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ON SOME RESULTS INVOLVING THE ČEBYŠEV FUNCTIONAL AND ITS GENERALISATIONS

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ABSTRACT. Recent results involving bounds of the Čebyšev functional to include means over different intervals are extended to a measurable space setting. Sharp bounds are obtained for the resulting expressions of the generalised Čebyšev functionals where the means are over different measurable sets.

Key words and phrases: Čebyšev functional, Grüss inequality, Measurable functions, Lebesgue integral, Perturbed rules.

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1. INTRODUCTION AND REVIEW OF SOME RECENT RESULTS

For two measurable functions $f, g : [a, b] \to \mathbb{R}$, define the functional, which is known in the literature as Čebyšev's functional, by

$$T(f, g) := \mathcal{M}(fg) - \mathcal{M}(f)\mathcal{M}(g), \tag{1.1}$$

where the integral mean is given by

$$\mathcal{M}(f) := \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.2}$$

The integrals in (1.1) are assumed to exist.

Further, the weighted Čebyšev functional is defined by

$$T(f, g; p) := \mathcal{M}(f, g; p) - \mathcal{M}(f; p)\mathcal{M}(g; p), \tag{1.3}$$

where the weighted integral mean is given by

$$\mathcal{M}(f; p) = \frac{\int_a^b p(x) f(x)\,dx}{\int_a^b p(x)\,dx}, \tag{1.4}$$

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with $0 < \int_a^b p(x)\,dx < \infty$.

We note that,

$$T(f,g;1) \equiv T(f,g)$$

and

$$\mathcal{M}(f;1) \equiv \mathcal{M}(f).$$

It is worthwhile noting that a number of identities relating to the Čebyšev functional already exist. The reader is referred to [17] Chapters IX and X. Korkine's identity is well known, see [17, p. 296] and is given by

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))\,dxdy. \tag{1.5}$$

It is identity (1.5) that is often used to prove an inequality due to Grüss for functions bounded above and below, [17].

The Grüss inequality is given by

$$|T(f,g)| \le \frac{1}{4}(\Phi_f - \phi_f)(\Phi_g - \phi_g), \tag{1.6}$$

where $\phi_f \le f(x) \le \Phi_f$ for $x \in [a,b]$.

If we let $S(f)$ be an operator defined by

$$S(f)(x) := f(x) - \mathcal{M}(f), \tag{1.7}$$

which shifts a function by its integral mean, then the following identity holds. Namely,

$$T(f,g) = T(S(f),g) = T(f,S(g)) = T(S(f),S(g)), \tag{1.8}$$

and so

$$T(f,g) = \mathcal{M}(S(f)g) = \mathcal{M}(fS(g)) = \mathcal{M}(S(f)S(g)) \tag{1.9}$$

since $\mathcal{M}(S(f)) = \mathcal{M}(S(g)) = 0$.

For the last term in (1.8) or (1.9) only one of the functions needs to be shifted by its integral mean. If the other were to be shifted by any other quantity, the identities would still hold. A weighted version of (1.9) related to

$$T(f,g) = \mathcal{M}((f(x)-\gamma)S(g)) \tag{1.10}$$

for γ arbitrary was given by Sonin [19] (see [17, p. 246]).

The interested reader is also referred to Dragomir [12] and Fink [14] for extensive treatments of the Grüss and related inequalities.

Identity (1.5) may also be used to prove the Čebyšev inequality which states that for $f(\cdot)$ and $g(\cdot)$ synchronous, namely $(f(x)-f(y))(g(x)-g(y)) \ge 0$, a.e. $x,y \in [a,b]$, then

$$T(f,g) \ge 0. \tag{1.11}$$

There are many identities involving the Čebyšev functional (1.1) or more generally (1.3). Recently, Cerone [2] obtained, for $f,g : [a,b] \to \mathbb{R}$ where f is of bounded variation and g continuous on $[a,b]$, the identity

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \psi(t)\,df(t), \tag{1.12}$$

where

$$\psi(t) = (t-a)G(t,b) - (b-t)G(a,t) \tag{1.13}$$

with

$$G(c,d)=\int_c^d g(x)\,dx. \tag{1.14}$$

The following theorem was proved in [2].

Theorem 1.1. *Let $f,g:[a,b]\to\mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$. Then*

$$(b-a)^2|T(f,g)|\le\begin{cases}\sup\limits_{t\in[a,b]}|\psi(t)|\bigvee\limits_a^b(f), & \\ L\int_a^b|\psi(t)|\,dt, & \text{for } f\ L-\text{Lipschitzian,}\\ \int_a^b|\psi(t)|\,df(t), & \text{for } f \text{ monotonic nondecreasing,}\end{cases} \tag{1.15}$$

where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

An equivalent identity and theorem were also obtained for the weighted Čebyšev functional (1.3).

The bounds for the Čebyšev functional were utilised to procure approximations to moments and moment generating functions.

In [8], bounds were obtained for the approximations of moments although the work in [2] places less stringent assumptions on the behaviour of the probability density function.

In a subsequent paper to [2], Cerone and Dragomir [6] obtained a refinement of the classical Čebyšev inequality (1.11).

Theorem 1.2. *Let $f:[a,b]\to\mathbb{R}$ be a monotonic nondecreasing function on $[a,b]$ and $g:[a,b]\to\mathbb{R}$ a continuous function on $[a,b]$ so that $\varphi(t)\ge 0$ for each $t\in(a,b)$. Then one has the inequality:*

$$T(f,g)\ge\frac{1}{(b-a)^2}\left|\int_a^b[(t-a)|G(t,b)|-(b-t)|G(a,t)|]\,df(t)\right|\ge 0, \tag{1.16}$$

where

$$\varphi(t)=\frac{G(t,b)}{b-t}-\frac{G(a,t)}{t-a} \tag{1.17}$$

and $G(c,d)$ is as defined in (1.14).

Bounds were also found for $|T(f,g)|$ in terms of the Lebesgue norms $\|\phi\|_p$, $p\ge 1$ effectively utilising (1.15) and noting that $\psi(t)=(t-a)(b-t)\varphi(t)$.

