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NEW INEQUALITIES BETWEEN ELEMENTARY SYMMETRIC POLYNOMIALS

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ABSTRACT. New families of sharp inequalities between elementary symmetric polynomials are proven. We estimate σ_{n-k} above and below by the elementary symmetric polynomials $\sigma_{n-k+1}, \ldots, \sigma_n$ in the case, when $x_1, \ldots, x_n$ are non-negative real numbers with sum equal to one.

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1. INTRODUCTION

Let $n \geq 2$ be an integer. As usual, we denote by $\sigma_0,\ \sigma_1, \ldots, \sigma_n$ the elementary symmetric polynomials of the variables $x_1, \ldots, x_n$.

In other words, $\sigma_0 = \sigma_0(x_1, \ldots, x_n) = 1$ and for $1 \leq k \leq n$

$$\sigma_k = \sigma_k(x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \ldots x_{i_k}.$$

The different $\sigma_0, \sigma_1, \ldots, \sigma_n$, are not comparable between them, but they are connected by nonlinear inequalities. To state them, it is more convenient to consider their averages $E_k = \sigma_k \Big/ \binom{n}{k}$, $k = 0, 1, \ldots, n$.

There are three basic types of inequalities between the symmetric functions with respect to the range of the variables $x_1, \ldots, x_n$.

For arbitrary real $x_1, \ldots, x_n$ the following inequalities are known:

$$E_k^2 \geq E_{k-1}E_{k+1}, \qquad 1 \leq k \leq n-1, \qquad \text{(Newton-Maclaurin)}, \tag{1.1}$$

$$4(E_{k+1}E_{k+3} - E_{k+2}^2)(E_kE_{k+2} - E_{k+1}^2) \geq (E_{k+1}E_{k+2} - E_kE_{k+3})^2,$$
$$k = 0, \ldots, n-3, \text{ (Rosset [4])},$$

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as well as the inequalities of Niculescu [2]. A complete description about their historical and contemporary stage of development can be found, for example, in [1] and [2].

Suppose now that all x_j, $j = 1, \dots, n$, are positive. Then the following general result (see [1, Theorem 77, p. 64]) is known:

Theorem 1.1 (Hardy, Littlewood, Pólya)**.** *For any positive* $x_1, \dots, x_n$ *and positive* $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ *the inequality*

$$E_1^{\alpha_1} \cdots E_n^{\alpha_n} \le E_1^{\beta_1} \cdots E_n^{\beta_n}$$

holds if and only if

$$\alpha_m + 2\alpha_{m+1} + \cdots + (n-m+1)\alpha_n \ge \beta_m + 2\beta_{m+1} + \cdots + (n-m+1)\beta_n$$

for each $1 \le m \le n$*.*

For other results in this direction see [1].

The aim of this paper is to obtain new inequalities between $\sigma_1, \dots, \sigma_n$ in the case when $x_1, \dots, x_n$ are non-negative, (Theorem 2.6 and Theorem 2.10 below). More precisely, we obtain the best possible estimates of $\sigma_1^k \sigma_{n-k}$ from below and above by linear functions of $\sigma_1^{k-1}\sigma_{n-k+1}, \dots, \sigma_1^0 \sigma_n$. Since all these functions are homogeneous with respect to $(x_1, \dots, x_n)$ of the same order, we can set $\sigma_1 = x_1 + \cdots + x_n = 1$, then our inequalities give the best possible estimates of σ_{n-k} by linear functions of $\sigma_{n-k+1}, \dots, \sigma_n$ for $k = 1, \dots, n-1$ in this case (Theorem 3.1 and Theorem 3.2 below). Inequalities of this type for $k = n-2$ have been recently obtained by Sato [4], which can be obtained as a consequence of Theorem 2.10 below.

2. NEW INEQUALITIES (THEOREM 2.6 AND THEOREM 2.10)

For the sake of completeness we give a straightforward proof of the following proposition, which is a consequence of Theorem 1.1, cited in the introduction. Here we suppose that $x_1, \dots, x_n$ are non-negative.

Proposition 2.1. *Let* $x_1, \dots, x_n$ *be non-negative real numbers,* $n \ge 2$*. Then for* $1 \le p \le n-1$ *we have*

$$\sigma_1 \sigma_p \ge \frac{n(p+1)}{n-p} \sigma_{p+1}. \tag{2.1}$$

Proof. Denote $\sigma_{l,n} = \sum_{1 \le i_1 < \cdots < i_l \le n} x_{i_1} x_{i_2} \cdots x_{i_l}$, $1 \le l \le n$. Note, that (2.1) is equivalent to

$$\sigma_{1,n} \sigma_{p,n} \ge \frac{n(p+1)}{n-p} \sigma_{p+1,n}. \tag{2.2}$$

First we shall check (2.2) for $p = 1$ and for $p = n-1$.

