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ON AN OPEN PROBLEM OF BAI-NI GUO AND FENG QI

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ABSTRACT. In this paper, an open problem posed respectively by B.-N. Guo and F. Qi in [4, 6, 7] is partially solved: an integral expression and a new double inequality of the generalized Mathieu's series $\sum_{n=1}^{\infty}\frac{2n}{(n^2+a^2)^{p+1}}$ are established by using some properties of gamma function and Fourier transform inequalities, where $a>0$, $p\in\mathbb{N}$.

Key words and phrases: Integral expression, Inequality, Mathieu's series, gamma function, Fourier transform inequality.

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1. INTRODUCTION

It is well-known that the following

$$S(a,1) \triangleq \sum_{n=1}^{\infty}\frac{2n}{(n^2+a^2)^2}, \quad a>0 \tag{1.1}$$

is called the Mathieu's series. The integral expression of Mathieu's series (1.1) was given in [3] as follows

$$S(a,1) = \frac{1}{a}\int_0^{\infty}\frac{x\sin ax}{e^x-1}dx. \tag{1.2}$$

The Mathieu' series (1.1) and related inequalities have been studied by many mathematicians for more than a century and there has been a vast amount of literature. Please refer to [4, 6, 7] and the references therein.

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The following Fourier transform inequalities can be found in [2, pp. 89–90]: If $f \in L([0,\infty))$ with $\lim_{t\to\infty} f(t) = 0$, then

$$\sum_{k=1}^{\infty}(-1)^k f(k\pi) < \int_0^{\infty} f(t)\cos t dt < \sum_{k=0}^{\infty}(-1)^k f(k\pi), \tag{1.3}$$

$$\sum_{k=0}^{\infty}(-1)^k f\left(\left(k+\frac{1}{2}\right)\pi\right) < \int_0^{\infty} f(t)\sin t dt < f(0) + \sum_{k=0}^{\infty}(-1)^k f\left(\left(k+\frac{1}{2}\right)\pi\right). \tag{1.4}$$

By using the integral expression (1.2) and Fourier transform inequality (1.4), Bai-Ni Guo established in [4] the following inequalities for Mathieu's series (1.1).

Theorem A ([4])**.** *for $a > 0$, then*

$$\frac{\pi}{a^3}\sum_{k=0}^{\infty}\frac{(-1)^k\left(k+\frac{1}{2}\right)}{\exp\left[\left(k+\frac{1}{2}\right)\frac{\pi}{a}\right]-1} < S(a,1) < \frac{1}{a^2}\left(1+\frac{\pi}{a}\sum_{k=0}^{\infty}\frac{(-1)^k\left(k+\frac{1}{2}\right)}{\exp\left[\left(k+\frac{1}{2}\right)\frac{\pi}{a}\right]-1}\right). \tag{1.5}$$

At the end of the short note [4], B.-N. Guo proposed an open problem: *Let*

$$S(a,p) = \sum_{n=1}^{\infty}\frac{2n}{(n^2+a^2)^{p+1}}, \tag{1.6}$$

where $p > 0$ and $a > 0$. Can one establish an integral expression of $S(a,p)$?

Soon after, Feng Qi further proposed in [6, 7] a similar open problem: *Let*

$$S(r,t,\alpha) = \sum_{n=1}^{\infty}\frac{2n^{\alpha/2}}{(n^{\alpha}+r^2)^{t+1}} \tag{1.7}$$

for $t > 0$, $r > 0$ and $\alpha > 0$. Can one obtain an integral expression of $S(r,t,\alpha)$? Give some sharp inequalities for the series $S(r,t,\alpha)$.

In this paper, using the well-known formula

$$\frac{1}{t^{a+1}} = \frac{1}{\Gamma(a+1)}\int_0^{\infty} x^a e^{-xt}dx, \tag{1.8}$$

which can be deduced from the definition of a gamma function, and Fourier transform inequalities (1.3) and (1.4), we will establish an integral expression and a new double inequality of the generalized Mathieu's series (1.6) for $p \in \mathbb{N}$, the set of all positive integers. Our results partially solve the open problems by B.-N. Guo and F. Qi in [4] and [6, 7] mentioned above.

2. The Integral Expressions

One of our main results is to establish an integral expression of $S(a,p)$ for $a > 0$ and $p \in \mathbb{N}$, which can be stated as the following.