It should be mentioned here that the author in [3] demonstrated relationships between the Čebyšev functional $T(f,g;a,b)$, the generalised trapezoidal functional $GT(f;a,x,b)$ and the Ostrowski functional $\Theta(f;a,x,b)$ defined by

$$T(f,g;a,b):=M(fg;a,b)-M(f;a,b)M(g;a,b)$$

$$GT(f;a,x,b):=\left(\frac{x-a}{b-a}\right)f(a)+\left(\frac{b-x}{b-a}\right)f(b)-M(f;a,b)$$

and

$$\Theta(f;a,x,b):=f(x)-M(f;a,b)$$

where the integral mean is defined by

$$M(f;a,b) := \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{1.18}$$

This was made possible through the fact that both $GT(f;a,x,b)$ and $\Theta(f;a,x,b)$ satisfy identities like (1.12) involving appropriate Peano kernels. Namely,

$$GT(f;a,x,b) = \int_a^b q(x,t)\,df(t), \quad q(x,t) = \frac{t-x}{b-a};\ x,t \in [a,b]$$

and

$$\Theta(f;a,x,b) = \int_a^b p(x,t)\,df(t), \quad (b-a)\,p(x,t) = \begin{cases} t-a, & t\in[a,x] \\ \\ t-b, & t\in(x,b] \end{cases}$$

respectively.

The reader is referred to [10], [13] and the references therein for applications of these to numerical quadrature.

For other Grüss type inequalities, see the books [17] and [18], and the papers [9] – [14], where further references are given.

Recently, Cerone and Dragomir [7] have pointed out generalisations of the above results for integrals defined on two different intervals $[a,b]$ and $[c,d]$.

Define the functional (generalised Čebyšev functional)

$$\begin{aligned} T(f,g;a,b,c,d) &:= M(fg;a,b) + M(fg;c,d) \\ &\quad - M(f;a,b)\,M(g;c,d) - M(f;c,d)\,M(g;a,b) \end{aligned} \tag{1.19}$$

then Cerone and Dragomir [7] proved the following result.

Theorem 1.3. *Let $f,g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d] \subset I$. Assume that the integrals involved in (1.19) exist. Then we have the inequality*

$$\begin{aligned} &|T(f,g;a,b,c,d)| \\ &\quad \le \left[T(f;a,b) + T(f;c,d) + (M(f;a,b) - M(f;c,d))^2\right]^{\frac{1}{2}} \\ &\qquad \times \left[T(g;a,b) + T(g;c,d) + (M(g;a,b) - M(g;c,d))^2\right]^{\frac{1}{2}} \end{aligned} \tag{1.20}$$

where

$$T(f;a,b) := \frac{1}{b-a}\int_a^b f^2(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)^2, \tag{1.21}$$

and the integrals involved in the right of (1.20) exist and $M(f;a,b)$ is as defined by (1.18).

They used a generalisation of the classical identity due to Korkine namely,

$$T(f,g;a,b,c,d) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d (f(x)-f(y))(g(x)-g(y))\,dydx \tag{1.22}$$

and the fact that

$$T(f,f;a,b,c,d) = T(f;a,b) + T(f;c,d) + (M(f;a,b) - M(f;c,d))^2. \tag{1.23}$$

From the Grüss inequality (1.6), then from (1.21) we obtain for f (and equivalent expressions for g)

$$T(f;a,b) \le \left(\frac{M_1 - m_1}{2}\right)^2 \text{ and } T(f;c,d) \le \left(\frac{M_2 - m_2}{2}\right)^2,$$

where $m_1 \leq f \leq M_1$ a.e. on $[a,b]$ and $m_2 \leq f \leq M_2$ a.e. on $[c,d]$.

Cerone and Dragomir [6] procured bounds for the generalised Čebyšev functional (1.19) in terms of the integral means and bounds, of f and g over the two intervals.

The following result was obtained in [1] for f and g of Hölder type involving the generalised Čebyšev functional (1.19) with (1.18).

Theorem 1.4. *Let $f,g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d] \subset I$. Further, suppose that f and g are of Hölder type so that for $x \in [a,b]$, $y \in [c,d]$*

$$|f(x) - f(y)| \leq H_1 |x-y|^r \quad \text{and} \quad |g(x) - g(y)| \leq H_2 |x-y|^s, \tag{1.24}$$

where H_1, $H_2 > 0$ and $r,s \in (0,1]$ are fixed. The following inequality then holds on the assumption that the integrals involved exist. Namely,

$$\begin{aligned}(\theta+1)(\theta+2)|T(f,g;a,b,c,d)| \\ \leq \frac{H_1 H_2}{(b-a)(d-c)}\left[|b-c|^{\theta+2} - |b-d|^{\theta+2} + |d-a|^{\theta+2} - |c-a|^{\theta+2}\right],\end{aligned} \tag{1.25}$$

where $\theta = r+s$ and $T(f,g;a,b,c,d)$ is as defined by (1.19) and (1.18).

Another generalised Čebyšev functional involving the mean of the product of two functions, and the product of the means of each of the functions, where one is over a different interval was examined in [7]. Namely,

$$\mathfrak{T}(f,g;a,b,c,d) := M(fg;a,b) - M(f;a,b)\,M(g;c,d), \tag{1.26}$$

which may be demonstrated to to satisfy the Körkine like identity

$$\mathfrak{T}(f,g;a,b,c,d) = \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x)\,(g(x) - g(y))\,dydx. \tag{1.27}$$

It may be noticed from (1.26) and (1.1) that $2\mathfrak{T}(f,g;a,b;a,b) = T(f,g;a,b)$.

It may further be noticed that (1.15) is related to (1.19) by the identity

$$T(f,g;a,b,c,d) = \mathfrak{T}(f,g;a,b,c,d) + \mathfrak{T}(g,f;c,d,a,b). \tag{1.28}$$

Theorem 1.5. *Let $f,g : I \subseteq \mathbb{R} \to \mathbb{R}$ be measurable on I and the intervals $[a,b]$, $[c,d] \subset I$. In addition, let $m_1 \leq f \leq M_1$ and $n_1 \leq g \leq N_1$ a.e. on $[a,b]$ with $n_2 \leq g \leq N_2$ a.e. on $[c,d]$. Then the following inequalities hold*

$$\begin{aligned}&|\mathfrak{T}(f,g;a,b,c,d)| \\ &\leq \left[T(f;a,b) + M^2(f;a,b)\right]^{\frac{1}{2}} \\ &\quad \times \left\{T(g;a,b) + T(g;c,d) + [M(g;a,b) - M(g;c,d)]^2\right\}^{\frac{1}{2}} \\ &\leq \left[\left(\frac{M_1 - m_1}{2}\right)^2 + M^2(f;a,b)\right]^{\frac{1}{2}} \\ &\quad \times \left\{\left(\frac{N_1 - n_1}{2}\right)^2 + \left(\frac{N_2 - n_2}{2}\right)^2 + [M(g;a,b) - M(g;c,d)]^2\right\}^{\frac{1}{2}},\end{aligned} \tag{1.29}$$

where $T(f;a,b)$ is as given by (1.21) and $M(f;a,b)$ by (1.18).