(i) For $p = 1$ the inequality (2.2) reads

$$\left(\sum_{i=1}^n x_i \right)^2 \ge \frac{2n}{n-1} \sum_{1 \le i < j \le n} x_i x_j,$$

which is equivalent to

$$(n-1)\left(\sum_{i=1}^n x_i \right)^2 \ge \frac{2n}{n-1} \sum_{1 \le i < j \le n} x_i x_j,$$

hence to $\sum_{1 \le i < j \le n} (x_i - x_j)^2 \ge 0$.

(ii) For $p = n-1$ (2.2) is equivalent to $\sigma_{1,n}\sigma_{n-1,n} \geq n^2\sigma_{n,n}$. If $\sigma_{n,n} = 0$, then (2.2) is obvious. Let $\sigma_{n,n} \neq 0$, then (2.2) is equivalent to $n^2 \leq \left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n \frac{1}{x_i}\right)$, which follow from AM-GM inequality.

We are going to prove (2.2) by recurrence with respect to $n \geq 2$.

(iii) We already proved that (2.2) is true for $n = 2$.

(iv) Let (2.2) be true for $n \geq 2$ and for each $p,\ 1 \leq p \leq n-1$. Fix p, $2 \leq p \leq n-1$. We will prove, that

$$\sigma_{1,n+1}\sigma_{p,n+1} \geq \frac{(n+1)(p+1)}{n+1-p}\sigma_{p+1,n+1}. \tag{2.3}$$

Since (2.3) is homogeneous, excluding the case $x_1 = \cdots = x_n = x_{n+1} = 0$, we may assume, that $\sigma_{1,n+1} = 1$.

Let $x_1 \leq x_2 \leq \cdots \leq x_{n+1}$. The following cases are possible:

1) Let $x_{n+1} = 1$. Then $x_1 = \cdots = x_n = 0$ and (2.3) becomes an equality.
2) Let $x_{n+1} = \frac{1}{n+1}$. Then $x_1 = \cdots = x_n = x_{n+1} = \frac{1}{n+1}$ and we obtain

$$\begin{aligned}\sigma_{p,n+1} - \frac{(n+1)(p+1)}{n+1-p}\sigma_{p+1,n+1}\\ &= \binom{n+1}{p}\frac{1}{(n+1)^p} - \frac{p+1}{n+1-p}\binom{n+1}{p+1}\frac{1}{(n+1)^p} = 0,\end{aligned}$$

hence (2.3) becomes again an equality.

3) Let $x_{n+1} \in \left(\frac{1}{n+1}; 1\right)$. Substitute $x_1+\cdots+x_n = 1-x_{n+1} = \sigma_{1,n} = s$, with $s \in \left(0; \frac{n}{n+1}\right)$. Then $\sigma_{p,n+1} = \sigma_{p,n} + (1-s)\sigma_{p-1,n}$ and $\sigma_{p+1,n+1} = \sigma_{p+1,n} + (1-s)\sigma_{p,n}$. Hence (2.3) is equivalent to

$$\sigma_{p,n} + (1-s)\sigma_{p-1,n} \geq \frac{(n+1)(p+1)}{n+1-p}\left[\sigma_{p+1,n} + (1-s)\sigma_{p,n}\right],$$

which is equivalent to

$$\left[\frac{n+1-p}{(n+1)(p+1)} - (1-s)\right]\sigma_{p,n} + \frac{(1-s)(n+1-p)}{(n+1)(p+1)}\sigma_{p-1,n} \geq \sigma_{p+1,n}. \tag{2.4}$$

From (iv) we obtain $\sigma_{p+1,n} \leq \frac{n-p}{n(p+1)}s\sigma_{p,n}$. Then (2.4) follows from the next inequality (if true):

$$\frac{n-p}{n(p+1)}s\sigma_{p,n} \leq \left[\frac{n+1-p}{(n+1)(p+1)} - (1-s)\right]\sigma_{p,n} + \frac{(1-s)(n+1-p)}{(n+1)(p+1)}\sigma_{p-1,n}, \tag{2.5}$$

which is equivalent to

$$\sigma_{p-1,n} \geq \frac{p[n(n+2)-(n+1)^2 s]}{n(n+1-p)(1-s)}\sigma_{p,n}. \tag{2.6}$$

It follows from (iv) that $\sigma_{p-1,n} \geq \frac{np}{(n+1-p)s}\sigma_{p,n}$.

