

# Journal of Inequalities in Pure and Applied Mathematics

## NEW INEQUALITIES BETWEEN ELEMENTARY SYMMETRIC POLYNOMIALS

TODOR P. MITEV

University of Rousse  
Department of Mathematics  
Rousse 7017, Bulgaria.  
E-Mail: [mitev@ami.ru.acad.bg](mailto:mitev@ami.ru.acad.bg)

©2000 Victoria University  
ISSN (electronic): 1443-5756  
116-02



---

volume 4, issue 2, article 48,  
2003.

*Received 7 November, 2002;  
accepted 21 February, 2003.*

*Communicated by: C.P. Niculescu*

---

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

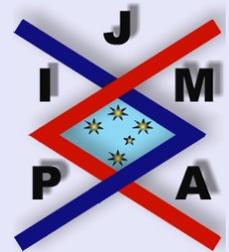
New families of sharp inequalities between elementary symmetric polynomials are proven. We estimate  $\sigma_{n-k}$  above and below by the elementary symmetric polynomials  $\sigma_{n-k+1}, \dots, \sigma_n$  in the case, when  $x_1, \dots, x_n$  are non-negative real numbers with sum equal to one.

*2000 Mathematics Subject Classification:* 26D05.

*Key words:* Elementary symmetric polynomials.

## Contents

1	Introduction .....	3
2	New Inequalities (Theorem 2.3 and Theorem 2.5) .....	5
3	The Sharpness of the Inequalities (2.8) and (2.14) .....	21



---

### New Inequalities Between Elementary Symmetric Polynomials

Todor P. Mitev

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 25

# 1. Introduction

Let  $n \geq 2$  be an integer. As usual, we denote by  $\sigma_0, \sigma_1, \dots, \sigma_n$  the elementary symmetric polynomials of the variables  $x_1, \dots, x_n$ .

In other words,  $\sigma_0 = \sigma_0(x_1, \dots, x_n) = 1$  and for  $1 \leq k \leq n$

$$\sigma_k = \sigma_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \dots x_{i_k}.$$

The different  $\sigma_0, \sigma_1, \dots, \sigma_n$ , are not comparable between them, but they are connected by nonlinear inequalities. To state them, it is more convenient to consider their averages  $E_k = \sigma_k / \binom{n}{k}$ ,  $k = 0, 1, \dots, n$ .

There are three basic types of inequalities between the symmetric functions with respect to the range of the variables  $x_1, \dots, x_n$ .

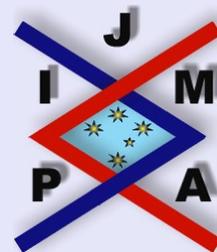
For arbitrary real  $x_1, \dots, x_n$  the following inequalities are known:

$$(1.1) \quad E_k^2 \geq E_{k-1}E_{k+1}, \quad 1 \leq k \leq n-1, \quad (\text{Newton-Maclaurin}),$$

$$4(E_{k+1}E_{k+3} - E_{k+2}^2)(E_kE_{k+2} - E_{k+1}^2) \geq (E_{k+1}E_{k+2} - E_kE_{k+3})^2, \\ k = 0, \dots, n-3, \quad (\text{Rosset [4]}),$$

as well as the inequalities of Niculescu [2]. A complete description about their historical and contemporary stage of development can be found, for example, in [1] and [2].

Suppose now that all  $x_j$ ,  $j = 1, \dots, n$ , are positive. Then the following general result (see [1, Theorem 77, p. 64]) is known:



**Theorem 1.1 (Hardy, Littlewood, Pólya).** For any positive  $x_1, \dots, x_n$  and positive  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$  the inequality

$$E_1^{\alpha_1} \dots E_n^{\alpha_n} \leq E_1^{\beta_1} \dots E_n^{\beta_n}$$

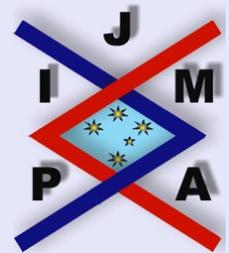
holds if and only if

$$\alpha_m + 2\alpha_{m+1} + \dots + (n - m + 1)\alpha_n \geq \beta_m + 2\beta_{m+1} + \dots + (n - m + 1)\beta_n$$

for each  $1 \leq m \leq n$ .

For other results in this direction see [1].

The aim of this paper is to obtain new inequalities between  $\sigma_1, \dots, \sigma_n$  in the case when  $x_1, \dots, x_n$  are non-negative, (Theorem 2.3 and Theorem 2.5 below). More precisely, we obtain the best possible estimates of  $\sigma_1^k \sigma_{n-k}$  from below and above by linear functions of  $\sigma_1^{k-1} \sigma_{n-k+1}, \dots, \sigma_1^0 \sigma_n$ . Since all these functions are homogeneous with respect to  $(x_1, \dots, x_n)$  of the same order, we can set  $\sigma_1 = x_1 + \dots + x_n = 1$ , then our inequalities give the best possible estimates of  $\sigma_{n-k}$  by linear functions of  $\sigma_{n-k+1}, \dots, \sigma_n$  for  $k = 1, \dots, n - 1$  in this case (Theorem 3.1 and Theorem 3.2 below). Inequalities of this type for  $k = n - 2$  have been recently obtained by Sato [4], which can be obtained as a consequence of Theorem 2.5 below.



**New Inequalities Between Elementary Symmetric Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 4 of 25

## 2. New Inequalities (Theorem 2.3 and Theorem 2.5)

For the sake of completeness we give a straightforward proof of the following proposition, which is a consequence of Theorem 1.1, cited in the introduction. Here we suppose that  $x_1, \dots, x_n$  are non-negative.