The generalised Čebyšev functional (1.26) and Theorem 1.5 was used in [4] to obtain bounds for a generalised Steffensen functional. It is also possible as demonstrated in [7] to recapture the Ostrowski functional (1.7) from (1.26) by using a limiting argument.

2. THE ČEBYŠEV FUNCTIONAL IN A MEASURABLE SPACE SETTING

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ – algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ – a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mathcal{A}, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is $\mu-$measurable and $\int_\Omega w(x) |f(x)| \, d\mu(x) < \infty\}$. Assume $\int_\Omega w(x) \, d\mu(x) > 0$.

If $f, g : \Omega \to \mathbb{R}$ are $\mu-$measurable functions and $f, g, fg \in L_w(\Omega, \mathcal{A}, \mu)$, then we may consider the Čebyšev functional

$$\begin{aligned} T_w(f,g) = T_w(f,g;\Omega) &:= \frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) f(x) g(x) \, d\mu(x) \\ &\quad - \frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) f(x) \, d\mu(x) \\ &\quad \times \frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) g(x) \, d\mu(x) . \end{aligned} \tag{2.1}$$

Remark 2.1. We note that a new measure $\nu(x)$ may be defined such that $d\nu(x) \equiv w(x) \, d\mu(x)$ however, in the current article the weight $w(x)$ and measure $\mu(x)$ are separated.

The following result is known in the literature as the Grüss inequality

$$|T_w(f,g)| \leq \frac{1}{4} (\Gamma - \gamma)(\Delta - \delta), \tag{2.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{2.3}$$

for μ – a.e. $x \in \Omega$.

The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

With the above assumptions and if $f \in L_w(\Omega, \mathcal{A}, \mu)$ then we may define

$$\begin{aligned} D_w(f) &:= D_{w,1}(f) \\ &:= \frac{1}{\int_\Omega w(x)\,d\mu(x)} \int_\Omega w(x) \\ &\quad \times \left| f(x) - \frac{1}{\int_\Omega w(y)\,d\mu(y)} \int_\Omega w(y) f(y) \, d\mu(y) \right| d\mu(x) . \end{aligned} \tag{2.4}$$

The following fundamental result was proved in [5].

Theorem 2.2. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$ $\mu-$ a.e. on Ω and $\int_\Omega w(y) \, d\mu(y) > 0$. If $f, g, fg \in L_w(\Omega, \mathcal{A}, \mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \ \text{ for } \ \mu - \text{a.e. } x \in \Omega, \tag{2.5}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2} (\Delta - \delta) D_w(f) . \tag{2.6}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

For $f \in L_{w,p}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w(x) |f(x)|^p d\mu(x) < \infty \right\}$, $1 \le p < \infty$ and $f \in L_\infty(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R}, \|f\|_{\Omega,\infty} := ess \sup_{x \in \Omega} |f(x)| < \infty \right\}$, we may also define

$$D_{w,p}(f) := \left[\frac{1}{\int_\Omega w(x) d\mu(x)} \int_\Omega w(x) \times \left| f(x) - \frac{1}{\int_\Omega w(y) d\mu(y)} \int_\Omega w(y) f(y) d\mu(y) \right|^p d\mu(x) \right]^{\frac{1}{p}} = \frac{\left\| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right\|_{\Omega,p}}{\left[\int_\Omega w(x) d\mu(x) \right]^{\frac{1}{p}}} \tag{2.7}$$

where $\|\cdot\|_{\Omega,p}$ is the usual $p-$norm on $L_{w,p}(\Omega, \mathcal{A}, \mu)$, namely,

$$\|h\|_{\Omega,p} := \left(\int_\Omega w |h|^p d\mu \right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$

and on $L_\infty(\Omega, \mathcal{A}, \mu)$

$$\|h\|_{\Omega,\infty} := ess \sup_{x \in \Omega} |h(x)| < \infty.$$

Cerone and Dragomir [5] produced the following result.

Corollary 2.3. *With the assumptions of Theorem 2.2, we have*

$$\begin{aligned} |T_w(f,g)| &\le \frac{1}{2}(\Delta - \delta) D_w(f) \\ &\le \frac{1}{2}(\Delta - \delta) D_{w,p}(f) \quad \textit{if } f \in L_{w,p}(\Omega, \mathcal{A}, \mu), \ 1 < p < \infty; \\ &\le \frac{1}{2}(\Delta - \delta) \left\| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right\|_{\Omega,\infty} \quad \textit{if } f \in L_\infty(\Omega, \mathcal{A}, \mu). \end{aligned} \tag{2.8}$$

Remark 2.4. The inequalities in (2.8) are in order of increasing coarseness. If we assume that $-\infty < \gamma \le f(x) \le \Gamma < \infty$ for μ – a.e. $x \in \Omega$, then by the Grüss inequality for $g = f$ we have for $p = 2$

$$\left[\frac{\int_\Omega w f^2 d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu} \right)^2 \right]^{\frac{1}{2}} \le \frac{1}{2}(\Gamma - \gamma). \tag{2.9}$$

By (2.8), we deduce the following sequence of inequalities

$$\begin{aligned} |T_w(f,g)| &\le \frac{1}{2}(\Delta - \delta) \frac{1}{\int_\Omega w d\mu} \int_\Omega w \left| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right| d\mu \\ &\le \frac{1}{2}(\Delta - \delta) \left[\frac{\int_\Omega w f^2 d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu} \right)^2 \right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma) \end{aligned} \tag{2.10}$$

for $f, g : \Omega \to \mathbb{R}$, μ – measurable functions and so that $-\infty < \gamma \le f(x) < \Gamma < \infty$, $-\infty < \delta \le g(x) \le \Delta < \infty$ for μ – a.e. $x \in \Omega$. Thus the first inequality in (2.10) or (2.6) is a refinement of the third which is the Grüss inequality (2.2). Further, (2.6) is also a refinement of the second inequality in (2.10). We note that all the inequalities in (2.8) – (2.10) are sharp.