Hence (2.6), and consequently (2.5) and (2.4), follow from

$$\frac{np}{(n+1-p)s}\sigma_{p,n} - \frac{p[n(n+2)-(n+1)^2 s]}{n(n+1-p)(1-s)}\sigma_{p,n} = \frac{p[(n+1)s-n]^2}{ns(n+1-p)(1-s)}\sigma_{p,n} \geq 0.$$

Since (2.2) is true for $p = 1$ and $p = n$ according to (i) and (ii), then (2.2) is fulfilled for n, $n \geq 2$. Hence the proposition is proved. □

Remark 2.2. It follows from the proof, that equality is achieved in the following two cases:

1) $x_1 = x_2 = \cdots = x_n = a \geq 0$.
2) $n - p + 1$ of $x_1, \ldots, x_n$ are equal to 0 and the rest of them are arbitrary non-negative real numbers.

Remark 2.3. (2.1) can be proven using Lemma 2.4 below, but in this way it will be difficult to see when (2.1) turns into an equality.

From now on n will be a fixed positive integer. It will be assumed that at least one of the non-negative numbers $x_1, \ldots, x_n$ differs from zero.

Lemma 2.4. *Let us assume that $x_1, \ldots, x_n$ are non-negative real numbers ($n \geq 2$) and $x_1 + \cdots + x_n = \sigma_1 = 1$. Then the function $f(x_1, \ldots, x_n) = a_1 + a_2\sigma_2 + \cdots + a_n\sigma_n$ ($a_1, \ldots, a_n$ are real numbers), achieves its maximum and minimum at least in some of the points $P_{k,n}\left(\frac{1}{k}, \ldots, \frac{1}{k}, 0, \ldots, 0\right)$, $1 \leq k \leq n$ (the first k coordinates of $P_{k,n}$ are equal to $\frac{1}{k}$, and the rest of them are equal to zero).*

Proof. The set $A_n = \{(x_1, \ldots, x_n)/x_i \geq 0,\ x_1 + \cdots + x_n = 1\}$ is compact and f is continuous in it, hence f achieves its minimum and maximum values. We rewrite f as follows:

$$f(x_1, \ldots, x_n) = x_1x_2g(x_3, \ldots x_n) + x_1h_1(x_3, \ldots, x_n) + x_2h_2(x_3, \ldots, x_n) + t(x_3, \ldots, x_n) + a_1.$$

As f is symmetric, then $h_1 \equiv h_2$ and therefore:

$$f(x_1, \ldots, x_n) = x_1x_2g(x_3, \ldots x_n) + (x_1 + x_2)h_1(x_3, \ldots, x_n) + t(x_3, \ldots, x_n) + a_1. \tag{2.7}$$

Let $P(x_1^0, \ldots, x_n^0)$ be a point in which f achieves its minimum value. We consider the function $F(x) = f(x, s - x, x_3^0, \ldots, x_n^0)$, $s = x_1^0 + x_2^0$, for $x \in [0; s]$ (we assume, that $s > 0$). Obviously the minimum values of F and f are equal and F achieves its minimum value for $x = x_1^0$. From (2.7) we obtain that $F(x) = \alpha x(s - x) + s\beta + \gamma = \alpha x(s - x) + \delta$, where α, δ depend on $x_1^0, x_2^0, x_3^0, \ldots, x_n^0,\ a_1, \ldots, a_n$.

The following three cases are possible:

(i) $\alpha = 0$. Then $F(x) = const$ and we may assume that $\min F = F(0)$ or $\min F = F\left(\frac{s}{2}\right)$.
(ii) $\alpha > 0$. Then $\min F = F(0)$.
(iii) $\alpha < 0$. Then $\min F = F\left(\frac{s}{2}\right)$.

Hence, as x_1^0 and x_2^0 were arbitrarily chosen then, for $\forall i \neq j$ we may assume that $x_i^0 = x_j^0$ or, at least one of them is equal to zero.

Let us choose a point $P(x_1^0, \ldots, x_n^0)$, for which the number of coordinates p which equal to zero is the highest possible and $x_1^0 \geq x_2^0 \geq \cdots \geq x_n^0$. If $p = n - 1$, then Lemma 2.4 is proven. Let $0 \leq p \leq n - 2$, i.e. $P(x_1^0, ..x_{n-p}^0, 0, \ldots, 0)$, $x_1^0 \cdots x_{n-p}^0 \neq 0$. Then for the pairs (x_i^0, x_j^0), $1 \leq i < j \leq n - p$ only case (iii) is valid, from which Lemma 2.4 follows. Lemma 2.4 is true also for the maximum value of f, since $\max f = \min(-f)$. □

Remark 2.5. A result similar to Lemma 2.4 is proved by Sato in [4].

