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EPI-DIFFERENTIABILITY AND OPTIMALITY CONDITIONS FOR AN EXTREMAL PROBLEM UNDER INCLUSION CONSTRAINTS

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ABSTRACT. In this paper, we establish first-order optimality conditions for the problem of minimizing a function f on the solution set of an inclusion $0 \in F(x)$ where f and the support function of a set-valued mapping F are epi-differentiable at $\overline{x}$.

Key words and phrases: Optimality Conditions, Epi-convergence, Epi-differentiability, Support function.

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1. INTRODUCTION

It is well known that epi-convergence of functions is coming to the fore as the correct concept for many situations in optimization. The strong feature of epi-convergence is that it corresponds to a geometric concept of approximation much like the one used in classical differential analysis (see [2]). Derivatives defined in terms of it therefore have a certain "robustness" that can be advantageous. Our principal objective is to give necessary and sufficient optimality conditions of type Ferma for the optimization problem

$$(P) \qquad \text{Maximize } f(x) \text{ subject to } 0 \in F(x)$$

where f is a function from a reflexive Banach space X into $\mathbb{R} \cup \{+\infty\}$ and F is a set valued map defined from X into another reflexive Banach space Y.

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The organization of the paper is as follows. Section 2 contains basic definitions and preliminaries that are widely used in the sequel. In Section 3 we study the epi-differentiability of the support function of F defined by $C_F(y^*,x) := \inf_{y\in F(x)} \langle y^*, y\rangle$ for every $y^* \in Y^*$. Section 4 is devoted to the optimality conditions and also for an application in mathematical programming problems.

2. PRELIMINARIES

Let F be a mapping defined in X with nonempty, closed and convex values in Y, and let X^* be the topological dual of X.

In all the sequel, we denote the domain of F and its graph by, respectively,

$$\mathrm{Dom}(F) := \{x \in X : F(x) \neq \emptyset\},$$

$$\mathrm{Gr}(F) := \{(x,y) \in X \times Y : y \in F(x)\}.$$

Let us recall some definitions.

Definition 2.1. A set-valued mapping F is said to be locally Lipschitz at $\overline{x}$ if there exists $\alpha > 0$ and $r > 0$ such that

$$F(x) \subset F(x') + \alpha \|x - x'\| \mathbb{B}_Y$$

for all x and x' in $\overline{x} + r\mathbb{B}_X$, where $\mathbb{B}_X$ indicates the unit ball centered at the origin in space X.

Let A be an arbitrary nonempty subset of X. The notions of contingent cone and tangent cone to A at a point $\overline{x} \in A$ will be used frequently in this paper.

The contingent cone to A at $\overline{x}$ is

$$K(S,\overline{x}) := \{v \in X : \exists\,(t_n) \searrow 0,\ \exists v_n \to v : \overline{x} + t_n v_n \in S,\ \forall n\}.$$

The tangent cone to A at $\overline{x}$ is

$$k(A,\overline{x}) := \left\{v \in X \ : \ \forall t_n \to 0^+, \exists v_n \to v \quad \overline{x} + t_n v_n \in A \ \ \forall n\right\}.$$

Definition 2.2. A set-valued map F is said to be proto-differentiable at $(\overline{x},\overline{y}) \in \mathrm{Gr}(F)$ if the contingent cone $K(\mathrm{Gr}(F),(\overline{x},\overline{y}))$ coincides with the tangent cone $k(\mathrm{Gr}(F),(\overline{x},\overline{y}))$.

The proto-derivative is thus the set-valued map denoted by $DF_{\overline{x},\overline{y},}$ the graph of which is the common set

$$\mathrm{Gr}(DF_{\overline{x},\overline{y}}) = K(\mathrm{Gr}F,(\overline{x},\overline{y})) = k(\mathrm{Gr}F,(\overline{x},\overline{y})).$$

For more details, see [1] and [9].

