond to a q from some functional class. We consider, as examples, the following classes: the usual scattering class $L_{1,1}$, for which the result was obtained earlier ([2] and [3]) by a more complicated argument, the class of compactly supported potentials which are locally in $L_{1,1}$, and the class of square integrable potentials. We also prove that each step in the scheme (1.3) is invertible. In Section 2 the estimates for F and A are obtained. These estimates and their applications are the main results of the paper. In Sections 3 – 6 applications to the inverse scattering problem are given. In [7] one finds a review of the author's results on one-dimensional inverse scattering problems and applications.

2. INEQUALITIES FOR A AND F

If one wants to study the characteristic properties of the scattering data (1.1), that is, a necessary and sufficient condition on these data to guarantee that the corresponding potential belongs to a prescribed functional class, then conditions A) and B) are always necessary for a real-valued q to be in $L_{1,1}$, the usual class in the scattering theory, or other class for which the scattering theory is constructed, and a condition of the type C) determines actually the class of potentials q. Conditions A) and B) are consequences of the unitarity of the selfadjointness of the Hamiltonian, finiteness of its negative spectrum, and the unitarity of the $S-$matrix. Our aim is to derive from equation (1.4) inequalities for F and A. This allows one to describe the set of q, defined by (1.5).

Let us assume:

$$\sup_{y \geq x}|F(y)| := \sigma_F(x) \in L^1, \quad F' \in L_{1,1}. \tag{2.1}$$

The function σ_F is monotone decreasing, $|F(x)| \leq \sigma_F(x)$. Equation (1.4) is of Fredholm type in $L^p_x := L^p(x,\infty)$ $\forall x \geq 0$ and $p = 1$. The norm of the operator in (1.4) can be estimated:

$$\|\mathbf{F}_x\| \leq \int_x^\infty \sigma_F(x+y)dy \leq \sigma_{1F}(2x), \quad \sigma_{1F}(x) := \int_x^\infty \sigma_F(y)dy. \tag{2.2}$$

Therefore (1.4) is uniquely solvable in L^1_x for any $x \geq x_0$ if

$$\sigma_{1F}(2x_0) < 1. \tag{2.3}$$

This conclusion is valid for any F satisfying (2.3), and conditions A), B), and C) are not used. Assuming (2.3) and (2.1) and taking $x \geq x_0$, let us derive inequalities for $A = A(x,y)$. Define

$$\sigma_A(x) := \sup_{y\geq x} |A(x,y)| := \|A\|.$$

From (1.4) one gets:

$$\sigma_A(x) \leq \sigma_F(2x) + \sigma_A(x) \sup_{y\geq x} \int_x^\infty \sigma_F(s+y)ds \leq \sigma_F(2x) + \sigma_A(x)\sigma_{1F}(2x).$$

Thus, if (2.3) holds, then

$$\sigma_A(x) \leq c\sigma_F(2x), \quad x \geq x_0. \tag{2.4}$$

By $c > 0$ different constants depending on x_0 are denoted. Let

$$\sigma_{1A}(x) := \|A\|_1 := \int_x^\infty |A(x,s)|ds.$$

Then (1.4) yields $\sigma_{1A}(x) \leq \sigma_{1F}(2x) + \sigma_{1A}(x)\sigma_{1F}(2x)$. So

$$\sigma_{1A}(x) \leq c\sigma_{1F}(2x), \quad x \geq x_0. \tag{2.5}$$

Differentiate (1.4) with respect to x and y to obtain:

$$(I + \mathbf{F}_x)A_x(x,y) = A(x,x)F(x+y) - F'(x+y), \quad y \geq x \geq 0, \tag{2.6}$$

and

$$A_y(x,y) + \int_x^\infty A(x,s)F'(s+y)ds = -F'(x+y), \quad y \geq x \geq 0. \tag{2.7}$$

Denote

$$\sigma_{2F}(x) := \int_x^\infty |F'(y)|dy, \quad \sigma_{2F}(x) \in L^1. \tag{2.8}$$