Theorem 2.6. *Let n, k be integer numbers, $1 \leq k \leq n - 1$. Then for arbitrary non-negative $x_1, \ldots, x_n$, the following inequality is true:*

$$\sigma_1^k\sigma_{n-k} \geq \sum_{i=1}^{k}(-1)^{i+1}\binom{n-k-1+i}{i}(n-k+i)^2(n-k)^{i-2}\sigma_1^{k-i}\sigma_{n-k+i}. \tag{2.8}$$

Proof. Since (2.8) is homogenous we may assume that $x_1 + \cdots + x_n = \sigma_1 = 1$. Then, according to Lemma 2.4 it suffices to prove, that $f(P_{m,n}) \geq 0$ for $1 \leq m \leq n$, where

$$f(x_1, \ldots, x_n) = \sigma_{n-k} + \sum_{i=1}^{k}\binom{n-k-1+i}{i}(n-k+i)^2(k-n)^{i-2}\sigma_{n-k+i}.$$

At the $P_{m,n}$ point we have $\sigma_{n-k+i} = \binom{m}{n-k+i} \frac{1}{m^{n-k+i}}$, hence

$$\sigma_{n-k+i} \neq 0 \quad \text{if and only if} \quad i \leq m-n+k. \tag{2.9}$$

We consider the following three possible cases for m:

(i) $m \leq n-k-1$, $k \leq n-2$. Then obviously $\sigma_{n-k} = \sigma_{n-k+1} = \cdots = \sigma_n = 0$, hence $f(P_{m,n}) = 0$.

(ii) $m = n-k$, $k \leq n-1$. From (2.9) we obtain $\sigma_{n-k} = \frac{1}{(n-k)^{n-k}}$ and $\sigma_{n-k+1} = \cdots = \sigma_n = 0$, hence $f(P_{m,n}) = \frac{1}{(n-k)^{n-k}} > 0$.

(iii) $m = n-k+p$, $1 \leq p \leq k$, $k \leq n-1$. From (2.9) and $m = n-k+p$ we obtain

$$\begin{aligned} f(P_{m,n}) &= \binom{n-k+p}{n-k} \frac{1}{(n-k+p)^{n-k}} + \sum_{i=1}^{k} \binom{n-k-1+i}{i} \\ &\qquad \times (n-k+i)^2 (k-n)^{i-2} \binom{n-k+p}{n-k+i} \frac{1}{(n-k+p)^{n-k+i}} \\ &= \binom{m}{p} \frac{1}{m^{m-p}} + \sum_{i=1}^{p} \binom{m-p-1+i}{i} \\ &\qquad \times (m-p+i)^2 (p-m)^{i-2} \binom{m}{m-p+i} \frac{1}{m^{m-p+i}}. \end{aligned}$$

Now from equality

$$\binom{m-p-1+i}{i} \binom{m}{m-p+i} (m-p+i) = \binom{m-1}{p} \binom{p}{i} m$$

we obtain

$$\begin{aligned} f(P_{m,n}) &= \binom{m}{p} \frac{1}{m^{m-p}} \\ &\qquad + \sum_{i=1}^{p} \binom{m-1}{p} \binom{p}{i} (m-p+i)^2 (p-m)^{i-2} \frac{1}{m^{m-p-1+i}}. \end{aligned}$$

This implies

$$\begin{aligned} &\frac{m^{m-p+1}}{\binom{m-1}{p}} f(P_{m,n}) \\ &\quad = \frac{m^2}{m-p} + p(m-p+1)\frac{m}{p-m} + \sum_{i=2}^{p} \binom{p}{i} (m-p+i) \left(\frac{p-m}{m}\right)^{i-2} \\ &\quad = m(1-p) + \sum_{i=2}^{p} \binom{p}{i} (m-p) \left(\frac{p-m}{m}\right)^{i-2} + \sum_{i=2}^{p} \binom{p}{i} i \left(\frac{p-m}{m}\right)^{i-2} \\ &\quad = m(1-p) + \frac{m^2}{m-p} \left[\left(1 + \frac{p-m}{m}\right)^p - \frac{p(p-m)}{m} - 1 \right] \\ &\qquad + \frac{mp}{p-m} \sum_{i=2}^{p} \binom{p-1}{i-1} \left(\frac{p-m}{m}\right)^{i-1}. \end{aligned}$$

Substituting $i = j + 1$ we obtain:

$$\frac{m^{m-p+1}}{\binom{m-1}{p}} f(P_{m,n})$$
$$= m(1-p) + \frac{m^2}{m-p}\left[\left(\frac{p}{m}\right)^p + \frac{p(m-p)}{m} - 1\right]$$
$$+ \frac{mp}{p-m}\sum_{j=1}^{p-1}\binom{p-1}{j}\left(\frac{p-m}{m}\right)^j$$
$$= m(1-p) + \frac{m^2}{m-p}\left(\frac{p}{m}\right)^p + mp - \frac{m^2}{m-p} + \frac{mp}{p-m}\left[\left(1+\frac{p-m}{m}\right)^{p-1} - 1\right]$$
$$= m + \frac{m^2}{m-p}\left(\frac{p}{m}\right)^p - \frac{m^2}{m-p} + \frac{mp}{p-m}\left(\frac{p}{m}\right)^{p-1} - \frac{mp}{p-m} = 0.$$

From (i) – (iii) it follows that Theorem 2.6 is true.