Lemma 2.1. *Let F be a Lipschitz set valued map at $\overline{x}$ and $\overline{y} \in F(\overline{x})$, one has*

i) $\displaystyle\limsup_{(t,v)\to(0^+,\overline{v})} D_tF_{\overline{x},\overline{y}}(v) = \limsup_{t\to 0^+} D_tF_{\overline{x},\overline{y}}(\overline{v}),$

ii) $\displaystyle\liminf_{(t,v)\to(0^+,\overline{v})} D_tF_{\overline{x},\overline{y}}(v) = \limsup_{t\to 0^+} D_tF_{\overline{x},\overline{y}}(\overline{v}),$

iii) $\displaystyle\lim_{(t,v)\to(0^+,\overline{v})} D_tF_{\overline{x},\overline{y}}(v) = \limsup_{t\to 0^+} D_tF_{\overline{x},\overline{y}}(\overline{v}),$

with $D_tF_{\overline{x},\overline{y}}(v) = t^{-1}(F(\overline{x}+tv) - \overline{y})$.

Proof. **i)** The inclusion " $\supset$ " is trivial. Let us prove the opposite inclusion. Consider any $w \in \displaystyle\limsup_{(t,v)\to(0^+,\overline{v})} D_tF_{\overline{x},\overline{y}}(v)$, there exists $(t_n,v_n,w_n) \to (0^+,\overline{v},w)$ such that $\overline{y} + t_n w_n \in F(\overline{x} + t_n v_n)$. As F is Lipschitz at $\overline{x}$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ one has

$$F(\overline{x} + t_n v_n) \subset F(\overline{x} + t_n \overline{v}) + \alpha t_n \|v_n - \overline{v}\| B_Y.$$

Since $\overline{y}+t_nw_n\in F(\overline{x}+t_nv_n)$, there exist $(\widetilde{w}_n)\subset Y$ and $(b_n)\subset B_Y$ such that for any $n\geq n_0$ we get $\overline{y}+t_nw_n=\overline{y}+t_n\widetilde{w}_n+\alpha t_n\|v_n-\overline{v}\|b_n$ and $\overline{y}+t_n\widetilde{w}_n\in F(\overline{x}+t_n\overline{v})$. This implies that $(t_n,\widetilde{w}_n)\to(0^+,w)$ and $\widetilde{w}_n\in D_{t_n}F_{\overline{x},\overline{y}}(\overline{v})$. Hence $w\in\limsup_{t\to 0^+}D_tF_{\overline{x},\overline{y}}(\overline{v})$.

ii) It suffices to prove that $\limsup_{(t,v)\to(0^+,\overline{v})}D_tF_{\overline{x},\overline{y}}(v)\subset\liminf_{t\to 0^+}D_tF_{\overline{x},\overline{y}}(\overline{v})$.

Let $w\in\limsup_{(t,v)\to(0^+,\overline{v})}D_tF_{\overline{x},\overline{y}}(v)$ and let $t_n\searrow 0^+$. Then there exists $w_n\to w$ such that $\overline{y}+t_nw_n\in F(\overline{x}+t_n\overline{v})$. Considering a sequence $v_n\to\overline{v}$; and using the Lipschitz property of F at $\overline{x}$, there exist $n_0\in\mathbb{N}$, $(\widetilde{w}_n)\subset Y$ and $(b_n)\subset B_Y$ such that $\overline{y}+t_nw_n=\overline{y}+t_n\widetilde{w}_n+\alpha t_n\|v_n-\overline{v}\|b_n$ and $\overline{y}+t_n\widetilde{w}_n\in F(\overline{x}+t_nv_n)$ for all $n\geq n_0$. Then $\widetilde{w}_n\in\limsup_{t\to 0^+}D_{t_n}F_{\overline{x},\overline{y}}(v_n)$ with $\widetilde{w}_n\to w$.

iii) It is a direct consequence of **i)** and **ii).**

□

Proposition 2.2. *Let F be a set valued map from X into Y and $(\overline{x},\overline{y})\in\mathrm{Gr}(F)$. If F is Lipschitz at $\overline{x}$ then F is proto-differentiable at $(\overline{x},\overline{y})$ with a proto-derivative $DF_{\overline{x},\overline{y}}(v)$ if and only if for every $v\in X$*

$$DF_{\overline{x},\overline{y}}(v)=\limsup_{t\to 0^+}D_tF_{\overline{x},\overline{y}}(v)\ \textit{exists.}$$

