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NEW INEQUALITIES BETWEEN ELEMENTARY SYMMETRIC POLYNOMIALS

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Abstract

New families of sharp inequalities between elementary symmetric polynomials are proven. We estimate σ_{n-k} above and below by the elementary symmetric polynomials $\sigma_{n-k+1}, \dots, \sigma_n$ in the case, when $x_1, \dots, x_n$ are non-negative real numbers with sum equal to one.

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Contents

1. Introduction

Let $n \geq 2$ be an integer. As usual, we denote by $\sigma_0, \sigma_1, \dots, \sigma_n$ the elementary symmetric polynomials of the variables $x_1, \dots, x_n$.

In other words, $\sigma_0 = \sigma_0(x_1, \dots, x_n) = 1$ and for $1 \leq k \leq n$

$$\sigma_k = \sigma_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \dots x_{i_k}.$$

The different $\sigma_0, \sigma_1, \dots, \sigma_n$, are not comparable between them, but they are connected by nonlinear inequalities. To state them, it is more convenient to consider their averages $E_k = \sigma_k \Big/ \binom{n}{k}$, $k = 0, 1, \dots, n$.

There are three basic types of inequalities between the symmetric functions with respect to the range of the variables $x_1, \dots, x_n$.

For arbitrary real $x_1, \dots, x_n$ the following inequalities are known:

$$E_k^2 \geq E_{k-1}E_{k+1}, \qquad 1 \leq k \leq n-1, \qquad \text{(Newton-Maclaurin)}, \tag{1.1}$$

$$4(E_{k+1}E_{k+3} - E_{k+2}^2)(E_kE_{k+2} - E_{k+1}^2) \geq (E_{k+1}E_{k+2} - E_kE_{k+3})^2,$$
$$k = 0, \dots, n-3, \quad \text{(Rosset [4])},$$

as well as the inequalities of Niculescu [2]. A complete description about their historical and contemporary stage of development can be found, for example, in [1] and [2].

Suppose now that all x_j, $j = 1, \dots, n$, are positive. Then the following general result (see [1, Theorem 77, p. 64]) is known:

Theorem 1.1 (Hardy, Littlewood, Pólya). *For any positive* $x_1, \dots, x_n$ *and positive* $\alpha_1, \dots, \alpha_n,\ \beta_1, \dots, \beta_n$ *the inequality*

$$E_1^{\alpha_1} \cdots E_n^{\alpha_n} \le E_1^{\beta_1} \cdots E_n^{\beta_n}$$

holds if and only if

$$\alpha_m + 2\alpha_{m+1} + \cdots + (n-m+1)\alpha_n \ge \beta_m + 2\beta_{m+1} + \cdots + (n-m+1)\beta_n$$

for each $1 \le m \le n$.

For other results in this direction see [1].

The aim of this paper is to obtain new inequalities between $\sigma_1, \dots, \sigma_n$ in the case when $x_1, \dots, x_n$ are non-negative, (Theorem 2.3 and Theorem 2.5 below). More precisely, we obtain the best possible estimates of $\sigma_1^k \sigma_{n-k}$ from below and above by linear functions of $\sigma_1^{k-1}\sigma_{n-k+1}, \dots, \sigma_1^0 \sigma_n$. Since all these functions are homogeneous with respect to $(x_1, \dots, x_n)$ of the same order, we can set $\sigma_1 = x_1 + \cdots + x_n = 1$, then our inequalities give the best possible estimates of σ_{n-k} by linear functions of $\sigma_{n-k+1}, \dots, \sigma_n$ for $k = 1, \dots, n-1$ in this case (Theorem 3.1 and Theorem 3.2 below). Inequalities of this type for $k = n-2$ have been recently obtained by Sato [4], which can be obtained as a consequence of Theorem 2.5 below.

2. New Inequalities (Theorem 2.3 and Theorem 2.5)

For the sake of completeness we give a straightforward proof of the following proposition, which is a consequence of Theorem 1.1, cited in the introduction. Here we suppose that $x_1, \ldots, x_n$ are non-negative.

Proposition 2.1. *Let* $x_1, \ldots, x_n$ *be non-negative real numbers,* $n \geq 2$. *Then for* $1 \leq p \leq n-1$ *we have*

$$\sigma_1 \sigma_p \geq \frac{n(p+1)}{n-p} \sigma_{p+1}. \tag{2.1}$$

Proof. Denote $\sigma_{l,n} = \sum_{1 \leq i_1 < \cdots < i_l \leq n} x_{i_1} x_{i_2} \cdots x_{i_l}$, $1 \leq l \leq n$. Note, that (2.1) is equivalent to

$$\sigma_{1,n} \sigma_{p,n} \geq \frac{n(p+1)}{n-p} \sigma_{p+1,n}. \tag{2.2}$$

First we shall check (2.2) for $p = 1$ and for $p = n-1$.