Then, using (2.7) and (2.4), one gets

$$\begin{aligned} ||A_y||_1 &\leq \int_x^\infty |F'(x+y)|dy + \sigma_{1A}(x) \sup_{s\geq x} \int_x^\infty |F'(s+y)|dy \\ &\leq \sigma_{2F}(2x)[1 + c\sigma_{1F}(2x)] \\ &\leq c\sigma_{2F}(2x), \end{aligned} \tag{2.9}$$

and using (2.6) one gets:

$$\|A_x\|_1 \leq A(x,x)\sigma_{1F}(2x) + \sigma_{2F}(2x) + \|A_x\|_1 \sigma_{1F}(2x),$$

so

$$\|A_x\|_1 \leq c[\sigma_{2F}(2x) + \sigma_{1F}(2x)\sigma_F(2x)]. \tag{2.10}$$

Let $y = x$ in (1.4), then differentiate (1.4) with respect to x and get:

$$\dot{A}(x,x) = -2F'(2x) + A(x,x)F(2x) - \int_x^\infty A_x(x,s)F(x+s)ds - \int_x^\infty A(x,s)F'(s+x)ds. \tag{2.11}$$

From (2.4), (2.5), (2.10) and (2.11) one gets:

$$|\dot{A}(x,x)| \le 2|F'(2x)| + c\sigma_F^2(2x) + c\sigma_F(2x)[\sigma_{2F}(2x) + \sigma_{1F}(2x)\sigma_F(2x)] + c\sigma_F(2x)\sigma_{2F}(2x). \tag{2.12}$$

Thus,

$$x|\dot{A}(x,x)| \in L^1, \tag{2.13}$$

provided that $xF'(2x) \in L^1$, $x\sigma_F^2(2x) \in L^1$, and $x\sigma_F(2x)\sigma_{2F}(2x) \in L^1$. Assumption (2.1) implies $xF'(2x) \in L^1$. If $\sigma_F(2x) \in L^1$, and $\sigma_F(2x) > 0$ decreases monotonically, then $x\sigma_F(x) \to 0$ as $x \to \infty$. Thus $x\sigma_F^2(2x) \in L^1$, and $\sigma_{2F}(2x) \in L^1$ because

$$\int_0^\infty dx \int_x^\infty |F'(y)|dy = \int_0^\infty |F'(y)|ydy < \infty,$$

due to (2.1). Thus, (2.1) implies (2.4), (2.5), (2.8), (2.9), and (2.12), while (2.12) and (1.5) imply $q \in \tilde{L}_{1,1}$ where $\tilde{L}_{1,1} = \left\{q : q = \overline{q},\ \int_{x_0}^\infty x|q(x)|dx < \infty\right\}$, and $x_0 \ge 0$ satisfies (2.3).

Let us assume now that (2.4), (2.5), (2.9), and (2.10) hold, where $\sigma_F \in L^1$ and $\sigma_{2F} \in L^1$ are some positive monotone decaying functions (which have nothing to do now with the function F, solving equation (1.4)), and derive estimates for this function F. Let us rewrite (1.4) as:

$$F(x+y) + \int_x^\infty A(x,s)F(s+y)ds = -A(x,y), \qquad y \ge x \ge 0. \tag{2.14}$$

Let $x + y = z,\ s + y = v$. Then,

$$F(z) + \int_z^\infty A(x, v+x-z)F(v)dv = -A(x, z-x), \qquad z \ge 2x. \tag{2.15}$$

From (2.15) one gets:

$$\sigma_F(2x) \le \sigma_A(x) + \sigma_F(2x) \sup_{z \ge 2x} \int_z^\infty |A(x, v+x-z)|dv \le \sigma_A(x) + \sigma_F(2x) \|A\|_1 .$$

Thus, using (2.5) and (2.3), one obtains:

$$\sigma_F(2x) \le c\sigma_A(x). \tag{2.16}$$

Also from (2.15) it follows that:

$$\begin{aligned} \sigma_{1F}(2x) := ||F||_1 := \int_{2x}^\infty |F(v)|dv &\le \int_{2x}^\infty |A(x, z-x)|dz + \int_{2x}^\infty \int_z^\infty |A(x, v+x-z)||F(v)|dvdz \\ &\le \|A\|_1 + ||F||_1 \|A\|_1 , \end{aligned} \tag{2.17}$$

so

$$\sigma_{1F}(2x) \le c\sigma_{1A}(x).$$

From (2.6) one gets:

$$\int_x^\infty |F'(x+y)|dy = \sigma_{2F}(2x) \le c\sigma_A(x)\sigma_{1A}(x) + \|A_x\| + c\,\|A_x\|_1\,\sigma_{1A}(x). \tag{2.18}$$