□

Remark 2.7. Theorem 2.6 for $k = 1$ is equivalent to Proposition 2.1 in the case when $p = n-1$.

Remark 2.8. It is easy to verify, that (2.8) is equivalent to

$$E_1^k E_{n-k} \geq \frac{1}{n}\sum_{i=1}^{k}\binom{k}{i}(n-k+i)\left(\frac{k-n}{n}\right)^{i-1} E_1^{k-i}E_{n-k+i}.$$

We define the sequence of real numbers $\{\alpha_{m,l}\}$, $m \in \mathbb{N}, l \in \mathbb{N}$ as follows:

$$\alpha_{1,l} = \frac{1}{l^l} \quad \text{for} \quad \forall l \in \mathbb{N}, \tag{2.10}$$

$$\alpha_{m,l} = 0 \quad \text{for} \quad m \geq l \geq 2 \quad \text{or} \quad m > 1, l = 1, \tag{2.11}$$

$$\binom{l}{m} l^m = l^l \alpha_{1,l-m} + \sum_{j=1}^{m}\binom{l}{m-j} l^{m-j}\alpha_{1+j,l-m+j} \quad \text{for} \quad 1 \leq m \leq l-1. \tag{2.12}$$

More precisely, the numbers $\alpha_{m,l}$ can be defined recurrently (excluding the cases when: $m > 1$, $l = 1$ or $m \geq l \geq 2$) as follows:

1) We get $\alpha_{1,l}$ for $l \geq 1$ from (2.10).
2) Then we determine $\alpha_{2,l}$ for $l \geq 3$ from $\binom{l}{1} l = l^l\alpha_{1,l-1} + \alpha_{2,l}$.
3) Then we determine $\alpha_{3,l}$ for $l \geq 4$ from $\binom{l}{2} l^2 = l^l\alpha_{1,l-2} + \binom{l}{1} l\alpha_{2,l-1} + \alpha_{3,l}$.
4) Then we determine $\alpha_{4,l}$ for $l \geq 5$ from $\binom{l}{3} l^3 = l^l\alpha_{1,l-3} + \binom{l}{2} l^2\alpha_{2,l-2} + \binom{l}{1} l\alpha_{3,l-1} + \alpha_{4,l}$ and so on.

For example, the values of $\alpha_{m,l}$ for $m \leq 5$, $l \leq 6$ are given in Table 3.1.

The sequence $\{\alpha_{m,l}\}$ has interesting properties. For example one can prove, that in the case when $\alpha_{m,l} \neq 0$: $\operatorname{sgn}\alpha_{m,l} = 1$ for m even and $\operatorname{sgn}\alpha_{m,l} = -1$ for m odd, $m \geq 3$.

We are going to prove the following property of the sequence $\{\alpha_{m,l}\}$:

Proposition 2.9. *For each integer number n, $n \geq 2$ we have:*

$$\alpha_{n,n+1} = (-1)^n\left(\frac{n+1}{2}\right)^2. \tag{2.13}$$

Proof. We will prove (2.12) by induction.

(i) We show, that $\alpha_{2,3} = (-1)^2 \left(\frac{2+1}{2}\right)^2$, (see Table 3.1).
(ii) Let (2.13) hold true for $\alpha_{2,3}, \dots, \alpha_{n-1,n}$.
(iii) Using (2.12) for $l = n+1$ and $m = n-1$, (2.10) for $l = 2$ and (ii) we obtain

$$\binom{n+1}{2}(n+1)^{n-1} = \frac{(n+1)^{n+1}}{4} + \sum_{j=1}^{n-2}\binom{n+1}{j+2}(-1)^{j+1}\left(\frac{j+2}{2}\right)^2(n+1)^{n-1-j} + \alpha_{n,n+1}.$$

Substituting $j = i-1$, this implies

$$\alpha_{n,n+1} = \binom{n+1}{2}(n+1)^{n-1} - \frac{(n+1)^{n+1}}{4} - \frac{1}{4}\sum_{i=2}^{n-1}\binom{n+1}{i+1}(-1)^i(i+1)^2(n+1)^{n-i}.$$