**Proposition 2.1.** *Let  $x_1, \dots, x_n$  be non-negative real numbers,  $n \geq 2$ . Then for  $1 \leq p \leq n - 1$  we have*

$$(2.1) \quad \sigma_1 \sigma_p \geq \frac{n(p+1)}{n-p} \sigma_{p+1}.$$

*Proof.* Denote  $\sigma_{l,n} = \sum_{1 \leq i_1 < \dots < i_l \leq n} x_{i_1} x_{i_2} \cdots x_{i_l}$ ,  $1 \leq l \leq n$ . Note, that (2.1) is equivalent to

$$(2.2) \quad \sigma_{1,n} \sigma_{p,n} \geq \frac{n(p+1)}{n-p} \sigma_{p+1,n}.$$

First we shall check (2.2) for  $p = 1$  and for  $p = n - 1$ .

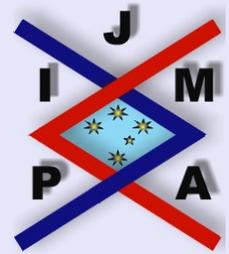
(i) For  $p = 1$  the inequality (2.2) reads

$$\left( \sum_{i=1}^n x_i \right)^2 \geq \frac{2n}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j,$$

which is equivalent to

$$(n-1) \left( \sum_{i=1}^n x_i \right)^2 \geq \frac{2n}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j,$$

hence to  $\sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq 0$ .



(ii) For  $p = n - 1$  (2.2) is equivalent to  $\sigma_{1,n}\sigma_{n-1,n} \geq n^2\sigma_{n,n}$ . If  $\sigma_{n,n} = 0$ , then (2.2) is obvious. Let  $\sigma_{n,n} \neq 0$ , then (2.2) is equivalent to  $n^2 \leq \left(\sum_{i=1}^n x_i\right) \left(\sum_{i=1}^n \frac{1}{x_i}\right)$ , which follow from AM-GM inequality.

We are going to prove (2.2) by recurrence with respect to  $n \geq 2$ .

(iii) We already proved that (2.2) is true for  $n = 2$ .

(iv) Let (2.2) be true for  $n \geq 2$  and for each  $p$ ,  $1 \leq p \leq n - 1$ . Fix  $p$ ,  $2 \leq p \leq n - 1$ . We will prove, that

$$(2.3) \quad \sigma_{1,n+1}\sigma_{p,n+1} \geq \frac{(n+1)(p+1)}{n+1-p}\sigma_{p+1,n+1}.$$

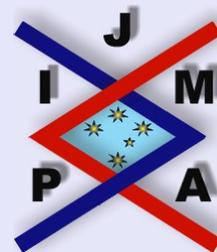
Since (2.3) is homogeneous, excluding the case  $x_1 = \dots = x_n = x_{n+1} = 0$ , we may assume, that  $\sigma_{1,n+1} = 1$ .

Let  $x_1 \leq x_2 \leq \dots \leq x_{n+1}$ . The following cases are possible:

- 1) Let  $x_{n+1} = 1$ . Then  $x_1 = \dots = x_n = 0$  and (2.3) becomes an equality.
- 2) Let  $x_{n+1} = \frac{1}{n+1}$ . Then  $x_1 = \dots = x_n = x_{n+1} = \frac{1}{n+1}$  and we obtain

$$\begin{aligned} & \sigma_{p,n+1} - \frac{(n+1)(p+1)}{n+1-p}\sigma_{p+1,n+1} \\ &= \binom{n+1}{p} \frac{1}{(n+1)^p} - \frac{p+1}{n+1-p} \binom{n+1}{p+1} \frac{1}{(n+1)^p} = 0, \end{aligned}$$

hence (2.3) becomes again an equality.



## New Inequalities Between Elementary Symmetric Polynomials

Todor P. Mitev

Title Page

Contents

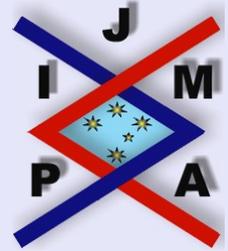


Go Back

Close

Quit

Page 6 of 25



Title Page

Contents



Go Back

Close

Quit

Page 7 of 25

3) Let  $x_{n+1} \in (\frac{1}{n+1}; 1)$ . Substitute  $x_1 + \dots + x_n = 1 - x_{n+1} = \sigma_{1,n} = s$ , with  $s \in (0; \frac{n}{n+1})$ . Then  $\sigma_{p,n+1} = \sigma_{p,n} + (1-s)\sigma_{p-1,n}$  and  $\sigma_{p+1,n+1} = \sigma_{p+1,n} + (1-s)\sigma_{p,n}$ . Hence (2.3) is equivalent to

$$\sigma_{p,n} + (1-s)\sigma_{p-1,n} \geq \frac{(n+1)(p+1)}{n+1-p} [\sigma_{p+1,n} + (1-s)\sigma_{p,n}],$$

which is equivalent to

$$(2.4) \quad \left[ \frac{n+1-p}{(n+1)(p+1)} - (1-s) \right] \sigma_{p,n} + \frac{(1-s)(n+1-p)}{(n+1)(p+1)} \sigma_{p-1,n} \geq \sigma_{p+1,n}.$$

From (iv) we obtain  $\sigma_{p+1,n} \leq \frac{n-p}{n(p+1)} s \sigma_{p,n}$ . Then (2.4) follows from the next inequality (if true):

$$(2.5) \quad \frac{n-p}{n(p+1)} s \sigma_{p,n} \leq \left[ \frac{n+1-p}{(n+1)(p+1)} - (1-s) \right] \sigma_{p,n} + \frac{(1-s)(n+1-p)}{(n+1)(p+1)} \sigma_{p-1,n},$$

which is equivalent to

$$(2.6) \quad \sigma_{p-1,n} \geq \frac{p[n(n+2) - (n+1)^2 s]}{n(n+1-p)(1-s)} \sigma_{p,n}.$$

It follows from (iv) that  $\sigma_{p-1,n} \geq \frac{np}{(n+1-p)s} \sigma_{p,n}$ .