Let us summarize the results:

Theorem 2.1. *If $x \ge x_0$ and (2.1) holds, then one has:*

$$\begin{gathered}\sigma_A(x) \le c\sigma_F(2x), \quad \sigma_{1A}(x) \le c\sigma_{1F}(2x), \quad ||A_y||_1 \le \sigma_{2F}(2x)(1 + c\sigma_{1F}(2x)),\\ \|A_x\|_1 \le c[\sigma_{2F}(2x) + \sigma_{1F}(2x)\sigma_F(2x)].\end{gathered} \tag{2.19}$$

Conversely, if $x \ge x_0$ and

$$\sigma_A(x) + \sigma_{1A}(x) + \|A_x\|_1 + ||A_y||_1 < \infty, \tag{2.20}$$

then

$$\begin{gathered}\sigma_F(2x) \le c\sigma_A(x), \quad \sigma_{1F}(2x) \le c\sigma_{1A}(x),\\ \sigma_{2F}(x) \le c[\sigma_A(x)\sigma_{1A}(x) + \|A_x\|_1\,(1 + \sigma_{1A}(x))].\end{gathered} \tag{2.21}$$

In Section 3 we replace the assumption $x \ge x_0 > 0$ by $x \ge 0$. The argument in this case is based on the Fredholm alternative. In [5] and [6] a characterization of the class of bounded and unbounded Fredholm operators of index zero is given.

3. APPLICATIONS

First, let us give *necessary and sufficient conditions on $\mathcal{S}$ for q to belong to the class $L_{1,1}$ of potentials*. These conditions are known [2], [3] and [4], but we give a short new argument using some ideas from [4]. We assume throughout that conditions A), B), and C) hold. These conditions are known to be necessary for $q \in L_{1,1}$. Indeed, conditions A) and B) are obvious, and C) is proved in Theorems 2.1 and 3.3. Conditions A), B), and C) are also sufficient for $q \in L_{1,1}$. Indeed if they hold, then we prove that equation (1.4) has a unique solution in L^1_x for all $x \ge 0$. This is a known fact [2], but we give a (new) proof because it is short. This proof combines some ideas from [2] and [4].

Theorem 3.1. *If A), B), and C) hold, then (1.4) has a solution in L^1_x for any $x \ge 0$ and this solution is unique.*

Proof. Since $\mathbf{F}_x$ is compact in L^1_x, $\forall x \ge 0$, by the Fredholm alternative it is sufficient to prove that

$$(I + \mathbf{F}_x)h = 0, \quad h \in L^1_x, \tag{3.1}$$

implies $h = 0$. Let us prove it for $x = 0$. The proof is similar for $x > 0$. If $h \in L^1$, then $h \in L^\infty$ because $\|h\|_\infty \le \|h\|_{L^1}\,\sigma_F(0)$. If $h \in L^1 \cap L^\infty$, then $h \in L^2$ because $\|h\|^2_{L^2} \le \|h\|_{L^\infty}\,\|h\|_{L^1}$. Thus, if $h \in L^1$ and solves (3.1), then $h \in L^2 \cap L^1 \cap L^\infty$.

Denote $\tilde{h} = \int_0^\infty h(x)e^{ikx}dx$, $h \in L^2$. Then,

$$\int_{-\infty}^\infty \tilde{h}^2 dk = 0. \tag{3.2}$$

Since $F(x)$ is real-valued, one can assume h to be real-valued. One has, using Parseval's equation:

$$0 = ((I + \mathbf{F}_0)h, h) = \frac{1}{2\pi} \|h\|^2 + \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 - S(k)]\tilde{h}^2(k)dk + \sum_{j=1}^{J} s_j h_j^2,$$

$$h_j := \int_0^\infty e^{-k_j x} h(x)dx.$$

Thus, using (3.2), one gets

$$h_j = 0,\ 1 \le j \le J, \quad (\tilde{h}, \tilde{h}) = (S(k)\tilde{h},\ \tilde{h}(-k)),$$

where we have used the real-valuedness of h, i.e. $\tilde{h}(-k) = \bar{\tilde{h}}(k), \forall k \in \mathbb{R}$.