Theorem 2.1. *Let $a > 0$ and $p \in \mathbb{N}$. Then we have*

$$\begin{aligned} S(a,p) &= \sum_{n=1}^{\infty}\frac{2n}{(n^2+a^2)^{p+1}} \\ &= \frac{2}{(2a)^p p!}\int_0^{\infty}\frac{t^p\cos\left(\frac{p\pi}{2}-at\right)}{e^t-1}dt \\ &\quad - 2\sum_{k=2}^{p}\frac{(k-1)(2a)^{k-2p-1}}{k!(p-k+1)}\binom{-(p+1)}{p-k}\int_0^{\infty}\frac{t^k\cos\left[\frac{\pi}{2}(2p-k+1)-at\right]}{e^t-1}dt. \end{aligned} \tag{2.1}$$

Proof. Let $a_n = \frac{2n}{(n^2+a^2)^{p+1}}$, where $a > 0$ and $p \in \mathbb{N}$. Then

$$a_n = \frac{n + ai + n - ai}{(n + ai)^{p+1}(n - ai)^{p+1}} = b_n + c_n,$$

where

$$b_n = \frac{1}{(n + ai)^p(n - ai)^{p+1}}, \quad c_n = \frac{1}{(n + ai)^{p+1}(n - ai)^p}.$$

By putting $n + ai = x$, we obtain

$$b_n = \frac{1}{x^p(x - 2ai)^{p+1}} = \sum_{k=1}^{p} \frac{A_k}{x^k} + \sum_{k=1}^{p+1} \frac{B_k}{(x - 2ai)^k},$$

where A_k and B_k are constants.

Applying the binomial expansion, we get

$$\begin{aligned}(x - 2ai)^{-(p+1)} &= (-2ai)^{-(p+1)} \left(1 - \frac{x}{2ai}\right)^{-(p+1)} \\ &= (-2ai)^{-(p+1)} \sum_{k=0}^{\infty} \binom{-(p+1)}{k} \left(-\frac{x}{2ai}\right)^k \\ &= (-2ai)^{-(p+1)} \sum_{k=0}^{\infty} \frac{1}{(-2ai)^k} \binom{-(p+1)}{k} x^k\end{aligned}$$

for $|x| < 2a$, i.e.

$$\begin{aligned}b_n &\sim (-2ai)^{-(p+1)} \sum_{k=0}^{p-1} \frac{1}{(-2ai)^k} \binom{-(p+1)}{k} x^{k-p} \\ &= (-2ai)^{-(p+1)} \sum_{k=1}^{p} \frac{1}{(-2ai)^{p-k}} \binom{-(p+1)}{p-k} \frac{1}{x^k}.\end{aligned}$$

Hence,

$$A_k = (-2ai)^{-2p+k-1} \binom{-(p+1)}{p-k}, \quad k = 1, 2, \dots, p.$$

Further, by putting $n - ai = y$ in b_n, we obtain

$$\begin{aligned}b_n &= \frac{1}{y^{p+1}(y + 2ai)^p} \\ &\sim \frac{(2ai)^{-p}}{y^{p+1}} \sum_{k=0}^{p} \binom{-p}{k} \left(\frac{y}{2ai}\right)^k \\ &= (2ai)^{-p} \sum_{k=1}^{p+1} \binom{-p}{p-k+1} \frac{1}{(2ai)^{p-k+1}} \frac{1}{y^k}.\end{aligned}$$

Hence,

$$B_k = (2ai)^{-2p+k-1} \binom{-p}{p-k+1}, \quad k = 1, 2, \dots, p+1.$$

Analogously,

$$c_n = \frac{1}{x^{p+1}(x - 2ai)^p} = \sum_{k=1}^{p+1} \frac{C_k}{x^k} + \sum_{k=1}^{p} \frac{D_k}{(x - 2ai)^k}.$$

Applying the same technique, for coefficients C_k, D_k we obtain

$$C_k = (-2ai)^{-2p+k-1}\binom{-p}{p-k+1}, \quad k = 1, 2, \dots, p+1,$$
$$D_k = (2ai)^{-2p+k-1}\binom{-(p+1)}{p-k}, \quad k = 1, 2, \dots, p.$$