Hence (2.6), and consequently (2.5) and (2.4), follow from

$$\begin{aligned} \frac{np}{(n+1-p)s} \sigma_{p,n} - \frac{p[n(n+2) - (n+1)^2s]}{n(n+1-p)(1-s)} \sigma_{p,n} \\ = \frac{p[(n+1)s - n]^2}{ns(n+1-p)(1-s)} \sigma_{p,n} \geq 0. \end{aligned}$$

Since (2.2) is true for  $p = 1$  and  $p = n$  according to (i) and (ii), then (2.2) is fulfilled for  $n, n \geq 2$ . Hence the proposition is proved. ■

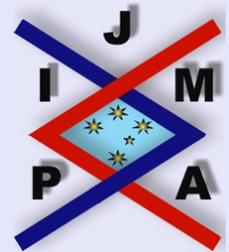
**Remark 2.1.** *It follows from the proof, that equality is achieved in the following two cases:*

- 1)  $x_1 = x_2 = \dots = x_n = a \geq 0$ .
- 2)  $n - p + 1$  of  $x_1, \dots, x_n$  are equal to 0 and the rest of them are arbitrary non-negative real numbers.

**Remark 2.2.** (2.1) can be proven using Lemma 2.2 below, but in this way it will be difficult to see when (2.1) turns into an equality.

From now on  $n$  will be a fixed positive integer. It will be assumed that at least one of the non-negative numbers  $x_1, \dots, x_n$  differs from zero.

**Lemma 2.2.** *Let us assume that  $x_1, \dots, x_n$  are non-negative real numbers ( $n \geq 2$ ) and  $x_1 + \dots + x_n = \sigma_1 = 1$ . Then the function  $f(x_1, \dots, x_n) = a_1 + a_2\sigma_2 + \dots + a_n\sigma_n$  ( $a_1, \dots, a_n$  are real numbers), achieves its maximum and minimum at least in some of the points  $P_{k,n}(\frac{1}{k}, \dots, \frac{1}{k}, 0, \dots, 0)$ ,  $1 \leq k \leq n$  (the first  $k$  coordinates of  $P_{k,n}$  are equal to  $\frac{1}{k}$ , and the rest of them are equal to zero).*



*Proof.* The set  $A_n = \{(x_1, \dots, x_n) / x_i \geq 0, x_1 + \dots + x_n = 1\}$  is compact and  $f$  is continuous in it, hence  $f$  achieves its minimum and maximum values. We rewrite  $f$  as follows:

$$f(x_1, \dots, x_n) = x_1 x_2 g(x_3, \dots, x_n) + x_1 h_1(x_3, \dots, x_n) + x_2 h_2(x_3, \dots, x_n) + t(x_3, \dots, x_n) + a_1.$$

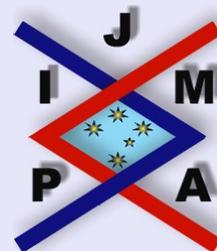
As  $f$  is symmetric, then  $h_1 \equiv h_2$  and therefore:

$$(2.7) \quad f(x_1, \dots, x_n) = x_1 x_2 g(x_3, \dots, x_n) + (x_1 + x_2) h_1(x_3, \dots, x_n) + t(x_3, \dots, x_n) + a_1.$$

Let  $P(x_1^0, \dots, x_n^0)$  be a point in which  $f$  achieves its minimum value. We consider the function  $F(x) = f(x, s - x, x_3^0, \dots, x_n^0)$ ,  $s = x_1^0 + x_2^0$ , for  $x \in [0; s]$  (we assume, that  $s > 0$ ). Obviously the minimum values of  $F$  and  $f$  are equal and  $F$  achieves its minimum value for  $x = x_1^0$ . From (2.7) we obtain that  $F(x) = \alpha x(s - x) + s\beta + \gamma = \alpha x(s - x) + \delta$ , where  $\alpha, \delta$  depend on  $x_1^0, x_2^0, x_3^0, \dots, x_n^0, a_1, \dots, a_n$ .

The following three cases are possible:

- (i)  $\alpha = 0$ . Then  $F(x) = \text{const}$  and we may assume that  $\min F = F(0)$  or  $\min F = F\left(\frac{s}{2}\right)$ .
- (ii)  $\alpha > 0$ . Then  $\min F = F(0)$ .
- (iii)  $\alpha < 0$ . Then  $\min F = F\left(\frac{s}{2}\right)$ .



**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 9 of 25

Hence, as  $x_1^0$  and  $x_2^0$  were arbitrarily chosen then, for  $\forall i \neq j$  we may assume that  $x_i^0 = x_j^0$  or, at least one of them is equal to zero.

Let us choose a point  $P(x_1^0, \dots, x_n^0)$ , for which the number of coordinates  $p$  which equal to zero is the highest possible and  $x_1^0 \geq x_2^0 \geq \dots \geq x_n^0$ . If  $p = n - 1$ , then Lemma 2.2 is proven. Let  $0 \leq p \leq n - 2$ , i.e.  $P(x_1^0, \dots, x_{n-p}^0, 0, \dots, 0)$ ,  $x_1^0 \cdots x_{n-p}^0 \neq 0$ . Then for the pairs  $(x_i^0, x_j^0)$ ,  $1 \leq i < j \leq n - p$  only case (iii) is valid, from which Lemma 2.2 follows. Lemma 2.2 is true also for the maximum value of  $f$ , since  $\max f = \min(-f)$ . ■

**Remark 2.3.** A result similar to Lemma 2.2 is proved by Sato in [4].

**Theorem 2.3.** Let  $n, k$  be integer numbers,  $1 \leq k \leq n - 1$ . Then for arbitrary non-negative  $x_1, \dots, x_n$ , the following inequality is true:

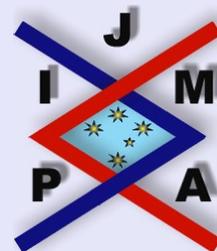
$$(2.8) \quad \sigma_1^k \sigma_{n-k} \geq \sum_{i=1}^k (-1)^{i+1} \binom{n-k-1+i}{i} (n-k+i)^2 (n-k)^{i-2} \sigma_1^{k-i} \sigma_{n-k+i}.$$

*Proof.* Since (2.8) is homogenous we may assume that  $x_1 + \dots + x_n = \sigma_1 = 1$ . Then, according to Lemma 2.2 it suffices to prove, that  $f(P_{m,n}) \geq 0$  for  $1 \leq m \leq n$ , where

$$f(x_1, \dots, x_n) = \sigma_{n-k} + \sum_{i=1}^k \binom{n-k-1+i}{i} (n-k+i)^2 (k-n)^{i-2} \sigma_{n-k+i}.$$

At the  $P_{m,n}$  point we have  $\sigma_{n-k+i} = \binom{m}{n-k+i} \frac{1}{m^{n-k+i}}$ , hence

$$(2.9) \quad \sigma_{n-k+i} \neq 0 \quad \text{if and only if} \quad i \leq m - n + k.$$



Title Page

Contents



Go Back

Close

Quit

Page 10 of 25

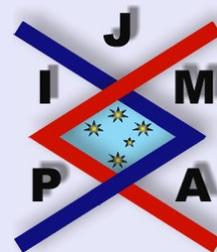
We consider the following three possible cases for  $m$ :

- (i)  $m \leq n - k - 1$ ,  $k \leq n - 2$ . Then obviously  $\sigma_{n-k} = \sigma_{n-k+1} = \dots = \sigma_n = 0$ , hence  $f(P_{m,n}) = 0$ .
- (ii)  $m = n - k$ ,  $k \leq n - 1$ . From (2.9) we obtain  $\sigma_{n-k} = \frac{1}{(n-k)^{n-k}}$  and  $\sigma_{n-k+1} = \dots = \sigma_n = 0$ , hence  $f(P_{m,n}) = \frac{1}{(n-k)^{n-k}} > 0$ .
- (iii)  $m = n - k + p$ ,  $1 \leq p \leq k$ ,  $k \leq n - 1$ . From (2.9) and  $m = n - k + p$  we obtain

$$\begin{aligned}
 f(P_{m,n}) &= \binom{n-k+p}{n-k} \frac{1}{(n-k+p)^{n-k}} + \sum_{i=1}^k \binom{n-k-1+i}{i} \\
 &\quad \times (n-k+i)^2 (k-n)^{i-2} \binom{n-k+p}{n-k+i} \frac{1}{(n-k+p)^{n-k+i}} \\
 &= \binom{m}{p} \frac{1}{m^{m-p}} + \sum_{i=1}^p \binom{m-p-1+i}{i} \\
 &\quad \times (m-p+i)^2 (p-m)^{i-2} \binom{m}{m-p+i} \frac{1}{m^{m-p+i}}.
 \end{aligned}$$

Now from equality

$$\binom{m-p-1+i}{i} \binom{m}{m-p+i} (m-p+i) = \binom{m-1}{p} \binom{p}{i} m$$



**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

Title Page

Contents

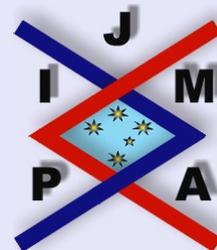


Go Back

Close

Quit

Page 11 of 25



**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

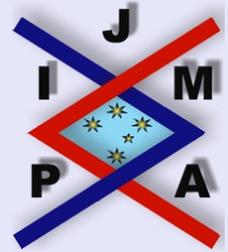
Page 12 of 25

we obtain

$$f(P_{m,n}) = \binom{m}{p} \frac{1}{m^{m-p}} + \sum_{i=1}^p \binom{m-1}{p} \binom{p}{i} (m-p+i)^2 (p-m)^{i-2} \frac{1}{m^{m-p-1+i}}.$$

This implies

$$\begin{aligned} & \frac{m^{m-p+1}}{\binom{m-1}{p}} f(P_{m,n}) \\ &= \frac{m^2}{m-p} + p(m-p+1) \frac{m}{p-m} + \sum_{i=2}^p \binom{p}{i} (m-p+i) \left(\frac{p-m}{m}\right)^{i-2} \\ &= m(1-p) + \sum_{i=2}^p \binom{p}{i} (m-p) \left(\frac{p-m}{m}\right)^{i-2} \\ & \quad + \sum_{i=2}^p \binom{p}{i} i \left(\frac{p-m}{m}\right)^{i-2} \\ &= m(1-p) + \frac{m^2}{m-p} \left[ \left(1 + \frac{p-m}{m}\right)^p - \frac{p(p-m)}{m} - 1 \right] \\ & \quad + \frac{mp}{p-m} \sum_{i=2}^p \binom{p-1}{i-1} \left(\frac{p-m}{m}\right)^{i-1}. \end{aligned}$$



Title Page

Contents



Go Back

Close

Quit

Page 13 of 25

Substituting  $i = j + 1$  we obtain:

$$\begin{aligned} & \frac{m^{m-p+1}}{\binom{m-1}{p}} f(P_{m,n}) \\ &= m(1-p) + \frac{m^2}{m-p} \left[ \left(\frac{p}{m}\right)^p + \frac{p(m-p)}{m} - 1 \right] \\ & \quad + \frac{mp}{p-m} \sum_{j=1}^{p-1} \binom{p-1}{j} \left(\frac{p-m}{m}\right)^j \\ &= m(1-p) + \frac{m^2}{m-p} \left(\frac{p}{m}\right)^p + mp - \frac{m^2}{m-p} \\ & \quad + \frac{mp}{p-m} \left[ \left(1 + \frac{p-m}{m}\right)^{p-1} - 1 \right] \\ &= m + \frac{m^2}{m-p} \left(\frac{p}{m}\right)^p - \frac{m^2}{m-p} + \frac{mp}{p-m} \left(\frac{p}{m}\right)^{p-1} - \frac{mp}{p-m} = 0. \end{aligned}$$

From (i) – (iii) it follows that Theorem 2.3 is true.