Now from the equalities $\binom{n+1}{i+1}(i+1) = \binom{n}{i}(n+1)$ and $\binom{n}{i} i = \binom{n-1}{i-1} n$ we obtain:

$$\begin{aligned}\alpha_{n,n+1} &= \binom{n+1}{2}(n+1)^{n-1} - \frac{(n+1)^{n-1}}{4} \\ &\quad - \frac{n+1}{4}\sum_{i=2}^{n-1}\binom{n}{i}(-1)^i(i+1)(n+1)^{n-i} \\ &= \frac{(n+1)^{n+1}}{4}\left[\frac{2n}{n+1} - 1 - \sum_{i=2}^{n-1}\binom{n}{i}(i+1)\left(\frac{-1}{n+1}\right)^i\right] \\ &= \frac{(n+1)^{n+1}}{4}\left[\frac{n-1}{n+1} - \sum_{i=2}^{n-1}\binom{n}{i}\left(\frac{-1}{n+1}\right)^i - n\sum_{i=2}^{n-1}\binom{n-1}{i-1}\left(\frac{-1}{n+1}\right)^i\right].\end{aligned}$$

Substituting $i = k+1$ we obtain

$$\begin{aligned}\alpha_{n,n+1} &= \frac{(n+1)^{n+1}}{4}\left[\frac{n-1}{n+1} - \left(1 + \frac{-1}{n+1}\right)^n + 1 + n\left(\frac{-1}{n+1}\right) + \left(\frac{-1}{n+1}\right)^n\right. \\ &\quad \left. - n\sum_{k=1}^{n-2}\binom{n-1}{k}\left(\frac{-1}{n+1}\right)^{k+1}\right] \\ &= \frac{(n+1)^{n+1}}{4}\left\{\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^n + \left(\frac{-1}{n+1}\right)^n\right. \\ &\quad \left. + \frac{n}{n+1}\left[\left(1 + \frac{-1}{n+1}\right)^{n-1} - 1 - \left(\frac{-1}{n+1}\right)^{n-1}\right]\right\} \\ &= \frac{(n+1)^{n+1}}{4}\left[\frac{n}{n+1} - \left(\frac{n}{n+1}\right)^n + \left(\frac{-1}{n+1}\right)^n\right. \\ &\quad \left. + \frac{n}{n+1}\left(\frac{n}{n+1}\right)^{n-1} - \frac{n}{n+1} - \frac{n}{n+1}\left(\frac{-1}{n+1}\right)^{n-1}\right] \\ &= \frac{(n+1)^{n+1}}{4}(-1)^n\left[\frac{1}{(n+1)^n} + \frac{n}{(n+1)^n}\right] = (-1)^n\left(\frac{n+1}{2}\right)^2.\end{aligned}$$

From (i), (ii) and (iii) it follows that (2.13) is true for each $n \geq 2$.

□

Theorem 2.10. *Let n and k be fixed integer numbers for which $1 \leq k \leq n-2$. Then for arbitrary non-negative $x_1, \dots, x_n$, the following inequality is fulfilled:*

$$\sigma_1^k \sigma_{n-k} \leq \alpha_{1,n-k}\sigma_1^n + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\sigma_1^{k-i}\sigma_{n-k+i}, \tag{2.14}$$

where $\{\alpha_{m,l}\}$ are defined from (2.10)-(2.12).

Proof. (2.14) is homogenous, therefore we may assume, that $x_1 + \dots + x_n = \sigma_1 = 1$. Then according to Lemma 2.4 it is sufficient to prove, that

$$f(P_{m,n}) \geq 0, \qquad \text{for each } m,\ 1 \leq m \leq n, \tag{2.15}$$

where

$$f(x_1, \dots, x_n) = \alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i}\sigma_{n-k+i} - \sigma_{n-k}.$$

Obviously at the point $P_{m,n}$ we have $\sigma_q = \binom{m}{q}\frac{1}{m^q}$ for $1 \leq q \leq n$, hence

$$\sigma_q \neq 0 \qquad \text{if and only if} \quad q \leq m. \tag{2.16}$$

We consider the following three possible cases for m:

(i) $m \leq n-k-1$. Then from (2.16) and (2.10) we obtain $f(P_{m,n}) = \alpha_{1,n-k} = \frac{1}{(n-k)^{n-k}} > 0$.

(ii) $m = n-k$. Then from (2.16) and (2.10) we obtain $f(P_{n-k,n}) = \alpha_{1,n-k} - \frac{1}{(n-k)^{n-k}} = 0$.