The second inequality in (2.10) under a less general setting was termed as a pre-Grüss inequality by Matić, Pečarić and Ujević [16]. Bounds for the Čebyšev functional have been put to good use by a variety of authors in providing perturbed numerical integration rules (see for example the book [13]).

3. GENERALISED ČEBYŠEV FUNCTIONAL IN A MEASURABLE SPACE SETTING

Let the conditions of the previous section hold. Further, let χ, κ be two measurable subsets of Ω and $f, g : \Omega \to \mathbb{R}$ be measurable functions such that $f, g, fg \in L_w(\Omega, \mathcal{A}, \mu)$ then consider the generalised Čebyšev functional

$$T_w^*(f, g; \chi, \kappa) := \mathcal{M}_w(fg; \chi) + \mathcal{M}_w(fg; \kappa) - \mathcal{M}_w(f; \chi) \cdot \mathcal{M}_w(g; \kappa) - \mathcal{M}_w(g; \chi) \cdot \mathcal{M}_w(f; \kappa), \tag{3.1}$$

where

$$\mathcal{M}_w(f; \chi) := \frac{1}{\int_\chi w(x)\, d\mu(x)} \int_\chi w(x) f(x)\, d\mu(x). \tag{3.2}$$

We note that if $\chi \equiv \kappa \equiv \Omega$ then, $T_w^*(f, g; \Omega, \Omega) = 2T_w(f, g; \Omega)$.

The following theorem providing bounds on (3.1) then holds.

Theorem 3.1. *Let $w, f, g : \Omega \to \mathbb{R}$ be $\mu-$measurable functions with $w \geq 0$, μ – a.e. on Ω and $\int_\chi w(x)\, d\mu(x) > 0$, $\int_\kappa w(x)\, d\mu(x) > 0$ for $\chi, \kappa \subset \Omega$. Further, let $f, g, f^2, g^2 \in L_w(\Omega, \mathcal{A}, \mu)$, then*

$$|T_w^*(f, g; \chi, \kappa)| \leq [B_w(f; \chi, \kappa)]^{\frac{1}{2}} [B_w(g; \chi, \kappa)]^{\frac{1}{2}}, \tag{3.3}$$

where

$$B_w(f; \chi, \kappa) = T_w(f; \chi) + T_w(f; \kappa) + [\mathcal{M}_w(f; \chi) - \mathcal{M}_w(f; \kappa)]^2 \tag{3.4}$$

which, from (2.1)

$$T_w(f; \chi) := T_w(f, f; \chi) = \mathcal{M}_w\left(f^2; \chi\right) - [\mathcal{M}_w(f; \chi)]^2 \tag{3.5}$$

and $\mathcal{M}_w(f; \chi)$ is as defined by (3.2).

Proof. It is a straight forward matter to demonstrate the following Korkine type identity for $T_w^*(f, g; \chi, \kappa)$ holds. Namely,

$$T_w^*(f, g; \chi, \kappa) = \frac{1}{\int_\chi w(x)\, d\mu(x) \int_\kappa w(y)\, d\mu(y)} \times \int_\chi \int_\kappa w(x) w(y) (f(x) - f(y)) (g(x) - g(y))\, d\mu(y)\, d\mu(x). \tag{3.6}$$

Now, using the Cauchy-Buniakowski-Schwartz inequality for double integrals, we have from (3.6)

$$\begin{aligned}
|T_w^*(f,g;\chi,\kappa)|^2 \le{} & \frac{1}{\int_\chi w(x)\,d\mu(x)\int_\kappa w(y)\,d\mu(y)} \\
& \times \int_\chi\int_\kappa w(x)\,w(y)\,(f(x)-f(y))^2\,d\mu(y)\,d\mu(x) \\
& \times \int_\chi\int_\kappa w(x)\,w(y)\,(g(x)-g(y))^2\,d\mu(y)\,d\mu(x) \\
={} & T_w(f,f;\chi,\kappa)\,T_w(g,g;\chi,\kappa).
\end{aligned}$$

However, by the Fubini theorem,

$$\begin{aligned}
T_w(f,f;\chi,\kappa) ={} & \frac{1}{\int_\chi w(x)\,d\mu(x)}\int_\chi w(x)\,f^2(x)\,d\mu(x) \\
& + \frac{1}{\int_\kappa w(y)\,d\mu(y)}\int_\kappa w(y)\,f^2(y)\,d\mu(y) \\
& - 2\frac{1}{\int_\chi w(x)\,d\mu(x)}\int_\chi w(x)\,f(x)\,d\mu(x)\int_\kappa w(y)\,f(y)\,d\mu(y) \\
={} & T_w(f;\chi)+T_w(f;\kappa)+[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)]^2
\end{aligned}$$

and a similar expression holds for g.

Hence (3.3) holds where from (3.4), $B_w(f;\chi,\kappa)=T_w(f,f;\chi,\kappa)$ and $T_w(f;\chi)$ is as given by (3.5). □

Corollary 3.2. *Let the conditions of Theorem 3.1 persist and in addition let*

$$m_1 \le f \le M_1 \text{ a.e. on } \chi \text{ and } m_2 \le f \le M_2 \text{ a.e. on } \kappa,$$
$$n_1 \le g \le N_1 \text{ a.e. on } \chi \text{ and } n_2 \le g \le N_2 \text{ a.e. on } \kappa.$$

Then we have the inequality

$$\begin{aligned}
(3.7)\quad & |T_w^*(f,g;\chi,\kappa)| \\
& \le \left[\left(\frac{M_1-m_1}{2}\right)^2+\left(\frac{M_2-m_2}{2}\right)^2+(\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa))^2\right]^{\frac{1}{2}} \\
& \quad \times \left[\left(\frac{N_1-n_1}{2}\right)^2+\left(\frac{N_2-n_2}{2}\right)^2+(\mathcal{M}_w(g;\chi)-\mathcal{M}_w(g;\kappa))^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Proof. The proof follows directly from (3.3) – (3.5), where by the Grüss inequality (2.2)

$$T_w(f;\chi)=T_w(f,f;\chi)\le\left(\frac{M_1-m_1}{2}\right)^2.$$

Similar inequalities for $T_w(f;\kappa)$, $T_w(g;\chi)$ and $T_w(g;\kappa)$ readily produce (3.7). □