(i) For $p = 1$ the inequality (2.2) reads

$$\left(\sum_{i=1}^{n} x_i \right)^2 \geq \frac{2n}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j,$$

which is equivalent to

$$(n-1) \left(\sum_{i=1}^{n} x_i \right)^2 \geq \frac{2n}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j,$$

hence to $\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq 0$.

(ii) For $p = n-1$ (2.2) is equivalent to $\sigma_{1,n}\sigma_{n-1,n} \geq n^2\sigma_{n,n}$. If $\sigma_{n,n} = 0$, then (2.2) is obvious. Let $\sigma_{n,n} \neq 0$, then (2.2) is equivalent to $n^2 \leq \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n \frac{1}{x_i}\right)$, which follow from AM-GM inequality.

We are going to prove (2.2) by recurrence with respect to $n \geq 2$.

(iii) We already proved that (2.2) is true for $n = 2$.

(iv) Let (2.2) be true for $n \geq 2$ and for each p, $1 \leq p \leq n-1$. Fix p, $2 \leq p \leq n-1$. We will prove, that

$$\sigma_{1,n+1}\sigma_{p,n+1} \geq \frac{(n+1)(p+1)}{n+1-p}\sigma_{p+1,n+1}. \tag{2.3}$$

Since (2.3) is homogeneous, excluding the case $x_1 = \cdots = x_n = x_{n+1} = 0$, we may assume, that $\sigma_{1,n+1} = 1$.

Let $x_1 \leq x_2 \leq \cdots \leq x_{n+1}$. The following cases are possible:

1) Let $x_{n+1} = 1$. Then $x_1 = \cdots = x_n = 0$ and (2.3) becomes an equality.

2) Let $x_{n+1} = \frac{1}{n+1}$. Then $x_1 = \cdots = x_n = x_{n+1} = \frac{1}{n+1}$ and we obtain

$$\begin{aligned}\sigma_{p,n+1} &- \frac{(n+1)(p+1)}{n+1-p}\sigma_{p+1,n+1} \\ &= \binom{n+1}{p}\frac{1}{(n+1)^p} - \frac{p+1}{n+1-p}\binom{n+1}{p+1}\frac{1}{(n+1)^p} = 0,\end{aligned}$$

hence (2.3) becomes again an equality.

3) Let $x_{n+1} \in \left(\frac{1}{n+1}; 1\right)$. Substitute $x_1 + \cdots + x_n = 1 - x_{n+1} = \sigma_{1,n} = s$, with $s \in \left(0; \frac{n}{n+1}\right)$. Then $\sigma_{p,n+1} = \sigma_{p,n} + (1-s)\sigma_{p-1,n}$ and $\sigma_{p+1,n+1} = \sigma_{p+1,n} + (1-s)\sigma_{p,n}$. Hence (2.3) is equivalent to

$$\sigma_{p,n} + (1-s)\sigma_{p-1,n} \geq \frac{(n+1)(p+1)}{n+1-p}\left[\sigma_{p+1,n} + (1-s)\sigma_{p,n}\right],$$

which is equivalent to

$$\left[\frac{n+1-p}{(n+1)(p+1)} - (1-s)\right]\sigma_{p,n} + \frac{(1-s)(n+1-p)}{(n+1)(p+1)}\sigma_{p-1,n} \geq \sigma_{p+1,n}. \tag{2.4}$$

From (iv) we obtain $\sigma_{p+1,n} \leq \frac{n-p}{n(p+1)} s\sigma_{p,n}$. Then (2.4) follows from the next inequality (if true):

$$\frac{n-p}{n(p+1)} s\sigma_{p,n} \leq \left[\frac{n+1-p}{(n+1)(p+1)} - (1-s)\right]\sigma_{p,n} + \frac{(1-s)(n+1-p)}{(n+1)(p+1)}\sigma_{p-1,n}, \tag{2.5}$$

which is equivalent to

$$\sigma_{p-1,n} \geq \frac{p[n(n+2) - (n+1)^2 s]}{n(n+1-p)(1-s)}\sigma_{p,n}. \tag{2.6}$$

It follows from (iv) that $\sigma_{p-1,n} \geq \frac{np}{(n+1-p)s}\sigma_{p,n}$.