Thus, $(\tilde{h}, \tilde{h}) = (\tilde{h}, S(-k)\tilde{h}(-k))$, where A) was used. Since $\|S(-k)\| = 1$, one has $\|h\|^2 = \left|(\tilde{h}, S(-k)\tilde{h}(-k))\right| \le \|h\|^2$, so the equality sign is attained in the Cauchy inequality. Therefore, $\tilde{h}(k) = S(-k)\tilde{h}(-k)$.

By condition B), the theory of Riemann problem (see [1]) guarantees existence and uniqueness of an analytic in $\mathbb{C}_+ := \{k : \Im k > 0\}$ function $f(k) := f_+(k),\ f(ik_j) = 0,\ \dot{f}(ik_j) \neq 0,\ 1 \le j \le J,\ f(\infty) = 1$, such that

$$f_+(k) = S(-k)f_-(k), \quad k \in \mathbb{R}, \tag{3.3}$$

and $f_-(k) = f(-k)$ is analytic in $\mathbb{C}_- := \{k : \operatorname{Im} k < 0\}$, $f_-(\infty) = 1$ in $\mathbb{C}_-$, $f_-(-ik_j) = 0,\ \dot{f}_-(-ik_j) \neq 0$. Here the property $S(-k) = S^{-1}(k),\ \forall k \in \mathbb{R}$ is used.

One has

$$\psi(k) := \frac{\tilde{h}(k)}{f(k)} = \frac{\tilde{h}(-k)}{f(-k)}, \quad k \in \mathbb{R}, \quad h_j = \tilde{h}(ik_j) = 0, \quad 1 \le j \le J.$$

The function $\psi(k)$ is analytic in $\mathbb{C}_+$ and $\psi(-k)$ is analytic in $\mathbb{C}_-$, they agree on $\mathbb{R}$, so $\psi(k)$ is analytic in $\mathbb{C}$. Since $f(\infty) = 1$ and $\tilde{h}(\infty) = 0$, it follows that $\psi \equiv 0$.

Thus, $\tilde{h} = 0$ and, consequently, $h(x) = 0$, as claimed. Theorem 3.1 is proved. □

The unique solution to equation (1.4) satisfies the estimates given in Theorem 2.1. In the proof of Theorem 2.1 the estimate $x|\dot{A}(x,x)| \in L^1(x_0, \infty)$ was established. So, by (1.5), $xq \in L^1(x_0, \infty)$.

The method developed in Section 2 gives accurate information about the behavior of q near infinity. An immediate consequence of Theorems 2.1 and 3.1 is:

Theorem 3.2. *If A), B), and C) hold, then q, obtained by the scheme (1.3), belongs to $L_{1,1}(x_0, \infty)$.*

Investigation of the behavior of $q(x)$ on $(0, x_0)$ requires additional argument. Instead of using the contraction mapping principle and inequalities, as in Section 2, one has to use the Fredholm theorem, which says that $\|(I + \mathbf{F}_x)^{-1}\| \le c$ for any $x \ge 0$, where the operator norm is for $\mathbf{F}_x$ acting in L^p_x, $p = 1$ and $p = \infty$, and the constant c does not depend on $x \ge 0$.

Such an analysis yields:

Theorem 3.3. *If and only if A), B), and C) hold, then $q \in L_{1,1}$.*

Proof. It is sufficient to check that Theorem 2.1 holds with $x \ge 0$ replacing $x \ge x_0$. To get (2.4) with $x_0 = 0$, one uses (1.4) and the estimate:

$$\|A(x,y)\| \le \left\|(I + \mathbf{F}_x)^{-1}\right\| \|F(x+y)\| \le c\sigma_F(2x), \quad \|\cdot\| = \sup_{y \ge x} |\cdot|,\ x \ge 0, \tag{3.4}$$

where the constant $c > 0$ does not depend on x. Similarly:

$$\|A(x,y)\|_1 \le c \sup_{s\ge x} \int_x^\infty |F(s+y)|dy \le c\sigma_{1F}(2x), \quad x \ge 0. \tag{3.5}$$

From (2.6) one gets:

$$\begin{aligned} ||A_x(x,y)||_1 &\le c[||F'(x+y)||_1 + A(x,x)||F(x+y)||_1] \\ &\le c\sigma_{2F}(2x) + c\sigma_F(2x)\sigma_{1F}(2x), \quad x \ge 0. \end{aligned} \tag{3.6}$$