Remark 3.3. If $\chi\equiv\kappa\equiv\Omega$ and $m_1=m_2=:m$ and $M_1=M_2=:M$ then $\mathcal{M}_w(f;\chi)=\mathcal{M}_w(f;\kappa)$. If $n_1=n_2=:n$ and $N_1=N_2=:N$ with $\chi\equiv\kappa\equiv\Omega$ we have $\mathcal{M}_w(g;\chi)=\mathcal{M}_w(g;\kappa)$. Thus we recapture the Grüss inequality

$$|T_w^*(f,g;\Omega,\Omega)|=2\,|T_w(f,g;\Omega)|\le 2\cdot\left(\frac{M-m}{2}\right)\left(\frac{N-n}{2}\right).$$

Following in the same spirit as (1.23) consider the generalised Čebyšev functional

$$T_w^{\dagger}(f,g;\chi,\kappa) := \mathcal{M}_w(fg;\chi) - \mathcal{M}_w(g;\chi)\mathcal{M}_w(f;\kappa), \tag{3.8}$$

where $\mathcal{M}_w(f;\chi)$ is as defined by (3.2) and $\chi,\kappa \subset \Omega$.

$T_w^{\dagger}(f,g;\chi,\kappa)$ may be shown to satisfy a Körkine type identity

$$T_w^{\dagger}(f,g;\chi,\kappa) = \frac{1}{\int_\chi w(x)\,d\mu(x)\int_\kappa w(y)\,d\mu(y)} \times \int_\chi\int_\kappa w(x)\,w(y)\,g(x)\,(f(x)-f(y))\,d\mu(y)\,d\mu(x). \tag{3.9}$$

The following theorem then provides bounds for (3.8) using (3.9), where the proof mimicks that used in obtaining bounds for $T_w^{*}(f,g;\chi,\kappa)$ and will thus be omitted.

Theorem 3.4. *Let $w,f,g:\Omega\to\mathbb{R}$ be $\mu-$measurable functions with $w\geq 0$, μ – a.e. on Ω and $\int_\chi w(x)\,d\mu(x)>0$ and $\int_\kappa w(x)\,d\mu(x)>0$ where $\chi,\kappa\subset\Omega$. Further, let $f,g,fg\in L_w(\Omega,\mathcal{A},\mu)$ then, for $m_1\leq g\leq M_1$ and $n_1\leq f\leq N_1$ a.e. on χ with $n_2\leq f\leq N_2$ a.e. on κ, the following inequalities hold. Namely,*

$$\begin{aligned} \left|T_w^{\dagger}(f,g;\chi,\kappa)\right| &\leq \left[T_w(g;\chi)+\mathcal{M}_w^2(g;\chi)\right]^{\frac12} \\ &\qquad\times\left\{T_w(f;\chi)+T_w(f;\kappa)+\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]^2\right\}^{\frac12} \\ &\leq \left[\left(\frac{M_1-m_1}{2}\right)^2+\mathcal{M}_w^2(g;\chi)\right]^{\frac12} \\ &\qquad\times\left\{\left(\frac{N_1-n_1}{2}\right)^2+\left(\frac{N_2-n_2}{2}\right)^2+\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]^2\right\}^{\frac12}, \end{aligned} \tag{3.10}$$

where $T_w(f;\chi)$ and $\mathcal{M}_w(f;\chi)$ are as defined in (3.5) and (3.2) respectively.

4. FURTHER GENERALISED ČEBYŠEV FUNCTIONAL BOUNDS

Let the conditions as described in Section 2 continue to hold. Let χ,κ be measurable subsets of Ω and define

$$\begin{aligned} D_w^{\dagger}(f;\chi,\kappa) &:= D_{w,1}^{\dagger}(f;\chi,\kappa) \\ &:= \mathcal{M}_w\left(\left|f(x)-\mathcal{M}_w(f;\kappa)\right|,\chi\right), \end{aligned} \tag{4.1}$$

where $\mathcal{M}_w(f;\chi)$ is as defined by (3.9).

The following theorem holds providing bounds for the generalised Čebyšev functional $T_w^{\dagger}(f,g;\chi,\kappa)$ defined by (3.4).

Theorem 4.1. *Let $w,f,g:\Omega\to\mathbb{R}$ be $\mu-$measurable functions with $w\geq 0$ $\mu-$a.e. on Ω. Further, let $\chi,\kappa\subset\Omega$ and $\int_\chi w(x)\,d\mu(x)>0$ and $\int_\kappa w(y)\,d\mu(y)>0$. If $f,g,fg\in L_w(\Omega,\mathcal{A},\mu)$ and there are constants δ,Δ such that*

$$-\infty<\delta\leq g(x)\leq\Delta<\infty \text{ for } \mu-\text{a.e. } x\in\chi,$$

then we have the inequality

$$\left|T_w^{\dagger}(f,g;\chi,\kappa)-\frac{\Delta+\delta}{2}\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]\right|\leq\frac{\Delta-\delta}{2}D_w^{\dagger}(f;\chi,\kappa), \tag{4.2}$$

where $D_w^{\dagger}(f;\chi,\kappa)$ is as defined by (4.1).

The constant $\frac{1}{2}$ is sharp in (4.2) in that it cannot be replaced by a smaller quantity.

Proof. From (3.4) we have the identity

$$T_w^{\dagger}(f,g;\chi,\kappa)=\frac{1}{\int_\chi w(x)\,d\mu(x)}\int_\chi w(x)\,g(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)\,. \tag{4.3}$$

Consider the measurable subsets χ_+ and χ_- of χ defined by

$$\chi_+ := \{x\in\chi | f(x)-\mathcal{M}_w(f;\kappa)\geq 0\} \tag{4.4}$$

and

$$\chi_- := \{x\in\chi | f(x)-\mathcal{M}_w(f;\kappa)< 0\} \tag{4.5}$$

so that $\chi=\chi_+\cup\chi_-$ and $\chi_+\cap\chi_-=\emptyset$.