Hence (2.6), and consequently (2.5) and (2.4), follow from

$$\frac{np}{(n+1-p)s}\sigma_{p,n} - \frac{p[n(n+2)-(n+1)^2 s]}{n(n+1-p)(1-s)}\sigma_{p,n} = \frac{p[(n+1)s-n]^2}{ns(n+1-p)(1-s)}\sigma_{p,n} \geq 0.$$

Since (2.2) is true for $p=1$ and $p=n$ according to (i) and (ii), then (2.2) is fulfilled for n, $n \geq 2$. Hence the proposition is proved. ∎

Remark 2.1. *It follows from the proof, that equality is achieved in the following two cases:*

1) $x_1 = x_2 = \cdots = x_n = a \geq 0.$

2) $n-p+1$ *of* $x_1, \ldots, x_n$ *are equal to* 0 *and the rest of them are arbitrary non-negative real numbers.*

Remark 2.2. *(2.1) can be proven using Lemma 2.2 below, but in this way it will be difficult to see when (2.1) turns into an equality.*

From now on n will be a fixed positive integer. It will be assumed that at least one of the non-negative numbers $x_1, \ldots, x_n$ differs from zero.

Lemma 2.2. *Let us assume that* $x_1, \ldots, x_n$ *are non-negative real numbers (*$n \geq 2$*) and* $x_1 + \cdots + x_n = \sigma_1 = 1$*. Then the function* $f(x_1, \ldots, x_n) = a_1 + a_2\sigma_2 + \cdots + a_n\sigma_n$ *(*$a_1, \ldots, a_n$ *are real numbers), achieves its maximum and minimum at least in some of the points* $P_{k,n}\left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0\right)$, $1 \leq k \leq n$ *(the first* k *coordinates of* $P_{k,n}$ *are equal to* $\frac{1}{k}$*, and the rest of them are equal to zero).*

Proof. The set $A_n = \{(x_1, \dots, x_n)/x_i \geq 0,\ x_1 + \cdots + x_n = 1\}$ is compact and f is continuous in it, hence f achieves its minimum and maximum values. We rewrite f as follows:

$$f(x_1, \dots, x_n) = x_1 x_2 g(x_3, \dots x_n) + x_1 h_1(x_3, \dots, x_n) + x_2 h_2(x_3, \dots, x_n) + t(x_3, \dots, x_n) + a_1.$$

As f is symmetric, then $h_1 \equiv h_2$ and therefore:

$$(2.7)\quad f(x_1, \dots, x_n) = x_1 x_2 g(x_3, \dots x_n) + (x_1 + x_2) h_1(x_3, \dots, x_n) + t(x_3, \dots, x_n) + a_1.$$

Let $P(x_1^0, \dots, x_n^0)$ be a point in which f achieves its minimum value. We consider the function $F(x) = f(x, s - x, x_3^0, \dots, x_n^0)$, $s = x_1^0 + x_2^0$, for $x \in [0; s]$ (we assume, that $s > 0$). Obviously the minimum values of F and f are equal and F achieves its minimum value for $x = x_1^0$. From (2.7) we obtain that $F(x) = \alpha x(s - x) + s\beta + \gamma = \alpha x(s - x) + \delta$, where α, δ depend on $x_1^0, x_2^0, x_3^0, \dots, x_n^0,\ a_1, \dots, a_n$.

The following three cases are possible:

(i) $\alpha = 0$. Then $F(x) = const$ and we may assume that $\min F = F(0)$ or $\min F = F\left(\frac{s}{2}\right)$.

(ii) $\alpha > 0$. Then $\min F = F(0)$.

(iii) $\alpha < 0$. Then $\min F = F\left(\frac{s}{2}\right)$.

Hence, as x_1^0 and x_2^0 were arbitrarily chosen then, for $\forall i \neq j$ we may assume that $x_i^0 = x_j^0$ or, at least one of them is equal to zero.

Let us choose a point $P(x_1^0, \dots, x_n^0)$, for which the number of coordinates p which equal to zero is the highest possible and $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$. If $p = n-1$, then Lemma 2.2 is proven. Let $0 \leq p \leq n-2$, i.e. $P(x_1^0, ..x_{n-p}^0, 0, \dots, 0)$, $x_1^0 \cdots x_{n-p}^0 \neq 0$. Then for the pairs (x_i^0, x_j^0), $1 \leq i < j \leq n-p$ only case (iii) is valid, from which Lemma 2.2 follows. Lemma 2.2 is true also for the maximum value of f, since $\max f = \min(-f)$. ∎