**Remark 2.4.** Theorem 2.3 for  $k = 1$  is equivalent to Proposition 2.1 in the case when  $p = n - 1$ .

**Remark 2.5.** It is easy to verify, that (2.8) is equivalent to

$$E_1^k E_{n-k} \geq \frac{1}{n} \sum_{i=1}^k \binom{k}{i} (n-k+i) \left(\frac{k-n}{n}\right)^{i-1} E_1^{k-i} E_{n-k+i}.$$

We define the sequence of real numbers  $\{\alpha_{m,l}\}$ ,  $m \in \mathbb{N}, l \in \mathbb{N}$  as follows:

$$(2.10) \quad \alpha_{1,l} = \frac{1}{l^l} \quad \text{for} \quad \forall l \in \mathbb{N},$$

$$(2.11) \quad \alpha_{m,l} = 0 \quad \text{for} \quad m \geq l \geq 2 \text{ or } m > 1, l = 1,$$

$$(2.12) \quad \binom{l}{m} l^m = l^l \alpha_{1,l-m} + \sum_{j=1}^m \binom{l}{m-j} l^{m-j} \alpha_{1+j,l-m+j}$$

for  $1 \leq m \leq l-1$ .

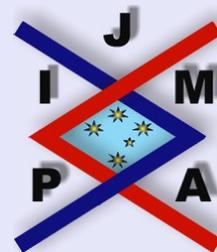
More precisely, the numbers  $\alpha_{m,l}$  can be defined recurrently (excluding the cases when:  $m > 1, l = 1$  or  $m \geq l \geq 2$ ) as follows:

- 1) We get  $\alpha_{1,l}$  for  $l \geq 1$  from (2.10).
- 2) Then we determine  $\alpha_{2,l}$  for  $l \geq 3$  from  $\binom{l}{1} l = l^l \alpha_{1,l-1} + \alpha_{2,l}$ .
- 3) Then we determine  $\alpha_{3,l}$  for  $l \geq 4$  from  $\binom{l}{2} l^2 = l^l \alpha_{1,l-2} + \binom{l}{1} l \alpha_{2,l-1} + \alpha_{3,l}$ .
- 4) Then we determine  $\alpha_{4,l}$  for  $l \geq 5$  from  $\binom{l}{3} l^3 = l^l \alpha_{1,l-3} + \binom{l}{2} l^2 \alpha_{2,l-2} + \binom{l}{1} l \alpha_{3,l-1} + \alpha_{4,l}$  and so on.

For example, the values of  $\alpha_{m,l}$  for  $m \leq 5, l \leq 6$  are given in Table 1.

The sequence  $\{\alpha_{m,l}\}$  has interesting properties. For example one can prove, that in the case when  $\alpha_{m,l} \neq 0$ :  $\text{sgn } \alpha_{m,l} = 1$  for  $m$  even and  $\text{sgn } \alpha_{m,l} = -1$  for  $m$  odd,  $m \geq 3$ .

We are going to prove the following property of the sequence  $\{\alpha_{m,l}\}$ :



**Proposition 2.4.** For each integer number  $n$ ,  $n \geq 2$  we have:

$$(2.13) \quad \alpha_{n,n+1} = (-1)^n \left( \frac{n+1}{2} \right)^2.$$

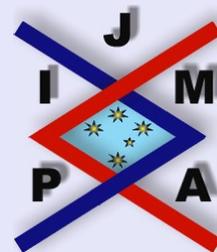
*Proof.* We will prove (2.12) by induction.

- (i) We show, that  $\alpha_{2,3} = (-1)^2 \left( \frac{2+1}{2} \right)^2$ , (see Table 1).
- (ii) Let (2.13) hold true for  $\alpha_{2,3}, \dots, \alpha_{n-1,n}$ .
- (iii) Using (2.12) for  $l = n + 1$  and  $m = n - 1$ , (2.10) for  $l = 2$  and (ii) we obtain

$$\begin{aligned} & \binom{n+1}{2} (n+1)^{n-1} \\ &= \frac{(n+1)^{n+1}}{4} + \sum_{j=1}^{n-2} \binom{n+1}{j+2} (-1)^{j+1} \left( \frac{j+2}{2} \right)^2 (n+1)^{n-1-j} + \alpha_{n,n+1}. \end{aligned}$$

Substituting  $j = i - 1$ , this implies

$$\begin{aligned} \alpha_{n,n+1} &= \binom{n+1}{2} (n+1)^{n-1} \\ &\quad - \frac{(n+1)^{n+1}}{4} - \frac{1}{4} \sum_{i=2}^{n-1} \binom{n+1}{i+1} (-1)^i (i+1)^2 (n+1)^{n-i}. \end{aligned}$$