Thus

$$\begin{aligned} a_n &= \frac{(2ai)^{-p}}{(n-ai)^{p+1}} + \frac{(-2ai)^{-p}}{(n+ai)^{p+1}} \\ &\quad + \sum_{k=1}^{p}(2ai)^{-2p+k-1}\left[\binom{-p}{p-k+1} + \binom{-(p+1)}{p-k}\right]\frac{1}{(n-ai)^k} \\ &\quad + \sum_{k=1}^{p}(-2ai)^{-2p+k-1}\left[\binom{-p}{p-k+1} + \binom{-(p+1)}{p-k}\right]\frac{1}{(n+ai)^k} \\ &= \frac{(2ai)^{-p}}{p!}\int_0^\infty t^p e^{-(n-ai)t}dt + \frac{(-2ai)^{-p}}{p!}\int_0^\infty t^p e^{-(n+ai)t}dt \\ &\quad + \sum_{k=1}^{p}(2ai)^{-2p+k-1}\left[\binom{-p}{p-k+1} + \binom{-(p+1)}{p-k}\right]\frac{1}{k!}\int_0^\infty t^k e^{-(n-ia)t}dt \\ &\quad + \sum_{k=1}^{p}(-2ai)^{-2p+k-1}\left[\binom{-p}{p-k+1} + \binom{-(p+1)}{p-k}\right]\frac{1}{k!}\int_0^\infty t^k e^{-(n+ia)t}dt. \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1}$$

and

$$\binom{-p}{p-k+1} + \binom{-(p+1)}{p-k} = \binom{-(p+1)}{p-k}\frac{1-k}{p-k+1},$$

we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &= \frac{(2ai)^{-p}}{p!}\int_0^\infty \frac{t^p}{e^t-1}e^{iat}dt + \frac{(-2ai)^{-p}}{p!}\int_0^\infty \frac{t^p}{e^t-1}e^{-iat}dt \\ &\quad + \sum_{k=1}^{p}(2ai)^{-2p+k-1}\binom{-(p+1)}{p-k}\frac{1-k}{k!(p-k+1)}\int_0^\infty \frac{t^k}{e^t-1}e^{iat}dt \\ &\quad + \sum_{k=1}^{p}(-2ai)^{-2p+k-1}\binom{-(p+1)}{p-k}\frac{1-k}{k!(p-k+1)}\int_0^\infty \frac{t^k}{e^t-1}e^{-iat}dt. \end{aligned}$$

Let $z = (2ai)^{-p}e^{iat}$ and $u = (2ai)^{-2p+k-1}e^{iat}$. Then

$$z + \bar{z} = \frac{2}{(2a)^p}\operatorname{Re}\left[\left(\cos\frac{p\pi}{2} - i\sin\frac{p\pi}{2}\right)(\cos at + i\sin at)\right] = \frac{2}{(2a)^p}\cos\left(\frac{p\pi}{2} - at\right)$$

and

$$u+\bar{u}=\frac{2\operatorname{Re}\left\{\left[\cos\frac{(2p-k+1)\pi}{2}-i\sin\frac{(2p-k+1)\pi}{2}\right](\cos at+i\sin at)\right\}}{(2a)^{2p+1-k}}$$
$$=\frac{2}{(2a)^{2p-k+1}}\cos\left[\frac{(2p-k+1)\pi}{2}-at\right].$$

Finally, we get

$$S(a,p)=\sum_{n=1}^{\infty}a_n$$
$$=\frac{2(2a)^{-p}}{p!}\int_0^\infty\frac{t^p}{e^t-1}\cos\left(\frac{p\pi}{2}-at\right)dt+\sum_{k=1}^{p}\frac{2}{(2a)^{2p-k+1}}\binom{-(p+1)}{p-k}\frac{1-k}{k!(p-k+1)}$$
$$\times\int_0^\infty\frac{t^k}{e^t-1}\cos\left[(2p-k+1)\frac{\pi}{2}-at\right]dt.$$