Remark 2.3. *A result similar to Lemma 2.2 is proved by Sato in [4].*

Theorem 2.3. *Let n, k be integer numbers, $1 \leq k \leq n-1$. Then for arbitrary non-negative $x_1, \dots, x_n$, the following inequality is true:*

$$(2.8)\quad \sigma_1^k \sigma_{n-k} \geq \sum_{i=1}^{k} (-1)^{i+1} \binom{n-k-1+i}{i} (n-k+i)^2 (n-k)^{i-2} \sigma_1^{k-i} \sigma_{n-k+i}.$$

Proof. Since (2.8) is homogenous we may assume that $x_1 + \cdots + x_n = \sigma_1 = 1$. Then, according to Lemma 2.2 it suffices to prove, that $f(P_{m,n}) \geq 0$ for $1 \leq m \leq n$, where

$$f(x_1, \dots, x_n) = \sigma_{n-k} + \sum_{i=1}^{k} \binom{n-k-1+i}{i} (n-k+i)^2 (k-n)^{i-2} \sigma_{n-k+i}.$$

At the $P_{m,n}$ point we have $\sigma_{n-k+i} = \binom{m}{n-k+i} \frac{1}{m^{n-k+i}}$, hence

$$(2.9)\qquad \sigma_{n-k+i} \neq 0 \quad \text{if and only if} \quad i \leq m-n+k.$$

We consider the following three possible cases for m:

(i) $m \leq n-k-1$, $k \leq n-2$. Then obviously $\sigma_{n-k} = \sigma_{n-k+1} = \cdots = \sigma_n = 0$, hence $f(P_{m,n}) = 0$.

(ii) $m = n-k$, $k \leq n-1$. From (2.9) we obtain $\sigma_{n-k} = \frac{1}{(n-k)^{n-k}}$ and $\sigma_{n-k+1} = \cdots = \sigma_n = 0$, hence $f(P_{m,n}) = \frac{1}{(n-k)^{n-k}} > 0$.

(iii) $m = n-k+p$, $1 \leq p \leq k$, $k \leq n-1$. From (2.9) and $m = n-k+p$ we obtain

$$\begin{aligned} f(P_{m,n}) &= \binom{n-k+p}{n-k} \frac{1}{(n-k+p)^{n-k}} + \sum_{i=1}^{k} \binom{n-k-1+i}{i} \\ &\quad \times (n-k+i)^2 (k-n)^{i-2} \binom{n-k+p}{n-k+i} \frac{1}{(n-k+p)^{n-k+i}} \\ &= \binom{m}{p} \frac{1}{m^{m-p}} + \sum_{i=1}^{p} \binom{m-p-1+i}{i} \\ &\quad \times (m-p+i)^2 (p-m)^{i-2} \binom{m}{m-p+i} \frac{1}{m^{m-p+i}}. \end{aligned}$$

Now from equality

$$\binom{m-p-1+i}{i} \binom{m}{m-p+i} (m-p+i) = \binom{m-1}{p} \binom{p}{i} m$$

we obtain

$$f(P_{m,n}) = \binom{m}{p}\frac{1}{m^{m-p}} + \sum_{i=1}^{p}\binom{m-1}{p}\binom{p}{i}(m-p+i)^2(p-m)^{i-2}\frac{1}{m^{m-p-1+i}}.$$

This implies

$$\begin{aligned}
&\frac{m^{m-p+1}}{\binom{m-1}{p}}f(P_{m,n})\\
&= \frac{m^2}{m-p} + p(m-p+1)\frac{m}{p-m} + \sum_{i=2}^{p}\binom{p}{i}(m-p+i)\left(\frac{p-m}{m}\right)^{i-2}\\
&= m(1-p) + \sum_{i=2}^{p}\binom{p}{i}(m-p)\left(\frac{p-m}{m}\right)^{i-2}\\
&\qquad + \sum_{i=2}^{p}\binom{p}{i}i\left(\frac{p-m}{m}\right)^{i-2}\\
&= m(1-p) + \frac{m^2}{m-p}\left[\left(1+\frac{p-m}{m}\right)^{p} - \frac{p(p-m)}{m} - 1\right]\\
&\qquad + \frac{mp}{p-m}\sum_{i=2}^{p}\binom{p-1}{i-1}\left(\frac{p-m}{m}\right)^{i-1}.
\end{aligned}$$

Substituting $i = j + 1$ we obtain:

$$\begin{aligned}
&\frac{m^{m-p+1}}{\binom{m-1}{p}} f(P_{m,n}) \\
&\quad = m(1-p) + \frac{m^2}{m-p}\left[\left(\frac{p}{m}\right)^p + \frac{p(m-p)}{m} - 1\right] \\
&\qquad + \frac{mp}{p-m}\sum_{j=1}^{p-1}\binom{p-1}{j}\left(\frac{p-m}{m}\right)^j \\
&\quad = m(1-p) + \frac{m^2}{m-p}\left(\frac{p}{m}\right)^p + mp - \frac{m^2}{m-p} \\
&\qquad + \frac{mp}{p-m}\left[\left(1+\frac{p-m}{m}\right)^{p-1} - 1\right] \\
&\quad = m + \frac{m^2}{m-p}\left(\frac{p}{m}\right)^p - \frac{m^2}{m-p} + \frac{mp}{p-m}\left(\frac{p}{m}\right)^{p-1} - \frac{mp}{p-m} = 0.
\end{aligned}$$

From (i) – (iii) it follows that Theorem 2.3 is true.