Proof. **i)** Fix $(v,w)\in K(\mathrm{Gr}\,F,(\overline{x},\overline{y}))$. There exists $(t_n,v_n,w_n)\to(0^+,v,w)$ such that $w_n\in D_{t_n}F_{\overline{x},\overline{y}}(v_n)$; that is $w\in\liminf_{(t,\widetilde{v})\to(0^+,v)}D_tF_{\overline{x},\overline{y}}(\widetilde{v})$. Using Lemma 2.1 one has $w\in\liminf_{(t,\widetilde{v})\to(0^+,v)}D_tF_{\overline{x},\overline{y}}(\widetilde{v})$. Consequently, for any $(t_n,\widetilde{v}_n)\to(0^+,v)$ there exists $\widetilde{w}_n\to w$ such that $(\overline{x},\overline{y})+t_n(\widetilde{v}_n,\widetilde{w}_n)\in\mathrm{Gr}(F)$. This implies that $(v,w)\in k(\mathrm{Gr}\,F,(\overline{x},\overline{y}))$, and hence the proto-differentiability of F at $(\overline{x},\overline{y})$.

ii) Fix $w\in\liminf_{t\to 0^+}D_tF_{\overline{x},\overline{y}}(v)$ and let $t_n\searrow 0$. From Lemma 2.1, one has $w\in\limsup_{(t,v')\to(0^+,v)}D_tF_{\overline{x},\overline{y}}(v')$ and consequently $(v,w)\in K(\mathrm{Gr}\,F,(\overline{x},\overline{y}))$. Thus by proto-differentiability $(v,w)\in k(\mathrm{Gr}\,F,(\overline{x},\overline{y}))$. Then there exists $(v_n,w_n)\to(v,w)$ such that $\overline{y}+t_nw_n\in F(\overline{x}+t_nv_n)$. Using the Lipschitz property of F at $\overline{x}$, there exists $\widetilde{w}_n\to w$ such that $\widetilde{w}_n\in D_{t_n}F_{\overline{x},\overline{y}}(v)$. Hence $w\in\limsup_{t\to 0^+}D_tF_{\overline{x},\overline{y}}(v)$; and the proof is finished.

□

In order to define the epi-derivative, as introduced by Rockafellar [8], let us recall the notion of epi-convergence and some of its main properties; for more details see [8].

A sequence of functions φ_n from X into $R\cup\{+\infty\}$ is said to be τ-epi-converging for a topology τ from X, and we denote by τ-elm φ_n its τ-epi-limit, if the two following conditions hold

$$\begin{cases}\forall\ x_n\xrightarrow{\tau}x & \varphi(x)\leq\limsup_{n\to+\infty}\varphi_n(x_n),\\ \exists\ x_n\xrightarrow{\tau}x & \varphi(x)\geq\limsup_{n\to+\infty}\varphi_n(x_n).\end{cases}$$

A family of functions $(\varphi_t)_{t>0}$ is said to be epi-converging to φ when $t\searrow 0$, if for every sequence $t_n\searrow 0$, the sequence of functions (φ_{t_n}) epi-converges to φ.

When s designs the strong topology of X and w its weak sequential topology, a sequence of functions (φ_n) is said to be Mosco-epi-converging to φ if (φ_n) s-epi-converges and w-epi-converges to φ, that is

$$\forall v_n\xrightarrow{w}v\quad\varphi(v)\leq\liminf\varphi_n(v_n),$$

$$\exists v_n\to v\quad\varphi(v)\geq\limsup\varphi_n(v_n)$$

Finally, recall that a sequence of subsets (C_n) τ-converges in the sense of Kuratawski-Painlevé to another subset C if the indicator functions δ_{C_n} of C_n τ -epi-converge to δ_C. Note, see [2], that if X is reflexive and $(C_n), C$ is a sequence of closed convex subsets, then (C_n) Mosco-converges to C if and only if $d(x, C_n) \to d(x, C)$ for all $x \in X$.