The proof is complete. □

Remark 2.2. Using the well-known formula for the polygamma function (see [1])

$$\psi^{(n)}(z)=(-1)^{n+1}\int_0^\infty\frac{t^ne^{-zt}}{1-e^{-t}}dt\qquad(n=1,2,3,\ldots,\operatorname{Re}z>0),$$

where $\psi(z)=\frac{d\ln\Gamma(z)}{dz}$, we obtain

$$\int_0^\infty\frac{t^p}{e^t-1}\cos\left(\frac{p\pi}{2}-at\right)dt$$
$$=\frac{e^{i\frac{p\pi}{2}}}{2}\int_0^\infty\frac{t^pe^{-t(1+ia)}}{1-e^{-t}}dt+\frac{e^{-i\frac{p\pi}{2}}}{2}\int_0^\infty\frac{t^pe^{-(1-ia)t}}{1-e^{-t}}dt$$
$$=\frac{e^{i\frac{p\pi}{2}}}{2}\psi^{(p)}(1+ia)+\frac{e^{-i\frac{p\pi}{2}}}{2}\psi^{(p)}(1-ia)$$
$$=\operatorname{Re}[e^{ip\pi/2}\psi^{(p)}(1+ia)].$$

Analogously,

$$\int_0^\infty\frac{t^p}{e^t-1}\cos\left[(2p-k+1)\frac{\pi}{2}-at\right]dt=\operatorname{Re}\left[e^{i(2p-k+1)\pi/2}\psi^{(p)}(1+ia)\right].$$

So for $S(a,p)$ we have the following expression

$$S(a,p)=\frac{2}{p!(2a)^p}\operatorname{Re}\left[e^{ip\pi/2}\psi^{(p)}(1+ia)\right]+\sum_{k=1}^{p}\frac{2(1-k)}{(2a)^{2p-k+1}k!(p-k+1)}\binom{-(p+1)}{p-k}\operatorname{Re}\left[e^{i(2p-k+1)\pi/2}\psi^{(p)}(1+ia)\right].\tag{2.2}$$

Remark 2.3. If $p>0$, $p\in\mathbb{R}$, then we have

$$\frac{2n}{(n^2+a^2)^{p+1}}=\frac{2}{\Gamma(p+1)}\int_0^\infty t^pne^{-(n^2+a^2)t}dt.$$

Using the Cauchy integration test, we obtain that $\sum_{n=1}^{\infty} ne^{-n^2 t}$ is convergent for all $t > 0$, i.e. $f(t) = \sum_{n=1}^{\infty} ne^{-n^2 t}$. Thus

$$S(a,p) = \frac{2}{\Gamma(p+1)} \int_0^{\infty} t^p e^{-a^2 t} \left(\sum_{n=1}^{\infty} ne^{-n^2 t} \right) dt = \frac{2}{\Gamma(p+1)} \int_0^{\infty} t^p e^{-a^2 t} f(t) dt. \tag{2.3}$$

Remark 2.4. In addition we set an open problem for summing up the functional series $\sum_{n=1}^{\infty} ne^{-n^2 t}$ for all $t > 0$.

3. THE INEQUALITY

Another one of our main results is to obtain a double inequality of $S(a,p)$ for $a > 0$ and $p \in \mathbb{N}$ by using Fourier transform inequalities (1.3) and (1.4).

Theorem 3.1. *For $a > 0$ and $p \in \mathbb{N}$, we have*

$$\begin{aligned} |S(a,p)| \le \frac{2}{a^{p+1}(2a)^p p!} &\left[\sum_{k=0}^{\infty} \frac{(-1)^k (k\pi)^p}{\exp \frac{k\pi}{a} - 1} + \sum_{k=0}^{\infty} (-1)^k \frac{((k+\frac{1}{2})\pi)^p}{\exp((k+\frac{1}{2})\frac{\pi}{a}) - 1} \right] \\ &+ \sum_{k=1}^{p} \frac{2(k-1)(2a)^{-2p+k-1}}{k!(p-k+1)a^{k+1}} \left| \binom{-(p+1)}{p-k} \right| \\ &\times \left[\sum_{j=0}^{\infty} \frac{(-1)^j (j\pi)^k}{\exp \frac{j\pi}{a} - 1} + \sum_{j=0}^{\infty} \frac{(-1)^j \left[\left(j+\frac{1}{2}\right) \pi \right]^k}{\exp \left[\left(j+\frac{1}{2}\right) \frac{\pi}{a} \right] - 1} \right]. \end{aligned} \tag{3.1}$$