If we define

$$I_+(f,g,w):=\int_{\chi_+} w(x)\,g(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x) \quad \text{and} \tag{4.6}$$
$$I_-(f,g,w):=\int_{\chi_-} w(x)\,g(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)$$

then we have from (4.3)

$$T_w^{\dagger}(f,g;\chi,\kappa)\int_\chi w(x)\,d\mu(x)=I_+(f,g,w)+I_-(f,g,w)\,. \tag{4.7}$$

Since $-\infty<\delta\leq g(x)\leq\Delta<\infty$ for μ -a.e. $x\in\chi$ and $\mu-$a.e. $x\in\Omega$ we may write

$$I_+(f,g,w)\leq\Delta\int_{\chi_+} w(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x) \tag{4.8}$$

and

$$I_-(f,g,w)\leq\delta\int_{\chi_-} w(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)\,. \tag{4.9}$$

Now, the identity

$$\begin{aligned}
[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)]&\int_\chi w(x)\,d\mu(x)\\
&=\int_\chi w(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)\\
&=\int_{\chi_+} w(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)\\
&\qquad+\int_{\chi_-} w(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)
\end{aligned} \tag{4.10}$$

holds so that we have from (4.9)

$$\begin{aligned}
I_-(f,g,w)\leq -\delta&\int_{\chi_+} w(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)\\
&+\delta\,[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)]\int_\chi w(x)\,d\mu(x)\,.
\end{aligned} \tag{4.11}$$

That is, combining (4.8) and (4.11) we have from (4.7)

$$T_w^{\dagger}(f,g;\chi,\kappa) \leq \frac{\Delta-\delta}{\int_{\chi} w(x)\,d\mu(x)} \int_{\chi_+} w(x)\left(f(x)-\mathcal{M}_w(f;\kappa)\right)d\mu(x) + \delta\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]. \tag{4.12}$$

Further, we have

$$\int_{\chi} w(x)\left|f(x)-\mathcal{M}_w(f;\kappa)\right|d\mu(x) = \int_{\chi_+} w(x)\left(f(x)-\mathcal{M}_w(f;\kappa)\right)d\mu(x) - \int_{\chi_-} w(x)\left(f(x)-\mathcal{M}_w(f;\kappa)\right)d\mu(x),$$

giving, from (4.10),

$$\int_{\chi} w(x)\left|f(x)-\mathcal{M}_w(f;\kappa)\right|d\mu(x) + \left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]\int_{\chi} w(x)\,d\mu(x) = 2\int_{\chi_+} w(x)\left(f(x)-\mathcal{M}_w(f;\kappa)\right)d\mu(x). \tag{4.13}$$

Substitution of (4.13) into (4.12) produces

$$T_w^{\dagger}(f,g;\chi,\kappa) \leq \frac{\Delta-\delta}{2}\cdot\frac{1}{\int_{\chi} w(x)\,d\mu(x)} \int_{\chi} w(x)\left|f(x)-\mathcal{M}_w(f;\kappa)\right|d\mu(x) + \frac{\Delta+\delta}{2}\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]. \tag{4.14}$$

Now, we may see from (4.14) that

$$T_w^{\dagger}(-f,g;\chi,\kappa) = -T_w^{\dagger}(f,g;\chi,\kappa)$$

and so

$$-T_w^{\dagger}(f,g;\chi,\kappa) \leq \frac{\Delta-\delta}{2}\cdot\frac{1}{\int_{\chi} w(x)\,d\mu(x)} \int_{\chi} w(x)\left|f(x)-\mathcal{M}_w(f;\kappa)\right|d\mu(x) - \frac{\Delta+\delta}{2}\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right]. \tag{4.15}$$

Combining (4.14) and (4.15) gives the result (4.2).

Now for the sharpness of the constant $\frac{1}{2}$.

To show this, it is perhaps easiest to let $\mathcal{M}_w(f;\chi) = \mathcal{M}_w(f;\kappa)$ in which instance the result of Theorem 2.2, namely, (2.6) is recaptured which was shown to be sharp in [5].

The proof is now complete. □

Remark 4.2. It should be noted that the result of Theorem 4.1 is a generalisation of Theorem 2.2 to involving means over different sets χ and κ. If we take $\chi = \kappa = \Omega$ in (4.2) then the result (2.6), which was proven in [5] is regained.

Following in the spirit of Section 2, we may define for χ, κ measurable subsets of Ω

$$D_{w,p}^{\dagger}(f;\chi,\kappa) := \left[\mathcal{M}_w\left(\left|f(\cdot)-\mathcal{M}_w(f;\kappa)\right|^p;\chi\right)\right]^{\frac{1}{p}}, \quad 1 \leq p < \infty \tag{4.16}$$

and

$$D_{w,\infty}^{\dagger}(f;\chi,\kappa) := ess\sup_{x\in\chi}|f(x) - \mathcal{M}_w(f;\kappa)|. \tag{4.17}$$

The following corollary then holds.

Corollary 4.3. *Let the conditions of Theorem 4.1 persist, then we have*

$$\begin{aligned} \left|T_w^{\dagger}(f,g;\chi,\kappa) - \frac{\Delta+\delta}{2}[\mathcal{M}_w(f;\chi) - \mathcal{M}_w(f;\kappa)]\right| \\ \leq \frac{\Delta-\delta}{2} D_{w,1}^{\dagger}(f;\chi,\kappa) \\ \leq \frac{\Delta-\delta}{2} D_{w,p}^{\dagger}(f;\chi,\kappa), \quad f \in L_{w,p}(\Omega,\mathcal{A},\mu), \quad 1 \leq p < \infty, \\ \leq \frac{\Delta-\delta}{2} D_{w,\infty}^{\dagger}(f;\chi,\kappa), \quad f \in L_{\infty}(\Omega,\mathcal{A},\mu), \end{aligned} \tag{4.18}$$

where $D_{w,p}^{\dagger}(f;\chi,\kappa)$ *and* $D_{w,\infty}^{\dagger}(f;\chi,\kappa)$ *are as defined in (4.16) and (4.17) respectively.*

The constant $\frac{1}{2}$ *is sharp in all the above inequalities.*

Proof. From the Sonin type identity (4.3) we have

$$\begin{aligned} T_w^{\dagger}(f;\chi,\kappa) - \frac{\Delta+\delta}{2}[\mathcal{M}_w(f;\chi) - \mathcal{M}_w(f;\kappa)] \\ = \frac{1}{\int_{\chi} w(x)\,d\mu(x)} \int_{\chi} w(x)\left(g(x) - \frac{\Delta+\delta}{2}\right)(f(x) - \mathcal{M}_w(f;\kappa))\,d\mu(x). \end{aligned} \tag{4.19}$$