Let f be a function defined from X into $R \cup \{+\infty\}$, finite at a point $\overline{x}$. The function f is (Mosco-) epi-differentiable at $\overline{x}$ if the difference quotient functions

$$(\Delta_t f)_{\overline{x}}(\cdot) := t^{-1}\left(f(\overline{x}+t\cdot) - f(\overline{x})\right);\ t > 0,$$

have the property that the (Mosco-) epi-limit function $f'_{\overline{x}}$ exists and $f'_{\overline{x}}(0) > -\infty$. Note that the epigraph of $f'_{\overline{x}}$ is the proto-derivative of the epigraph of f at $(\overline{x}, f(\overline{x}))$.

By $\partial_e f(\overline{x})$, the epi-gradient of f at $\overline{x}$, we denote the set of all vectors $x^* \in X^*$ satisfying $f'_{\overline{x}}(v) \geq \langle x^*, v\rangle$ for all $v \in X$.

3. EPI-DIFFERENTIABILITY OF THE SUPPORT FUNCTION OF A SET-VALUED MAPPING

In the remainder of this paper we assume that X is reflexive and we denote by $\delta^*_A(\cdot)$ the support function of a subset A of X

$$\delta^*_A(x^*) := \sup_{x \in A} \langle x^*, x\rangle \qquad \text{for every } x^* \in X^*.$$

It is easy to see that for two subsets A and B of X

$$\delta^*_{A+B}(\cdot) = \delta^*_A(\cdot) + \delta^*_B(\cdot),$$
$$A \subset B \Longrightarrow \delta^*_A(\cdot) \leq \delta^*_B(\cdot),$$

and if A and B are closed convex then the last implication is an equivalence.

For the following,

- F will be a set-valued mapping from X into Y with a closed convex set-values.
- $N_{F(\overline{x})}(\overline{y})$ designs the normal cone to $F(\overline{x})$ at $\overline{y} \in F(\overline{x})$, i.e.

$$N_{F(\overline{x})}(\overline{y}) := \{y^* \in Y^* : \langle y^*, y - \overline{y}\rangle \leq 0 \quad \text{for all } y \in F(\overline{x})\}$$
$$= \left\{y^* \in Y^* : \langle y^*, \overline{y}\rangle = \delta^*_{F(\overline{x})}(y^*)\right\}.$$

- We denote by $Y^*_F := \left\{y^* \in Y^* : \delta^*_{F(\overline{x})}(y^*) < +\infty\right\}$ the barrier cone of F. It is easy to prove that when F is locally Lipschitz, see [5], the set Y^*_F does not depend on $\overline{x}$.

Definition 3.1. A function $f : X \to R \cup \{+\infty\}$ is said to be (Mosco-) epi-regular at $\overline{x}$ if the (Mosco-) epi-derivative of f exists at $\overline{x}$ and coincides with the directional derivative of f at $\overline{x}$.

Lemma 3.1. *Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. Suppose that F is Lipschitz at $\overline{x}$, then the function $\psi_t(v) := \delta^*_{D_t F_{\overline{x},\overline{y}}(v)}(y^*)$ is equi-locally Lipschitz.*

Proof. Fix $\overline{v} \in X$. As F is Lipschitz at $\overline{x}$, there exist $\alpha > 0$ and $r > 0$ such that for any $t \in]0, r]$ and any $v, v' \in \overline{v} + rB_X$

$$F(\overline{x} + tv) \subset F(\overline{x} + tv') + \alpha t \|v - v'\| B_Y.$$

Consequently $(D_t F)_{\overline{x},\overline{y}}(v) \subset (D_t F)_{\overline{x},\overline{y}}(v') + \alpha \|v - v'\| B_Y$. Hence

$$|\psi_t(v) - \psi_t(v')| \leq \alpha \|y^*\| \|v - v'\|$$

for any $t \in]0, r]$ and any $v, v' \in \overline{v} + rB_X$.