Proof. For all $k = 1, 2, \ldots, p$, let

$$I(a,k) = \int_0^{\infty} \frac{t^k \cos at}{e^t - 1} dt \quad \text{and} \quad J(a,k) = \int_0^{\infty} \frac{t^k \sin at}{e^t - 1} dt.$$

Then

$$\begin{aligned} S(a,p) = \frac{2}{(2a)^p p!} &\left[I(a,p) \cos \frac{p\pi}{2} + J(a,p) \sin \frac{p\pi}{2} \right] \\ &+ \sum_{k=1}^{p} \frac{2(1-k)(2a)^{k-2p-1}}{k!(p-k+1)} \binom{-(p+1)}{p-k} \\ &\times \left[I(a,k) \cos \frac{(2p-k+1)\pi}{2} + J(a,k) \sin \frac{(2p-k+1)\pi}{2} \right]. \end{aligned}$$

Since

$$\begin{aligned} I(a,k) &= \frac{1}{a^{k+1}} \int_0^{\infty} \frac{t^k \cos t}{e^{t/a} - 1} dt, \\ J(a,k) &= \frac{1}{a^{k+1}} \int_0^{\infty} \frac{t^k \sin t}{e^{t/a} - 1} dt \end{aligned}$$

for fixed $a > 0$ and $k = 1, 2, \ldots, p$, and

$$f_k \in L([0,\infty)), \quad \lim_{t \to \infty} f_k(t) = \lim_{t \to \infty} \frac{t^k}{e^{t/a} - 1} = 0, \quad \lim_{t \to 0} f_k(t) = 0,$$

where $f_k(t) = \frac{t^k}{e^{t/a}-1}$, then, using inequalities (1.3) and (1.4), we have

$$
\begin{aligned}
|S(a,p)| &\le \frac{2}{p!(2a)^p}[I(a,p)+J(a,p)] \\
&\quad + \sum_{k=1}^{p} \frac{2(k-1)}{k!(p-k+1)(2a)^{2p-k+1}} \left| \binom{-(p+1)}{p-k} \right| [I(a,k)+J(a,k)] \\
&\le \frac{2(2a)^{-p}}{p!a^{p+1}} \left[\sum_{k=0}^{\infty} \frac{(-1)^k (k\pi)^p}{\exp\frac{k\pi}{a}-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \left[\left(k+\frac{1}{2}\right)\pi\right]^p}{\exp\left[\left(k+\frac{1}{2}\right)\frac{\pi}{a}\right]-1} \right] \\
&\quad + \sum_{k=1}^{p} \frac{2(k-1)(2a)^{-2p+k-1}}{k!a^{k+1}(p-k+1)} \left| \binom{-(p+1)}{p-k} \right| \\
&\quad \times \left[\sum_{j=0}^{\infty} \frac{(-1)^j (j\pi)^k}{\exp\frac{j\pi}{a}-1} + \sum_{j=0}^{\infty} \frac{(-1)^j \left[\left(j+\frac{1}{2}\right)\pi\right]^k}{\exp\left[\left(j+\frac{1}{2}\right)\frac{\pi}{a}\right]-1} \right].
\end{aligned}
$$

The proof is complete. □

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REFERENCES

[1] M. ABRAMOWITZ AND I.A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, Washington, 1965.

[2] P.S. BULLEN, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics **97**, Addison Wesley Longman Limited, 1998.

[3] O.E. EMERSLEBEN, Über die reihe $\sum_{k=1}^{\infty} k(k^2+c^2)^{-2}$, *Math. Ann.*, **125** (1952), 165–171.

[4] B.-N. GUO, Note on Mathieu's inequality, *RGMIA Res. Rep. Coll.*, **3**(3) (2000), Art. 5, 389–392. Available online at `http://rgmia.vu.edu.au/v3n3.html`.

[5] E. MATHIEU, *Traité de physique mathématique, VI–VII: Théorie de l'élasticité des corps solides*, Gauthier-Villars, Paris, 1890.

[6] F. QI, *Inequalities for Mathieu's series*, *RGMIA Res. Rep. Coll*., **4**(2) (2001), Art. 3, 187–193. Available online at `http://rgmia.vu.edu.au/v4n2.html`.

[7] F. QI AND Ch.-P. CHEN, Notes on double inequalities of Mathieu's series, *Internat. J. Pure Appl. Math.*, (2003), in press.