∎

Remark 2.4. *Theorem 2.3 for $k = 1$ is equivalent to Proposition 2.1 in the case when $p = n - 1$.*

Remark 2.5. *It is easy to verify, that (2.8) is equivalent to*

$$E_1^k E_{n-k} \geq \frac{1}{n}\sum_{i=1}^{k}\binom{k}{i}(n-k+i)\left(\frac{k-n}{n}\right)^{i-1} E_1^{k-i} E_{n-k+i}.$$

We define the sequence of real numbers $\{\alpha_{m,l}\}$, $m \in \mathbb{N}, l \in \mathbb{N}$ as follows:

$$\alpha_{1,l} = \frac{1}{l^l} \qquad \text{for} \qquad \forall l \in \mathbb{N}, \tag{2.10}$$

$$\alpha_{m,l} = 0 \qquad \text{for} \qquad m \geq l \geq 2 \text{ or } \quad m > 1, l = 1, \tag{2.11}$$

$$\binom{l}{m} l^m = l^l \alpha_{1,l-m} + \sum_{j=1}^{m} \binom{l}{m-j} l^{m-j} \alpha_{1+j,l-m+j} \quad \text{for } 1 \leq m \leq l-1. \tag{2.12}$$

More precisely, the numbers $\alpha_{m,l}$ can be defined recurrently (excluding the cases when: $m > 1$, $l = 1$ or $m \geq l \geq 2$) as follows:

1) We get $\alpha_{1,l}$ for $l \geq 1$ from (2.10).
2) Then we determine $\alpha_{2,l}$ for $l \geq 3$ from $\binom{l}{1} l = l^l \alpha_{1,l-1} + \alpha_{2,l}$.
3) Then we determine $\alpha_{3,l}$ for $l \geq 4$ from $\binom{l}{2} l^2 = l^l \alpha_{1,l-2} + \binom{l}{1} l \alpha_{2,l-1} + \alpha_{3,l}$.
4) Then we determine $\alpha_{4,l}$ for $l \geq 5$ from $\binom{l}{3} l^3 = l^l \alpha_{1,l-3} + \binom{l}{2} l^2 \alpha_{2,l-2} + \binom{l}{1} l \alpha_{3,l-1} + \alpha_{4,l}$ and so on.

For example, the values of $\alpha_{m,l}$ for $m \leq 5$, $l \leq 6$ are given in Table 1.

The sequence $\{\alpha_{m,l}\}$ has interesting properties. For example one can prove, that in the case when $\alpha_{m,l} \neq 0$: $\operatorname{sgn} \alpha_{m,l} = 1$ for m even and $\operatorname{sgn} \alpha_{m,l} = -1$ for m odd, $m \geq 3$.

We are going to prove the following property of the sequence $\{\alpha_{m,l}\}$:

Proposition 2.4. *For each integer number n, $n \geq 2$ we have:*

$$\alpha_{n,n+1} = (-1)^n \left(\frac{n+1}{2}\right)^2. \tag{2.13}$$

Proof. We will prove (2.12) by induction.

(i) We show, that $\alpha_{2,3} = (-1)^2 \left(\frac{2+1}{2}\right)^2$, (see Table 1).

(ii) Let (2.13) hold true for $\alpha_{2,3}, \dots, \alpha_{n-1,n}$.

(iii) Using (2.12) for $l = n+1$ and $m = n-1$, (2.10) for $l = 2$ and (ii) we obtain

$$\begin{aligned} &\binom{n+1}{2}(n+1)^{n-1} \\ &= \frac{(n+1)^{n+1}}{4} + \sum_{j=1}^{n-2} \binom{n+1}{j+2} (-1)^{j+1} \left(\frac{j+2}{2}\right)^2 (n+1)^{n-1-j} + \alpha_{n,n+1}. \end{aligned}$$

Substituting $j = i-1$, this implies

$$\begin{aligned} \alpha_{n,n+1} &= \binom{n+1}{2}(n+1)^{n-1} \\ &\quad - \frac{(n+1)^{n+1}}{4} - \frac{1}{4}\sum_{i=2}^{n-1} \binom{n+1}{i+1} (-1)^i (i+1)^2 (n+1)^{n-i}. \end{aligned}$$