Now, the first result in (4.18) was obtained in Theorem 4.1 in the guise of (4.2). However, it may be obtained directly from the identity (4.19) since

$$\begin{aligned} \left|T_w^{\dagger}(f;\chi,\kappa) - \frac{\Delta+\delta}{2}[\mathcal{M}_w(f;\chi) - \mathcal{M}_w(f;\kappa)]\right| \\ \leq \frac{1}{\int_{\chi} w(x)\,d\mu(x)} \int_{\chi} w(x)\left|g(x) - \frac{\Delta+\delta}{2}\right||f(x) - \mathcal{M}_w(f;\kappa)|\,d\mu(x) \\ \leq ess\sup_{x\in\chi}\left|g(x) - \frac{\Delta+\delta}{2}\right| D_{w,1}^{\dagger}(f;\chi,\kappa). \end{aligned} \tag{4.20}$$

Now, for $-\infty < \delta \leq g(x) \leq \Delta < \infty$ for $x \in \chi$, then

$$ess\sup_{x\in\chi}\left|g(x) - \frac{\Delta+\delta}{2}\right| = \frac{\Delta-\delta}{2} \tag{4.21}$$

and so the first inequality in (4.17) results.

Further, we have, using Hölder's inequality

$$\begin{aligned} D_{w,1}^{\dagger}(f;\chi,\kappa) &= \frac{1}{\int_{\chi} w(x)\,d\mu(x)} \int_{\chi} w(x)\,|f(x) - \mathcal{M}_w(f;\kappa)|\,d\mu(x) \\ &\leq D_{w,p}^{\dagger}(f;\chi,\kappa) \\ &\leq D_{w,\infty}^{\dagger}(f;\chi,\kappa), \end{aligned}$$

where we have used (4.16) and (4.17) producing the remainder of the results in (4.18) from (4.20) and (4.21).

The sharpness of the constants follows from Hölder's inequality and the sharpness of the first inequality proven earlier. □

Remark 4.4. We note that

$$T_w^{\dagger}(f,g;\chi,\kappa)-\frac{\Delta+\delta}{2}\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right] = T_w(f,g;\chi)+\left[\mathcal{M}_w(g;\chi)-\frac{\Delta+\delta}{2}\right]\left[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)\right] \tag{4.22}$$

so that

$$T_w^{\dagger}(f,g;\chi,\kappa)=T_w(f,g;\chi)$$

if either or both $\mathcal{M}_w(g;\chi)\equiv\frac{\Delta+\delta}{2}$ and $\mathcal{M}_w(f;\chi)\equiv\mathcal{M}_w(f;\kappa)$ hold.

Thus Theorem 4.1 and Corollary 4.3 are generalisations of Theorem 2.2 and Corollary 2.3 respectively.

Corollary 4.5. *Let the conditions in Theorem 4.1 hold and further assume that κ is chosen in such a way that $\mathcal{M}_w(f;\kappa)=0$, then*

$$\begin{aligned}\left|\mathcal{M}_w(fg;\chi)-\frac{\Delta+\delta}{2}\mathcal{M}_w(f;\chi)\right| &\le \frac{\Delta-\delta}{2}\mathcal{M}_w(|f|;\chi)\\ &\le \frac{\Delta-\delta}{2}\left[\mathcal{M}_w(|f|^p;\chi)\right]^{\frac{1}{p}},\quad f\in L_{w,p}(\Omega,\mathcal{A},\mu),\\ &\le \frac{\Delta-\delta}{2}ess\sup_{x\in\chi}|f(x)|,\quad f\in L_\infty(\Omega,\mathcal{A},\mu),\end{aligned} \tag{4.23}$$

The constant $\frac{1}{2}$ is sharp in the above inequalities.

Proof. Taking $\mathcal{M}_w(f;\kappa)=0$ in (4.18) and , using (3.8), (4.16) and (4.17) readily produces the stated result. □

Remark 4.6. The result (4.23) provides a Čebyšev-like expression in which the arithmetic average of the upper and lower bounds of the function $g(\cdot)$ is in place of the traditional integral mean. The above formulation may be advantageous if the norms of $f(\cdot)$ are known or are more easily calculated than the shifted norms.

Remark 4.7. Similar results as procured for $T_w^{\dagger}(f,g;\chi,\kappa)$ may be obtained for the generalised Čebyšev functional $T_w^{*}(f,g;\chi,\kappa)$ as defined by (3.1). We note that

$$\begin{aligned}T_w^{*}(f,g;\chi,\kappa)&=T_w^{\dagger}(f,g;\chi,\kappa)+T_w^{\dagger}(f,g;\kappa,\chi)\\ &=\frac{1}{\int_\chi w(x)\,d\mu(x)}\int_\chi w(x)\,g(x)\,(f(x)-\mathcal{M}_w(f;\kappa))\,d\mu(x)\\ &\quad+\frac{1}{\int_\kappa w(y)\,d\mu(y)}\int_\kappa w(y)\,g(y)\,(f(y)-\mathcal{M}_w(f;\chi))\,d\mu(y).\end{aligned} \tag{4.24}$$

As an example, we consider a result corresponding to (4.2). Assume that the conditions of Theorem 4.1 hold and let

$$-\infty<\delta_1\le g(x)\le\Delta_1<\infty\ \text{ for } \mu-\text{a.e. } x\in\chi$$

with

$$-\infty<\delta_2\le g(x)\le\Delta_2<\infty\ \text{ for } \mu-\text{a.e. } x\in\kappa.$$

Then from (4.24), we have

$$\left|T_w^*(f,g;\chi,\kappa)-\left(\frac{\Delta_2+\delta_2}{2}+\frac{\Delta_1+\delta_1}{2}\right)[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)]\right| \leq \frac{\Delta_1-\delta_1}{2}D_w^{\dagger}(f;\chi,\kappa)+\frac{\Delta_2-\delta_2}{2}D_w^{\dagger}(f;\kappa,\chi). \tag{4.25}$$

where $D_w^{\dagger}(f;\chi,\kappa)$ is as defined in (4.1). We notice from (4.25) that

$$|T_w^*(f,g;\chi,\kappa)-(\Delta+\delta)[\mathcal{M}_w(f;\chi)-\mathcal{M}_w(f;\kappa)]| \leq \frac{\Delta-\delta}{2}\left[D_w^{\dagger}(f;\chi,\kappa)+D_w^{\dagger}(f;\kappa,\chi)\right],$$

where $\delta_1=\delta_2=\delta$ and $\Delta_1=\Delta_2=\Delta$.