The proof is thus complete. □

Proposition 3.2. *Let* $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. *Suppose that* Y *is reflexive and that for every sequence* $t_n \searrow 0$

$$d\left(\cdot, D_{t_n}F_{\overline{x},\overline{y}}\left(v\right)\right) \longrightarrow d\left(\cdot, DF_{\overline{x},\overline{y}}\left(v\right)\right). \tag{3.1}$$

Then:

i) *the set-valued map* $N_{D_{t_n}F_{\overline{x},\overline{y}}(v)}$ *graph-converges to* $N_{DF_{\overline{x},\overline{y}}(v)}$,

ii) *there exists* $w \in DF_{\overline{x},\overline{y}}\left(v\right)$, $y^* \in N_{DF_{\overline{x},\overline{y}}(v)}\left(w\right)$, $w_n \in D_{t_n}F_{\overline{x},\overline{y}}\left(v\right)$ *and* $y_n^* \in N_{D_{t_n}F_{\overline{x},\overline{y}}(v)}\left(w_n\right)$ *such that* $(w_n, y_n^*) \longrightarrow (w, y^*)$ *and* $\delta^*_{D_{t_n}F_{\overline{x},\overline{y}}(v)}(y_n^*) \longrightarrow \delta^*_{DF_{\overline{x},\overline{y}}(v)}(y^*)$.

Motivated by the article of Demyanov, Lemaréchal and Zowe [4], where the authors approximate F under the assumption that

$$\delta^*_{DF_{\overline{x},\overline{y}}(\cdot)}(y^*) := \lim_{t\searrow 0, h\to d} t^{-1}\left(\delta^*_{F(x+th)}(y^*) - \delta^*_{F(x)}(y^*)\right)$$

exists (as an element of $\mathbb{R}$) for every $y^* \in \mathbb{R}^p$, we give in Theorem 3.3 sufficient conditions insuring the existence of this derivative.

Theorem 3.3. *Let* $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. *Suppose that* Y *is reflexive,* F *is directionally Lipschitz at* $\overline{x}$ *and that* $N_{F(\overline{x})}(\overline{y}) \neq \emptyset$. *Suppose also that the condition (3.1) is satisfied for each* v *and that the function* $\delta^*_{D_{t_n}F_{\overline{x},\overline{y}}(v)}(\cdot)$ *is equi-Lipschitz. i.e.* $\exists \beta \geq 0$ *such that*

$$\delta^*_{C_n(v)}(y_n^*) \geq \delta^*_{C_n(v)}(y^*) - \beta \left\| y_n^* - y^* \right\|.$$

Then for all $y^* \in N_{F(\overline{x})}(\overline{y})$, *the function* $f(x) := \delta^*_{F(x)}(y^*) = \sup_{y\in F(x)} \langle y^*, y\rangle$ *is epi-regular at* $\overline{x}$, *with* $\delta^*_{DF_{\overline{x},\overline{y}}(\cdot)}(y^*)$ *as its epi-derivative*.

Proof. Let $t_n \searrow 0$. By definition of f one has

$$t_n^{-1}[f(\overline{x} + t_n v) - f(\overline{x})] = \delta^*_{D_{t_n}F_{\overline{x},\overline{y}}(v)}(y^*).$$

Setting $C_n\left(v\right) := D_{t_n}F_{\overline{x},\overline{y}}\left(v\right)$, $C\left(v\right) := DF_{\overline{x},\overline{y}}\left(v\right)$ and $\Psi_n\left(v\right) := \delta^*_{C_n(v)}(y^*)$, then condition (3.1) permits us to conclude that $\delta_{C_n(v)}$ Mosco-converges to $\delta_{C(v)}$. Using Attouch's theorem [2], we conclude that $\delta^*_{C_n(v)}$ Mosco-converges to δ^*_{C}. Hence

a) for any $y_n^* \xrightarrow{w^*} y^*$ one has $\delta^*_{C(v)}(y^*) \leq \liminf \delta^*_{C_n(v)}(y_n^*)$,

b) there exists $y_n^* \xrightarrow{s} y^*$ such that $\delta^*_{C(v)}(y^*) \geq \limsup \delta^*_{C_n(v)}(y_n^*)$.