Now from the equalities $\binom{n+1}{i+1}(i+1) = \binom{n}{i}(n+1)$ and $\binom{n}{i} i = \binom{n-1}{i-1} n$ we obtain:

$$\begin{aligned}
\alpha_{n,n+1} &= \binom{n+1}{2}(n+1)^{n-1} - \frac{(n+1)^{n-1}}{4} \\
&\quad - \frac{n+1}{4}\sum_{i=2}^{n-1}\binom{n}{i}(-1)^i(i+1)(n+1)^{n-i} \\
&= \frac{(n+1)^{n+1}}{4}\left[\frac{2n}{n+1} - 1 - \sum_{i=2}^{n-1}\binom{n}{i}(i+1)\left(\frac{-1}{n+1}\right)^i\right] \\
&= \frac{(n+1)^{n+1}}{4}\left[\frac{n-1}{n+1} - \sum_{i=2}^{n-1}\binom{n}{i}\left(\frac{-1}{n+1}\right)^i\right. \\
&\quad \left. -n\sum_{i=2}^{n-1}\binom{n-1}{i-1}\left(\frac{-1}{n+1}\right)^i\right].
\end{aligned}$$

Substituting $i = k+1$ we obtain

$$\begin{aligned}
\alpha_{n,n+1} &= \frac{(n+1)^{n+1}}{4}\left[\frac{n-1}{n+1} - \left(1+\frac{-1}{n+1}\right)^n + 1 + n\left(\frac{-1}{n+1}\right)\right. \\
&\quad \left. + \left(\frac{-1}{n+1}\right)^n - n\sum_{k=1}^{n-2}\binom{n-1}{k}\left(\frac{-1}{n+1}\right)^{k+1}\right] \\
&= \frac{(n+1)^{n+1}}{4}\left\{\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^n + \left(\frac{-1}{n+1}\right)^n\right.
\end{aligned}$$

$$+\frac{n}{n+1}\left[\left(1+\frac{-1}{n+1}\right)^{n-1}-1-\left(\frac{-1}{n+1}\right)^{n-1}\right]\Bigg\}$$

$$=\frac{(n+1)^{n+1}}{4}\left[\frac{n}{n+1}-\left(\frac{n}{n+1}\right)^{n}+\left(\frac{-1}{n+1}\right)^{n}\right.$$

$$\left.+\frac{n}{n+1}\left(\frac{n}{n+1}\right)^{n-1}-\frac{n}{n+1}-\frac{n}{n+1}\left(\frac{-1}{n+1}\right)^{n-1}\right]$$

$$=\frac{(n+1)^{n+1}}{4}(-1)^{n}\left[\frac{1}{(n+1)^{n}}+\frac{n}{(n+1)^{n}}\right]=(-1)^{n}\left(\frac{n+1}{2}\right)^{2}.$$

From (i), (ii) and (iii) it follows that (2.13) is true for each $n \geq 2$.

∎

Theorem 2.5. *Let n and k be fixed integer numbers for which $1 \leq k \leq n-2$. Then for arbitrary non-negative $x_1, \dots, x_n$, the following inequality is fulfilled:*

$$\sigma_1^k \sigma_{n-k} \leq \alpha_{1,n-k}\sigma_1^n + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\sigma_1^{k-i}\sigma_{n-k+i}, \tag{2.14}$$

where $\{\alpha_{m,l}\}$ are defined from (2.10)-(2.12).

Proof. (2.14) is homogenous, therefore we may assume, that $x_1 + \cdots + x_n = \sigma_1 = 1$. Then according to Lemma 2.2 it is sufficient to prove, that

$$f(P_{m,n}) \geq 0, \qquad \text{for each } m,\ 1 \leq m \leq n, \tag{2.15}$$

where

$$f(x_1,\dots,x_n) = \alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\sigma_{n-k+i} - \sigma_{n-k}.$$

Obviously at the point $P_{m,n}$ we have $\sigma_q = \binom{m}{q}\frac{1}{m^q}$ for $1 \le q \le n$, hence

(2.16) $$\sigma_q \neq 0 \qquad \text{if and only if} \quad q \le m.$$

We consider the following three possible cases for m:

(i) $m \le n-k-1$. Then from (2.16) and (2.10) we obtain $f(P_{m,n}) = \alpha_{1,n-k} = \frac{1}{(n-k)^{n-k}} > 0$.