From (2.7) one gets:

$$||A_y(x,y)||_1 \le c[\sigma_{2F}(2x) + \sigma_{1F}(2x)\sigma_{2F}(2x)] \le \sigma_{2F}(2x). \tag{3.7}$$

Similarly, from (2.11) and (3.3) – (3.6) one gets (2.12). Then one checks (2.13) as in the proof of Theorem 2.1. Consequently Theorem 2.1 holds with $x_0 = 0$. Theorem 3.3 is proved. □

4. COMPACTLY SUPPORTED POTENTIALS

In this section, *necessary and sufficient conditions are given for q to belong to the class*

$$L^a_{1,1} := \left\{ q : q = \overline{q}, q = 0 \text{ if } x > a, \int_0^a x|q|dx < \infty \right\}.$$

Recall that the Jost solution is:

$$f(x,k) = e^{ikx} + \int_x^\infty A(x,y)e^{iky}dy, \quad f(0,k) := f(k). \tag{4.1}$$

Lemma 4.1. *If $q \in L^a_{1,1}$, then $f(x,k) = e^{ikx}$ for $x > a$, $A(x,y) = 0$ for $y \ge x \ge a$, $F(x+y) = 0$ for $y \ge x \ge a$ (cf. (1.4)), and $F(x) = 0$ for $x \ge 2a$.*

Thus, (1.4) with $x = 0$ yields $A(0,y) := A(y) = 0$ for $x \ge 2a$. The Jost function

$$f(k) = 1 + \int_0^{2a} A(y)e^{iky}dy, \quad A(y) \in W^{1,1}(0,a), \tag{4.2}$$

is an entire function of exponential type $\le 2a$, that is, $|f(k)| \le ce^{2a|k|}$, $k \in \mathbb{C}$, and $S(k) = f(-k)/f(k)$ is a meromorphic function in $\mathbb{C}$. In (4.2) $W^{l,p}$ is the Sobolev space, and the inclusion (4.2) follows from Theorem 2.1.

Let us formulate the assumption D):

D) the Jost function $f(k)$ is an entire function of exponential type $\le 2a$.

Theorem 4.2. *Assume A),B), C) and D). Then $q \in L^a_{1,1}$. Conversely, if $q \in L^a_{1,1}$, then A),B), C) and D) hold.*

Proof. Necessity. If $q \in L_{1,1}$, then A), B) and C) hold by Theorem 3.3, and D) is proved in Lemma 4.1. The necessity is proved.

Sufficiency. If A), B) and C) hold, then $q \in L_{1,1}$. One has to prove that $q = 0$ for $x > a$. If D) holds, then from the proof of Lemma 4.1 it follows that $A(y) = 0$ for $y \ge 2a$.

We claim that $F(x) = 0$ for $x \ge 2a$.

If this is proved, then (1.4) yields $A(x,y) = 0$ for $y \ge x \ge a$, and so $q = 0$ for $x > a$ by (1.5).

Let us prove the claim.

Take $x > 2a$ in (1.2). The function $1 - S(k)$ is analytic in $\mathbb{C}_+$ except for J simple poles at the points ik_j. If $x > 2a$ then one can use the Jordan lemma and residue theorem to obtain:

$$F_S(x) = \frac{1}{2\pi} \int_{-\infty}^\infty [1 - S(k)]e^{ikx}dk = -i \sum_{j=1}^J \frac{f(-ik_j)}{\dot{f}(ik_j)} e^{-k_j x}, \quad x > 2a. \tag{4.3}$$

Since $f(k)$ is entire, the Wronskian formula

$$f'(0,k)f(-k) - f'(0,-k)f(k) = 2ik$$

is valid on $\mathbb{C}$, and at $k = ik_j$ it yields:

$$f'(0,ik_j)f(-ik_j) = -2k_j,$$

because $f(ik_j) = 0$. This and (4.3) yield

$$F_s(x) = \sum_{j=1}^{J} \frac{2ik_j}{f'(0,ik_j)\dot{f}(ik_j)} e^{-k_j x} = -\sum_{j=1}^{J} s_j e^{-k_j x} = -F_d(x), \quad x > 2a.$$

Thus, $F(x) = F_s(x) + F_d(x) = 0$ for $x > 2a$. The sufficiency is proved.
Theorem 4.2 is proved. □

In [2] a condition on $\mathcal{S}$, which guarantees that $q = 0$ for $x > a$, is given under the assumption that there is no discrete spectrum, that is $F = F_s$.