Title Page

Contents

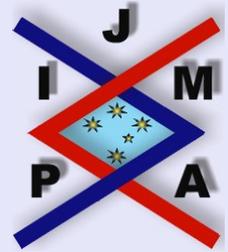


Go Back

Close

Quit

Page 15 of 25



**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 16 of 25

Now from the equalities  $\binom{n+1}{i+1} (i+1) = \binom{n}{i} (n+1)$  and  $\binom{n}{i} i = \binom{n-1}{i-1} n$  we obtain:

$$\begin{aligned} \alpha_{n,n+1} &= \binom{n+1}{2} (n+1)^{n-1} - \frac{(n+1)^{n-1}}{4} \\ &\quad - \frac{n+1}{4} \sum_{i=2}^{n-1} \binom{n}{i} (-1)^i (i+1) (n+1)^{n-i} \\ &= \frac{(n+1)^{n+1}}{4} \left[ \frac{2n}{n+1} - 1 - \sum_{i=2}^{n-1} \binom{n}{i} (i+1) \left(\frac{-1}{n+1}\right)^i \right] \\ &= \frac{(n+1)^{n+1}}{4} \left[ \frac{n-1}{n+1} - \sum_{i=2}^{n-1} \binom{n}{i} \left(\frac{-1}{n+1}\right)^i \right. \\ &\quad \left. - n \sum_{i=2}^{n-1} \binom{n-1}{i-1} \left(\frac{-1}{n+1}\right)^i \right]. \end{aligned}$$

Substituting  $i = k + 1$  we obtain

$$\begin{aligned} \alpha_{n,n+1} &= \frac{(n+1)^{n+1}}{4} \left[ \frac{n-1}{n+1} - \left(1 + \frac{-1}{n+1}\right)^n + 1 + n \left(\frac{-1}{n+1}\right) \right. \\ &\quad \left. + \left(\frac{-1}{n+1}\right)^n - n \sum_{k=1}^{n-2} \binom{n-1}{k} \left(\frac{-1}{n+1}\right)^{k+1} \right] \\ &= \frac{(n+1)^{n+1}}{4} \left\{ \frac{n}{n+1} - \left(\frac{n}{n+1}\right)^n + \left(\frac{-1}{n+1}\right)^n \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{n}{n+1} \left[ \left( 1 + \frac{-1}{n+1} \right)^{n-1} - 1 - \left( \frac{-1}{n+1} \right)^{n-1} \right] \Big\} \\
= & \frac{(n+1)^{n+1}}{4} \left[ \frac{n}{n+1} - \left( \frac{n}{n+1} \right)^n + \left( \frac{-1}{n+1} \right)^n \right. \\
& \left. + \frac{n}{n+1} \left( \frac{n}{n+1} \right)^{n-1} - \frac{n}{n+1} - \frac{n}{n+1} \left( \frac{-1}{n+1} \right)^{n-1} \right] \\
= & \frac{(n+1)^{n+1}}{4} (-1)^n \left[ \frac{1}{(n+1)^n} + \frac{n}{(n+1)^n} \right] = (-1)^n \left( \frac{n+1}{2} \right)^2.
\end{aligned}$$

From (i), (ii) and (iii) it follows that (2.13) is true for each  $n \geq 2$ .

■

**Theorem 2.5.** *Let  $n$  and  $k$  be fixed integer numbers for which  $1 \leq k \leq n - 2$ . Then for arbitrary non-negative  $x_1, \dots, x_n$ , the following inequality is fulfilled:*

$$(2.14) \quad \sigma_1^k \sigma_{n-k} \leq \alpha_{1,n-k} \sigma_1^n + \sum_{i=1}^k \alpha_{1+i,n-k+i} \sigma_1^{k-i} \sigma_{n-k+i},$$

where  $\{\alpha_{m,l}\}$  are defined from (2.10)-(2.12).

*Proof.* (2.14) is homogenous, therefore we may assume, that  $x_1 + \dots + x_n = \sigma_1 = 1$ . Then according to Lemma 2.2 it is sufficient to prove, that

$$(2.15) \quad f(P_{m,n}) \geq 0, \quad \text{for each } m, \quad 1 \leq m \leq n,$$



**New Inequalities Between Elementary Symmetric Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 17 of 25

where

$$f(x_1, \dots, x_n) = \alpha_{1,n-k} + \sum_{i=1}^k \alpha_{1+i,n-k+i} \sigma_{n-k+i} - \sigma_{n-k}.$$

Obviously at the point  $P_{m,n}$  we have  $\sigma_q = \binom{m}{q} \frac{1}{m^q}$  for  $1 \leq q \leq n$ , hence

$$(2.16) \quad \sigma_q \neq 0 \quad \text{if and only if} \quad q \leq m.$$

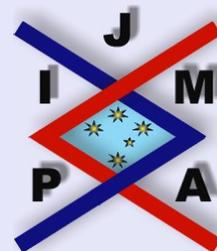
We consider the following three possible cases for  $m$ :

(i)  $m \leq n - k - 1$ . Then from (2.16) and (2.10) we obtain  $f(P_{m,n}) = \alpha_{1,n-k} = \frac{1}{(n-k)^{n-k}} > 0$ .

(ii)  $m = n - k$ . Then from (2.16) and (2.10) we obtain  $f(P_{n-k,n}) = \alpha_{1,n-k} - \frac{1}{(n-k)^{n-k}} = 0$ .

(iii)  $m = n - k + p$ , where  $1 \leq p \leq k$ . From (2.16) it follows

$$\begin{aligned} f(P_{m,n}) &= \alpha_{1,n-k} + \sum_{i=1}^k \alpha_{1+i,n-k+i} \binom{n-k+p}{n-k+i} \frac{1}{(n-k+p)^{n-k+i}} \\ &\quad - \binom{n-k+p}{n-k} \frac{1}{(n-k+p)^{n-k}} \\ &= \frac{1}{(n-k+p)^{n-k+p}} \left[ (n-k+p)^{n-k+p} \alpha_{1,n-k} \right. \\ &\quad \left. + \sum_{i=1}^k \binom{n-k+p}{n-k+i} (n-k+p)^{p-i} \alpha_{1+i,n-k+i} \right] \end{aligned}$$