(ii) $m = n-k$. Then from (2.16) and (2.10) we obtain $f(P_{n-k,n}) = \alpha_{1,n-k} - \frac{1}{(n-k)^{n-k}} = 0$.

(iii) $m = n-k+p$, where $1 \le p \le k$. From (2.16) it follows

$$\begin{aligned} f(P_{m,n}) &= \alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\binom{n-k+p}{n-k+i}\frac{1}{(n-k+p)^{n-k+i}} \\ &\quad - \binom{n-k+p}{n-k}\frac{1}{(n-k+p)^{n-k}} \\ &= \frac{1}{(n-k+p)^{n-k+p}}\Bigg[(n-k+p)^{n-k+p}\alpha_{1,n-k} \\ &\quad + \sum_{i=1}^{k}\binom{n-k+p}{n-k+i}(n-k+p)^{p-i}\alpha_{1+i,n-k+i} \end{aligned}$$

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$$- \binom{n-k+p}{n-k} (n-k+p)^p \Bigg] .$$

However, $\binom{n-k+p}{n-k+i} \neq 0$ for $i \leq p$, and $\frac{1}{(n-k+p)^{n-k+p}} = \alpha_{1,n-k+p}$ according to (2.10), and we get

$$(2.17) \quad f(P_{m,n}) = \alpha_{1,n-k+p} \Bigg[(n-k+p)^{n-k+p} \alpha_{1,n-k} + \sum_{i=1}^{p} \binom{n-k+p}{p-i} (n-k+p)^{p-i} \alpha_{1+i,n-k+i} - \binom{n-k+p}{p} (n-k+p)^p \Bigg] .$$

Obviously $\alpha_{1,n-k} = \alpha_{1,(n-k+p)-p}$ and $\alpha_{1+i,n-k+i} = \alpha_{1+i,(n-k+p)-p+i}$. Then the right hand side of (2.17) is equal to zero according (2.12) for $l = n-k+p$ and $m = p$.

Therefore $f(P_{m,n}) = 0$ in this case.

It follows from (i), (ii) and (iii) that (2.15) is true, and hence (2.14) is also true. ∎

Remark 2.6. *Theorem 2.5 is true as well for $k = n-1$, since both sides of (2.14) are equal in this case, which follows from (2.11).*

Remark 2.7. *An analogue of Theorem 2.5 for $k = 0$ is the inequality between the arithmetic and geometric means.*

Corollary 2.6. *Let A_n, G_n, H_n be the classical averages of the positive real numbers $x_1, \dots, x_n$ $(n \geq 2)$. Then the following inequality is true:*

$$\left[\frac{nA_n}{(n-1)G_n}\right]^{n-1}\frac{1}{G_n}+\left[n-\left(1+\frac{1}{n-1}\right)^{n-1}\right]\frac{1}{A_n}\geq\frac{n}{H_n}. \tag{2.18}$$

Proof. (2.18) follows from:

$$\sigma_1 = nA_n,\ \sigma_{n-1}=\frac{nG_n^n}{H_n},\ \sigma_n = G_n^n,$$
$$\alpha_{1,n-1}=\frac{1}{(n-1)^{n-1}},\ \alpha_{2,n}=n^2-\frac{n^n}{(n-1)^{n-1}}$$

and from Theorem 2.5 for $k = 1$. ∎

Corollary 2.7 (Explicit expression of Theorem 2.5 for $k = n-2$). *For each integer number n ($n \geq 3$) we have:*

$$\sigma_1^{n-2}\sigma_2 \leq \frac{1}{4}\sigma_1^n + \sum_{i=1}^{n-2}(-1)^{i+1}\left(\frac{i+2}{2}\right)^2\sigma_1^{n-2-i}\sigma_{2+i}.$$

Proof. It follows from Proposition 2.4 and from Theorem 2.5 for $k = n-2$. ∎

Remark 2.8. *Corollary 2.7 is the principle result in [4].*

Remark 2.9. *Corollary 2.7 shows that Theorem 2.5 for $k = n-2$ is equivalent to Theorem 2.3 in the case when $k = n-1$.*

3. The Sharpness of the Inequalities (2.8) and (2.14)

The following two theorems prove that the estimates in Theorem 2.3 and Theorem 2.5 are, in a certain sense, the best possible.