Similar results for $T_w^*(f,g;\chi,\kappa)$ to those expounded in Corollary 4.3 for $T_w^{\dagger}(f,g;\chi,\kappa)$ may be obtained, however these will not be considered any further here.

5. Some Specific Inequalities

Some particular specialisation of the results in the previous sections will now be examined. New results are provided by these specialisations.

A. Let $w,f,g:I\to\mathbb{R}$ be Lebesgue integrable functions with $w\geq 0$ a.e. on the interval I and $\int_I w(x)\,dx>0$. If $f,g,fg\in L_{w,1}(I)$, where

$$L_{w,p}(I):=\left\{f:I\to\mathbb{R}\,\middle|\,\int_I w(x)|f(x)|^p\,dx<\infty\right\}$$

and

$$L_\infty(I):=ess\sup_{x\in I}|f(x)|$$

and

$$-\infty<\delta\leq g(x)\leq\Delta<\infty \quad \text{for } x\in[a,b]\subset I,$$

then we have the inequality, for $[c,d]\subset I$,

$$\begin{aligned}\left|T_w^{\dagger}(f,g;[a,b],[c,d])-\frac{\Delta+\delta}{2}[\mathcal{M}_w(f;[a,b])-\mathcal{M}_w(f;[c,d])]\right| &\leq \frac{\Delta-\delta}{2}\mathcal{M}_w(|f(\cdot)-\mathcal{M}_w(f;[c,d])|;[a,b]) \\ &\leq \frac{\Delta-\delta}{2}[\mathcal{M}_w(|f(\cdot)-\mathcal{M}_w(f;[c,d])|^p;[a,b])]^{\frac{1}{p}}, \quad f\in L_{w,p}[I] \\ &\leq \frac{\Delta-\delta}{2}ess\sup_{x\in[a,b]}|f(x)-\mathcal{M}_w(f;[c,d])|, \quad f\in L_\infty[I],\end{aligned} \tag{5.1}$$

where

$$T_w^{\dagger}(f,g;[a,b],[c,d])=\mathcal{M}_w(fg;[a,b])-\mathcal{M}_w(g;[a,b])\mathcal{M}_w(f;[c,d])$$

and

$$\mathcal{M}_w(f;[a,b]):=\frac{1}{\int_a^b w(x)\,dx}\int_a^b w(x)f(x)\,dx.$$

The constant $\frac{1}{2}$ is sharp for all the inequalities in (5.1).

To obtain the result (5.1), we have identified $[a,b]$ with χ and $[c,d]$ with κ in the preceding work specifically in (4.2).

If we take $[a,b]=[c,d]$ then results obtained in [5] are captured. Further, taking $w(x)=1$, $x\in I$ produces a result obtained in [11] from the first inequality in (5.1).

B. Let $\bar{\mathbf{a}}=(a_1,\dots,a_n)$, $\bar{\mathbf{b}}=(b_1,\dots,b_n)$, $\bar{\mathbf{p}}=(p_1,\dots,p_n)$ be $n-$tuples of real numbers with $p_i\geq 0$, $i\in\{1,2,\dots,n\}$ and with $P_k=\sum_{i=1}^{k}p_i$, $P_n=1$. Further, if

$$b\leq b_i\leq B,\quad i\in\{1,2,\dots,n\}$$

then for $m\leq n$

$$\begin{aligned}(5.2)\qquad &\left|\sum_{i=1}^{n}p_ia_ib_i-\frac{B+b}{2}\left[\sum_{i=1}^{n}p_ia_i-\frac{1}{P_m}\sum_{j=1}^{m}p_ja_j\right]-\frac{1}{P_m}\sum_{j=1}^{m}p_ja_j\cdot\sum_{i=1}^{n}p_ib_i\right|\\ &\leq\frac{B-b}{2}\sum_{i=1}^{n}p_i\left|a_i-\frac{1}{P_m}\sum_{j=1}^{m}p_ja_j\right|\\ &\leq\frac{B-b}{2}\left[\sum_{i=1}^{n}p_i\left|a_i-\frac{1}{P_m}\sum_{j=1}^{m}p_ja_j\right|^{\alpha}\right]^{\frac{1}{\alpha}},\quad 1<\alpha<\infty\\ &\leq\frac{B-b}{2}\max_{i\in\overline{1,n}}\left|a_i-\frac{1}{P_m}\sum_{j=1}^{m}p_ja_j\right|.\end{aligned}$$

If $\sum_{j=1}^{m}p_ja_j=0$, then the above results simplify.

The constant $\frac{1}{2}$ is sharp for all the inequalities in (5.1).

If $p_i=1,\ i\in\{1,\dots,n\}$ then the following unweighted inequalities may be stated from (5.2). Namely,

$$\begin{aligned}(5.3)\qquad &\left|\frac{1}{n}\sum_{i=1}^{n}a_ib_i-\frac{1}{m}\sum_{i=1}^{m}a_i\cdot\frac{1}{n}\sum_{i=1}^{n}b_i-\frac{B+b}{2}\left[\frac{1}{n}\sum_{i=1}^{n}a_i-\frac{1}{m}\sum_{j=1}^{m}a_j\right]\right|\\ &\leq\frac{B-b}{2}\frac{1}{n}\sum_{i=1}^{n}\left|a_i-\frac{1}{m}\sum_{j=1}^{m}a_j\right|\\ &\leq\frac{B-b}{2}\left(\frac{1}{n}\sum_{i=1}^{n}\left|a_i-\frac{1}{m}\sum_{j=1}^{m}a_j\right|^{\alpha}\right)^{\frac{1}{\alpha}}\\ &\leq\frac{B-b}{2}\max_{i\in\overline{1,n}}\left|a_i-\frac{1}{m}\sum_{j=1}^{m}a_j\right|.\end{aligned}$$

For $m=n$ and $a_i=b_i$ for each $i\in\{1,2,\dots,n\}$ then from (5.2),

$$0\leq\sum_{i=1}^{n}p_ib_i^2-\left(\sum_{i=1}^{n}p_ib_i\right)^2\leq\frac{B-b}{2}\sum_{i=1}^{n}p_i\left|b_i-\sum_{j=1}^{n}p_jb_j\right|\leq\left(\frac{B-b}{2}\right)^2,$$

providing a counterpart to the Schwartz inequality.

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