**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 18 of 25

$$- \binom{n-k+p}{n-k} (n-k+p)^p \Big].$$

However,  $\binom{n-k+p}{n-k+i} \neq 0$  for  $i \leq p$ , and  $\frac{1}{(n-k+p)^{n-k+p}} = \alpha_{1,n-k+p}$  according to (2.10), and we get

$$(2.17) \quad f(P_{m,n}) = \alpha_{1,n-k+p} \left[ (n-k+p)^{n-k+p} \alpha_{1,n-k} + \sum_{i=1}^p \binom{n-k+p}{p-i} (n-k+p)^{p-i} \alpha_{1+i,n-k+i} - \binom{n-k+p}{p} (n-k+p)^p \right].$$

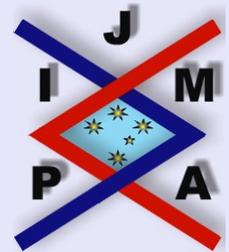
Obviously  $\alpha_{1,n-k} = \alpha_{1,(n-k+p)-p}$  and  $\alpha_{1+i,n-k+i} = \alpha_{1+i,(n-k+p)-p+i}$ . Then the right hand side of (2.17) is equal to zero according (2.12) for  $l = n - k + p$  and  $m = p$ .

Therefore  $f(P_{m,n}) = 0$  in this case.

It follows from (i), (ii) and (iii) that (2.15) is true, and hence (2.14) is also true. ■

**Remark 2.6.** Theorem 2.5 is true as well for  $k = n - 1$ , since both sides of (2.14) are equal in this case, which follows from (2.11).

**Remark 2.7.** An analogue of Theorem 2.5 for  $k = 0$  is the inequality between the arithmetic and geometric means.



**New Inequalities Between Elementary Symmetric Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 19 of 25

**Corollary 2.6.** Let  $A_n, G_n, H_n$  be the classical averages of the positive real numbers  $x_1, \dots, x_n$  ( $n \geq 2$ ). Then the following inequality is true:

$$(2.18) \quad \left[ \frac{nA_n}{(n-1)G_n} \right]^{n-1} \frac{1}{G_n} + \left[ n - \left( 1 + \frac{1}{n-1} \right)^{n-1} \right] \frac{1}{A_n} \geq \frac{n}{H_n}.$$

*Proof.* (2.18) follows from:

$$\begin{aligned} \sigma_1 &= nA_n, \quad \sigma_{n-1} = \frac{nG_n^n}{H_n}, \quad \sigma_n = G_n^n, \\ \alpha_{1,n-1} &= \frac{1}{(n-1)^{n-1}}, \quad \alpha_{2,n} = n^2 - \frac{n^n}{(n-1)^{n-1}} \end{aligned}$$

and from Theorem 2.5 for  $k = 1$ . ■

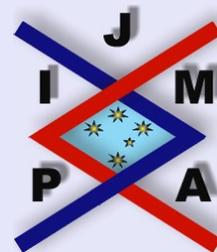
**Corollary 2.7 (Explicit expression of Theorem 2.5 for  $k = n - 2$ ).** For each integer number  $n$  ( $n \geq 3$ ) we have:

$$\sigma_1^{n-2} \sigma_2 \leq \frac{1}{4} \sigma_1^n + \sum_{i=1}^{n-2} (-1)^{i+1} \left( \frac{i+2}{2} \right)^2 \sigma_1^{n-2-i} \sigma_{2+i}.$$

*Proof.* It follows from Proposition 2.4 and from Theorem 2.5 for  $k = n - 2$ . ■

**Remark 2.8.** Corollary 2.7 is the principle result in [4].

**Remark 2.9.** Corollary 2.7 shows that Theorem 2.5 for  $k = n - 2$  is equivalent to Theorem 2.3 in the case when  $k = n - 1$ .



Title Page

Contents



Go Back

Close

Quit

Page 20 of 25

### 3. The Sharpness of the Inequalities (2.8) and (2.14)

The following two theorems prove that the estimates in Theorem 2.3 and Theorem 2.5 are, in a certain sense, the best possible.

**Theorem 3.1.** *Let  $n$  and  $k$ ,  $1 \leq k \leq n - 1$  be integers. Let the real numbers  $\beta_1, \dots, \beta_k$  have the property (3.1). We say that the real numbers  $\beta_1, \dots, \beta_k$  have the property (3.1) if for any non-negative real numbers  $x_1, \dots, x_n$  with a sum equal to one the following inequality is fulfilled:*

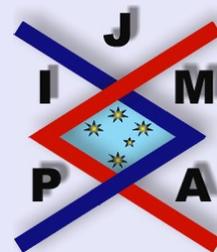
$$(3.1) \quad \sigma_{n-k} \geq \sum_{i=1}^k \beta_i \sigma_{n-k+i}.$$

*Then for arbitrary non-negative real numbers  $x_1, \dots, x_n$  with sum equal to one the following inequality is fulfilled:*

$$(3.2) \quad \sum_{i=1}^k \beta_i \sigma_{n-k+i} \leq \sum_{i=1}^k (-1)^{i+1} \binom{n-k-1+i}{i} (n-k+i)^2 (n-k)^{i-2} \sigma_{n-k+i}$$

*Proof.* Set  $f_1 = f_1(x_1, \dots, x_n) = \sigma_{n-k} - \sum_{i=1}^k \beta_i \sigma_{n-k+i}$  and

$$f_2 = f_2(x_1, \dots, x_n) = \sigma_{n-k} + \sum_{i=1}^k \binom{n-k+i}{i} (n-k+i)^2 (k-n)^{i-2} \sigma_{n-k+i}.$$



New Inequalities Between  
Elementary Symmetric  
Polynomials

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 21 of 25

Then (3.2) is equivalent to  $f_1 - f_2 \geq 0$ . On the other hand, according to Lemma 2.2, it is sufficient to verify this inequality at the points  $P_{m,n}$ . We have at these points:

- (i) For  $1 \leq m \leq n - k - 1$ ,  $k \leq n - 2$  apparently  $f_1 = f_2 = 0$ , hence  $f_1 - f_2 = 0$ .
- (ii) For  $m = n - k$ ,  $k \leq n - 1$  we obtain  $f_1 = f_2 = \frac{1}{(n-k)^{n-k}}$ , hence  $f_1 - f_2 = 0$ .
- (iii) For  $1 \leq n - k < m \leq n$  from the proof of Theorem 2.3 it follows, that  $f_2 = 0$ . As  $f_1 \geq 0$  according to (3.1), hence  $f_1 - f_2 \geq 0$ .