Theorem 3.1. *Let n and k, $1 \le k \le n-1$ be integers. Let the real numbers $\beta_1, \dots, \beta_k$ have the property (3.1). We say that the real numbers $\beta_1, \dots, \beta_k$ have the property (3.1) if for any non-negative real numbers $x_1, \dots, x_n$ with a sum equal to one the following inequality is fulfilled:*

$$\sigma_{n-k} \ge \sum_{i=1}^{k} \beta_i \sigma_{n-k+i}. \tag{3.1}$$

Then for arbitrary non-negative real numbers $x_1, \dots, x_n$ with sum equal to one the following inequality is fulfilled:

$$\sum_{i=1}^{k} \beta_i \sigma_{n-k+i} \le \sum_{i=1}^{k} (-1)^{i+1} \binom{n-k-1+i}{i} (n-k+i)^2 (n-k)^{i-2} \sigma_{n-k+i} \tag{3.2}$$

Proof. Set $f_1 = f_1(x_1, \dots, x_n) = \sigma_{n-k} - \sum_{i=1}^{k} \beta_i \sigma_{n-k+i}$ and

$$f_2 = f_2(x_1, \dots, x_n) = \sigma_{n-k} + \sum_{i=1}^{k} \binom{n-k+i}{i} (n-k+i)^2 (k-n)^{i-2} \sigma_{n-k+i}.$$

Then (3.2) is equivalent to $f_1 - f_2 \geq 0$. On the other hand, according to Lemma 2.2, it is sufficient to verify this inequality at the points $P_{m,n}$. We have at these points:

(i) For $1 \leq m \leq n-k-1$, $k \leq n-2$ apparently $f_1 = f_2 = 0$, hence $f_1 - f_2 = 0$.

(ii) For $m = n-k$, $k \leq n-1$ we obtain $f_1 = f_2 = \frac{1}{(n-k)^{n-k}}$, hence $f_1 - f_2 = 0$.

(iii) For $1 \leq n-k < m \leq n$ from the proof of Theorem 2.3 it follows, that $f_2 = 0$. As $f_1 \geq 0$ according to (3.1), hence $f_1 - f_2 \geq 0$.

From (i), (ii) and (iii) it follows that $f_1 - f_2 \geq 0$ in each point $P_{m,n}$ and we complete the proof of the theorem. ∎

Theorem 3.2. *Let n and k be integers, $1 \leq k \leq n-2$. Let the real numbers $\gamma_1, \dots, \gamma_{k+1}$ have the property (3.3). We say that the real numbers $\gamma_1, \dots, \gamma_{k+1}$ have the property (3.3) if for any non-negative real numbers $x_1, \dots, x_n$ with sum equal to one, the following inequality is fulfilled:*

$$\sigma_{n-k} \leq \gamma_1 + \sum_{i=1}^{k} \gamma_{i+1}\sigma_{n-k+i}. \tag{3.3}$$

Then for any non-negative real numbers $x_1, \dots, x_n$ with sum equal to one the following inequality is fulfilled:

$$\alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\sigma_{n-k+i} \leq \gamma_1 + \sum_{i=1}^{k} \gamma_{1+i}\sigma_{n-k+i}. \tag{3.4}$$

Proof. Set

$$f_1 = f_1(x_1, \dots, x_n) = \gamma_1 + \sum_{i=1}^{k} \gamma_{1+i}\sigma_{n-k+i} - \sigma_{n-k}$$

and

$$f_2 = f_2(x_1, \dots, x_n) = \alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\sigma_{n-k+i} - \sigma_{n-k}.$$

Then (3.4) is equivalent to $f_1 - f_2 \geq 0$. We are going to check this inequality at the points $P_{m,n}$. From (3.3) at $P_{n-k,n}$ it follows, that

$$\gamma_1 \geq \frac{1}{(n-k)^{n-k}} = \alpha_{1,n-k}. \tag{3.5}$$

We consider the possible cases for m:

(i) $1 \leq m \leq n-k-1$. Then $f_1 - f_2 = \gamma_1 - \alpha_{1,n-k} \geq 0$ at $P_{m,n}$ according to (3.5).

(ii) $n-k \leq m \leq n$. Then $f_1 \geq 0$ at $P_{m,n}$ according to (3.3) and from the proof of Theorem 2.5 it follows that $f_2 = 0$, therefore $f_1 - f_2 \geq 0$.

From (i) and (ii) we obtain, that $f_1 - f_2 \geq 0$ in each point $P_{m,n}$ $(1 \leq m \leq n)$. Applying Lemma 2.2 we complete the proof of Theorem 3.2. ∎

Table 1:

l	$\alpha_{1,l}$	$\alpha_{2,l}$	$\alpha_{3,l}$	$\alpha_{4,l}$	$\alpha_{5,l}$
1	1	0	0	0	0
2	1/4	0	0	0	0
3	1/27	9/4	0	0	0
4	1/256	176/27	-4	0	0
5	1/3125	3275/256	-775/27	25/4	0
6	1/46656	65844/3125	-6579/64	316/3	- 9

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