(iii) $m = n-k+p$, where $1 \leq p \leq k$. From (2.16) it follows

$$\begin{aligned} f(P_{m,n}) = \alpha_{1,n-k} &+ \sum_{i=1}^{k} \alpha_{1+i,n-k+i} \binom{n-k+p}{n-k+i} \frac{1}{(n-k+p)^{n-k+i}} \\ &- \binom{n-k+p}{n-k} \frac{1}{(n-k+p)^{n-k}} \\ = &\frac{1}{(n-k+p)^{n-k+p}} \Bigg[(n-k+p)^{n-k+p}\alpha_{1,n-k} \\ &+ \sum_{i=1}^{k} \binom{n-k+p}{n-k+i} (n-k+p)^{p-i}\alpha_{1+i,n-k+i} - \binom{n-k+p}{n-k} (n-k+p)^p \Bigg]. \end{aligned}$$

However, $\binom{n-k+p}{n-k+i} \neq 0$ for $i \leq p$, and $\frac{1}{(n-k+p)^{n-k+p}} = \alpha_{1,n-k+p}$ according to (2.10), and we get

$$\begin{aligned} f(P_{m,n}) = \alpha_{1,n-k+p} \Bigg[&(n-k+p)^{n-k+p}\alpha_{1,n-k} \\ &+ \sum_{i=1}^{p} \binom{n-k+p}{p-i} (n-k+p)^{p-i}\alpha_{1+i,n-k+i} \\ &- \binom{n-k+p}{p} (n-k+p)^p \Bigg]. \end{aligned} \tag{2.17}$$

Obviously $\alpha_{1,n-k} = \alpha_{1,(n-k+p)-p}$ and $\alpha_{1+i,n-k+i} = \alpha_{1+i,(n-k+p)-p+i}$. Then the right hand side of (2.17) is equal to zero according (2.12) for $l = n-k+p$ and $m = p$.

Therefore $f(P_{m,n}) = 0$ in this case.

It follows from (i), (ii) and (iii) that (2.15) is true, and hence (2.14) is also true. □

Remark 2.11. Theorem 2.10 is true as well for $k = n-1$, since both sides of (2.14) are equal in this case, which follows from (2.11).

Remark 2.12. An analogue of Theorem 2.10 for $k = 0$ is the inequality between the arithmetic and geometric means.

Corollary 2.13. *Let A_n, G_n, H_n be the classical averages of the positive real numbers $x_1, \dots, x_n$ $(n \geq 2)$. Then the following inequality is true:*

$$\left[\frac{nA_n}{(n-1)G_n}\right]^{n-1}\frac{1}{G_n} + \left[n - \left(1+\frac{1}{n-1}\right)^{n-1}\right]\frac{1}{A_n} \geq \frac{n}{H_n}. \tag{2.18}$$

Proof. (2.18) follows from:

$$\sigma_1 = nA_n,\ \sigma_{n-1} = \frac{nG_n^n}{H_n},\ \sigma_n = G_n^n,\ \alpha_{1,n-1} = \frac{1}{(n-1)^{n-1}},\ \alpha_{2,n} = n^2 - \frac{n^n}{(n-1)^{n-1}}$$

and from Theorem 2.10 for $k = 1$. □

Corollary 2.14 (Explicit expression of Theorem 2.10 for $k = n-2$)**.** *For each integer number n ($n \geq 3$) we have:*

$$\sigma_1^{n-2}\sigma_2 \leq \frac{1}{4}\sigma_1^n + \sum_{i=1}^{n-2}(-1)^{i+1}\left(\frac{i+2}{2}\right)^2\sigma_1^{n-2-i}\sigma_{2+i}.$$

Proof. It follows from Proposition 2.9 and from Theorem 2.10 for $k = n-2$. □

Remark 2.15. Corollary 2.14 is the principle result in [4].

Remark 2.16. Corollary 2.14 shows that Theorem 2.10 for $k = n-2$ is equivalent to Theorem 2.6 in the case when $k = n-1$.

3. THE SHARPNESS OF THE INEQUALITIES (2.8) AND (2.14)

The following two theorems prove that the estimates in Theorem 2.6 and Theorem 2.10 are, in a certain sense, the best possible.