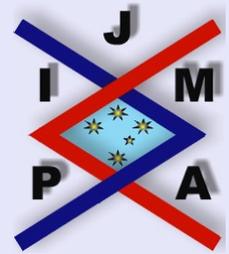
From (i), (ii) and (iii) it follows that  $f_1 - f_2 \geq 0$  in each point  $P_{m,n}$  and we complete the proof of the theorem. ■

**Theorem 3.2.** *Let  $n$  and  $k$  be integers,  $1 \leq k \leq n - 2$ . Let the real numbers  $\gamma_1, \dots, \gamma_{k+1}$  have the property (3.3). We say that the real numbers  $\gamma_1, \dots, \gamma_{k+1}$  have the property (3.3) if for any non-negative real numbers  $x_1, \dots, x_n$  with sum equal to one, the following inequality is fulfilled:*

$$(3.3) \quad \sigma_{n-k} \leq \gamma_1 + \sum_{i=1}^k \gamma_{i+1} \sigma_{n-k+i}.$$

*Then for any non-negative real numbers  $x_1, \dots, x_n$  with sum equal to one the following inequality is fulfilled:*

$$(3.4) \quad \alpha_{1,n-k} + \sum_{i=1}^k \alpha_{1+i,n-k+i} \sigma_{n-k+i} \leq \gamma_1 + \sum_{i=1}^k \gamma_{1+i} \sigma_{n-k+i}.$$



**New Inequalities Between Elementary Symmetric Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 22 of 25

*Proof.* Set

$$f_1 = f_1(x_1, \dots, x_n) = \gamma_1 + \sum_{i=1}^k \gamma_{1+i} \sigma_{n-k+i} - \sigma_{n-k}$$

and

$$f_2 = f_2(x_1, \dots, x_n) = \alpha_{1,n-k} + \sum_{i=1}^k \alpha_{1+i,n-k+i} \sigma_{n-k+i} - \sigma_{n-k}.$$

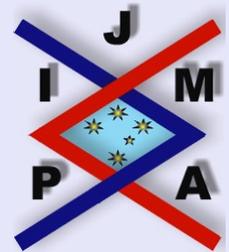
Then (3.4) is equivalent to  $f_1 - f_2 \geq 0$ . We are going to check this inequality at the points  $P_{m,n}$ . From (3.3) at  $P_{n-k,n}$  it follows, that

$$(3.5) \quad \gamma_1 \geq \frac{1}{(n-k)^{n-k}} = \alpha_{1,n-k}.$$

We consider the possible cases for  $m$ :

- (i)  $1 \leq m \leq n - k - 1$ . Then  $f_1 - f_2 = \gamma_1 - \alpha_{1,n-k} \geq 0$  at  $P_{m,n}$  according to (3.5).
- (ii)  $n - k \leq m \leq n$ . Then  $f_1 \geq 0$  at  $P_{m,n}$  according to (3.3) and from the proof of Theorem 2.5 it follows that  $f_2 = 0$ , therefore  $f_1 - f_2 \geq 0$ .

From (i) and (ii) we obtain, that  $f_1 - f_2 \geq 0$  in each point  $P_{m,n}$  ( $1 \leq m \leq n$ ). Applying Lemma 2.2 we complete the proof of Theorem 3.2. ■



**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

Title Page

Contents



Go Back

Close

Quit

Page 23 of 25

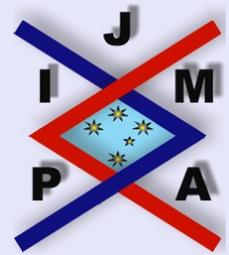


Table 1:

$l$	$\alpha_{1,l}$	$\alpha_{2,l}$	$\alpha_{3,l}$	$\alpha_{4,l}$	$\alpha_{5,l}$
1	1	0	0	0	0
2	1/4	0	0	0	0
3	1/27	9/4	0	0	0
4	1/256	176/27	-4	0	0
5	1/3125	3275/256	-775/27	25/4	0
6	1/46656	65844/3125	-6579/64	316/3	-9

---

**New Inequalities Between  
Elementary Symmetric  
Polynomials**

Todor P. Mitev

---

Title Page

Contents



Go Back

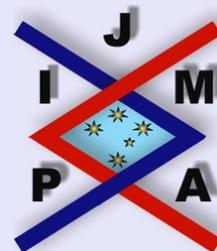
Close

Quit

Page 24 of 25

## References

- [1] G. HARDY, J.E. LITTLEWOOD AND G.PÓLYA, *Inequalities*, Cambridge Mathematical Library 2nd ed., 1952.
- [2] C.P. NICULESCU, A new look at Newton's inequalities, *J. Inequal. Pure and Appl. Math.*, **1**(2) (2000), Article 17. [ONLINE: [http://jipam.vu.edu.au/v1n2/014\\_99.html](http://jipam.vu.edu.au/v1n2/014_99.html)]
- [3] S. ROSSET, Normalized symmetric functions, Newton inequalities and a new set of stronger inequalities. *Amer. Math. Soc.*, **96** (1989), 815–820.
- [4] N. SATO, Symmetric polynomial inequalities, *Crux Mathematicorum with Mathematical Mayhem*, **27** (2001), 529–533.



---

### New Inequalities Between Elementary Symmetric Polynomials

Todor P. Mitev

---

Title Page

Contents



Go Back

Close

Quit

Page 25 of 25