5. SQUARE INTEGRABLE POTENTIALS

Let us introduce conditions (5.1) – (5.3):

(5.1) $$2ik\left[f(k) - 1 + \frac{Q}{2ik}\right] \in L^2(\mathbb{R}_+) := L^2, \quad Q := \int_0^\infty q ds,$$

(5.2) $$k\left[1 - S(k) + \frac{Q}{ik}\right] \in L^2,$$

(5.3) $$k[|f(k)|^2 - 1] \in L^2.$$

Theorem 5.1. *If A), B), C), and any one of the conditions (5.1) – (5.3) hold, then* $q \in L^2$.

Proof. We refer to [3] for the proof. □

6. INVERTIBILITY OF THE STEPS IN THE INVERSION PROCEDURE

We assume A), B), and C) and prove:

Theorem 6.1. *The steps in (1.3) are invertible:*

(6.1) $$\mathcal{S} \Longleftrightarrow F \Longleftrightarrow A \Longleftrightarrow q.$$

Proof.

(1) Step $\mathcal{S} \Rightarrow F$ is done by formula (1.2). Step $F \Rightarrow \mathcal{S}$ is done by taking $x \to -\infty$ in (1.2). The asymptotics of $F(x)$, as $x \to -\infty$, yields $J, s_j, k_j, \quad 1 \le j \le J$, that is, $F_d(x)$. Then $F_s = F - F_d$ is calculated, and $1 - S(k)$ is calculated by taking the inverse Fourier transform of $F_s(x)$. Thus,

(2) Step $F \Rightarrow A$ is done by solving (1.4), which has one and only one solution in L^1_x for any $x \ge 0$ by Theorem 3.1. Step $A \Rightarrow F$ is done by solving equation (1.4) for F. Let $x + y = z$ and $s + y = v$. Write (1.4) as

(6.2) $$(I + B)F := F(z) + \int_z^\infty A(x, v + x - z)F(v)dv = -A(x, z - x), \quad z \ge 2x \ge 0.$$

The norm of the integral operator B in L^1_{2x} is estimated as follows:

$$||B|| \leq \sup_{v>0} \int_0^v |A(x, v+x-z)|dz \leq c \sup_{v>0} \int_0^v \sigma\left(x + \frac{v-z}{2}\right) dz \leq 2\int_0^\infty \sigma(x+w)dw = 2\int_x^\infty \sigma(t)dt, \tag{6.3}$$

where the known estimate [2] was used: $|A(x,y)| \leq c\sigma\left(\frac{x+y}{2}\right), \quad \sigma(x) := \int_x^\infty |q|dt$. It follows from (6.3) that $||B|| < 1$ if $x > x_0$, where x_0 is large enough. Indeed, $\int_x^\infty \sigma(s)ds \to 0$ as $x \to \infty$ if $q \in L_{1,1}$. Therefore, for $x > x_0$ equation (6.2) is uniquely solvable in $L^1_{2x_0}$ by the contraction mapping principle.

(3) Step $A \Rightarrow q$ is done by formula (1.5). Step $q \Rightarrow A$ is done by solving the known Volterra equation (see [2] or [3]):

$$A(x,y) = \frac{1}{2}\int_{\frac{x+y}{2}}^\infty q(t)dt + \int_{\frac{x+y}{2}}^\infty ds \int_0^{\frac{y-x}{2}} dtq(s-t)A(s-t, s+t). \tag{6.4}$$

Thus, Theorem 6.1 is proved.

□

Note that Theorem 6.1 implies that if one starts with a $q \in L_{1,1}$, computes the scattering data (1.1) corresponding to this q, and uses the inversion scheme (1.3), then the potential obtained by the formula (1.5) is equal to the original potential q.

If $F(z)$ is known for $x \geq 2x_0$, then (6.2) can be written as a Volterra equation with a finite region of integration.

$$F(z) + \int_z^{2x_0} A(x, v+x-z)F(v)dv = -A(x, z-x) - \int_{2x_0}^\infty A(x, v+x-z)F(v)dv, \tag{6.5}$$

where the right-hand side in (6.5) is known. This Volterra integral equation on the interval $z \in (0, 2x_0)$ is uniquely solvable by iterations. Thus, $F(z)$ is uniquely determined on $(0, 2x_0)$, and, consequently, on $(0, \infty)$.

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