Theorem 3.1. *Let n and k, $1 \leq k \leq n-1$ be integers. Let the real numbers $\beta_1, \dots, \beta_k$ have the property (3.1). We say that the real numbers $\beta_1, \dots, \beta_k$ have the property (3.1) if for any non-negative real numbers $x_1, \dots, x_n$ with a sum equal to one the following inequality is fulfilled:*

$$\sigma_{n-k} \geq \sum_{i=1}^{k}\beta_i\sigma_{n-k+i}. \tag{3.1}$$

Then for arbitrary non-negative real numbers $x_1, \dots, x_n$ with sum equal to one the following inequality is fulfilled:

$$\sum_{i=1}^{k}\beta_i\sigma_{n-k+i} \leq \sum_{i=1}^{k}(-1)^{i+1}\binom{n-k-1+i}{i}(n-k+i)^2(n-k)^{i-2}\sigma_{n-k+i} \tag{3.2}$$

Proof. Set $f_1 = f_1(x_1, \dots, x_n) = \sigma_{n-k} - \sum_{i=1}^{k} \beta_i \sigma_{n-k+i}$ and

$$f_2 = f_2(x_1, \dots, x_n) = \sigma_{n-k} + \sum_{i=1}^{k} \binom{n-k+i}{i} (n-k+i)^2 (k-n)^{i-2} \sigma_{n-k+i}.$$

Then (3.2) is equivalent to $f_1 - f_2 \geq 0$. On the other hand, according to Lemma 2.4, it is sufficient to verify this inequality at the points $P_{m,n}$. We have at these points:

(i) For $1 \leq m \leq n-k-1$, $k \leq n-2$ apparently $f_1 = f_2 = 0$, hence $f_1 - f_2 = 0$.
(ii) For $m = n-k$, $k \leq n-1$ we obtain $f_1 = f_2 = \frac{1}{(n-k)^{n-k}}$, hence $f_1 - f_2 = 0$.
(iii) For $1 \leq n-k < m \leq n$ from the proof of Theorem 2.6 it follows, that $f_2 = 0$. As $f_1 \geq 0$ according to (3.1), hence $f_1 - f_2 \geq 0$.

From (i), (ii) and (iii) it follows that $f_1 - f_2 \geq 0$ in each point $P_{m,n}$ and we complete the proof of the theorem. □

Theorem 3.2. *Let n and k be integers, $1 \leq k \leq n-2$. Let the real numbers $\gamma_1, \dots, \gamma_{k+1}$ have the property (3.3). We say that the real numbers $\gamma_1, \dots, \gamma_{k+1}$ have the property (3.3) if for any non-negative real numbers $x_1, \dots, x_n$ with sum equal to one, the following inequality is fulfilled:*

$$\sigma_{n-k} \leq \gamma_1 + \sum_{i=1}^{k} \gamma_{i+1} \sigma_{n-k+i}. \tag{3.3}$$

Then for any non-negative real numbers $x_1, \dots, x_n$ with sum equal to one the following inequality is fulfilled:

$$\alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i} \sigma_{n-k+i} \leq \gamma_1 + \sum_{i=1}^{k} \gamma_{1+i} \sigma_{n-k+i}. \tag{3.4}$$

Proof. Set

$$f_1 = f_1(x_1, \dots, x_n) = \gamma_1 + \sum_{i=1}^{k} \gamma_{1+i} \sigma_{n-k+i} - \sigma_{n-k}$$

and

$$f_2 = f_2(x_1, \dots, x_n) = \alpha_{1,n-k} + \sum_{i=1}^{k} \alpha_{1+i,n-k+i} \sigma_{n-k+i} - \sigma_{n-k}.$$

Then (3.4) is equivalent to $f_1 - f_2 \geq 0$. We are going to check this inequality at the points $P_{m,n}$. From (3.3) at $P_{n-k,n}$ it follows, that

$$\gamma_1 \geq \frac{1}{(n-k)^{n-k}} = \alpha_{1,n-k}. \tag{3.5}$$

We consider the possible cases for m:

(i) $1 \leq m \leq n-k-1$. Then $f_1 - f_2 = \gamma_1 - \alpha_{1,n-k} \geq 0$ at $P_{m,n}$ according to (3.5).
(ii) $n-k \leq m \leq n$. Then $f_1 \geq 0$ at $P_{m,n}$ according to (3.3) and from the proof of Theorem 2.10 it follows that $f_2 = 0$, therefore $f_1 - f_2 \geq 0$.

From (i) and (ii) we obtain, that $f_1 - f_2 \geq 0$ in each point $P_{m,n}$ $(1 \leq m \leq n)$. Applying Lemma 2.4 we complete the proof of Theorem 3.2. □

Table 3.1:

l	$\alpha_{1,l}$	$\alpha_{2,l}$	$\alpha_{3,l}$	$\alpha_{4,l}$	$\alpha_{5,l}$
1	1	0	0	0	0
2	1/4	0	0	0	0
3	1/27	9/4	0	0	0
4	1/256	176/27	-4	0	0
5	1/3125	3275/256	-775/27	25/4	0
6	1/46656	65844/3125	-6579/64	316/3	- 9

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