Let us prove that

i) for any $v_n \to v$ one has $\Psi\left(v\right) := \delta^*_{C(v)}(y^*) \leq \liminf \Psi_n(v_n)$,

ii) there exists $v_n \to v$ such that $\Psi\left(v\right) \geq \limsup \Psi_n(v_n)$.

i) Let $v_n \to v$. From Lemma 3.1, there exists $n_0 \in N$ such that for any $n \geq n_0$

$$\Psi_n(v_n) \geq \Psi_n(v) - \alpha \left\| y^* \right\| \left\| v_n - v \right\|.$$

Letting $n \to +\infty$ we get $\liminf \Psi_n(v_n) \geq \liminf \Psi_n(v)$. Finally, using a), we have $\liminf \Psi_n(v_n) \geq \Psi\left(v\right)$. The result is thus proved.

ii) Considering b), there exists $y_n^* \xrightarrow{s} y^*$ such that $\Psi\left(v\right) \geq \limsup \delta^*_{C_n(v)}(y_n^*)$. Since $\delta^*_{C_n(v)}(\cdot)$ is equi-Lipschitz, there exists $\beta \geq 0$ such that

$$\delta^*_{C_n(v)}(y_n^*) \geq \delta^*_{C_n(v)}(y^*) - \beta \left\| y_n^* - y^* \right\|.$$

Hence $\limsup \Psi_n(v_n) \leq \Psi(v)$. Moreover, since F is directionally Lipschitz at $\overline{x}$ on has

$$\lim_{t\to 0^+} \delta^*_{(D_tF)_{\overline{x},\overline{y}}(v)}(y^*) = \lim_{(t,v')\to(0^+,v)} \delta^*_{(D_tF)_{\overline{x},\overline{y}}(v')}(y^*)$$

Thus

$$\delta^*_{(DF)_{\overline{x},\overline{y}}(v)}(y^*) = \lim_{t\to 0^+} \delta^*_{(D_tF)_{\overline{x},\overline{y}}(v)}(y^*) = f'_{\overline{x}}(v) = \lim_{t\to 0^+} t^{-1}[f(\overline{x}+tv) - f(\overline{x})].$$

The proof of the theorem is complete. □

Remark 3.4. When, instead of (3.1), we assume the Mosco-Proto-differentiability of F at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$; we can justify the Mosco-epi-regularity of f at $\overline{x}$.

Indeed, suppose that there exists $v_n \xrightarrow{w} v$ such that $D_{t_n}F_{\overline{x},\overline{y}}(v_n)$ does not Mosco-converge to $DF_{\overline{x},\overline{y}}(v)$. Consequently, there exists $z_n \xrightarrow{w} z$ such that $z_n \in t_n^{-1}(F(\overline{x}+t_nv_n) - \overline{y})$ and $z \notin DF_{\overline{x},\overline{y}}(v)$. Thus $(v_n, z_n) \xrightarrow{w} (v,z)$, $(v_n,z_n) \in t_n^{-1}(\operatorname{Gr} F - (\overline{x},\overline{y}))$ and $z \notin DF_{\overline{x},\overline{y}}(v)$; which contradicts F Mosco-Proto-differentiable at $\overline{x}$.

Theorem 3.5. *Let $f : X \to R \cup \{+\infty\}$ be a function epi-differentiable at $\overline{x}$ and let $g : X \to R \cup \{+\infty\}$ be a function epi-regular at $\overline{x}$. Then $f+g$ is epi-differentiable at $\overline{x}$.*

Proof. Setting

$$a_n(v) := t^{-1}[f(\overline{x}+t_nv) - f(\overline{x})], b_n(v) := t^{-1}[g(\overline{x}+t_nv) - g(\overline{x})]$$

and

$$c_n(v) := t^{-1}[(f+g)(\overline{x}+t_nv) - (f+g)(\overline{x})].$$

i) Let $v_n \xrightarrow{w} v$. Since f and g are epi-differentiable at $\overline{x}$, we have

$$\liminf_{n\to\infty} c_n(v_n) \geq \liminf_{n\to\infty} a_n(v_n) + \liminf_{n\to\infty} b_n(v_n) \geq a(v) + b(v).$$

ii) Since f and g are epi-differentiable at $\overline{x}$, there exist $v_n^1 \xrightarrow{s} v$ and $v_n^2 \xrightarrow{s} v$ such that

$$b(v) \geq \liminf_{n\to\infty} b_n(v_n^2) \text{ and } a(v) \geq \liminf_{n\to\infty} a_n(v_n^1).$$

Consequently

$$\liminf_{n\to\infty} c_n(v_n^1) \leq \liminf_{n\to\infty} a_n(v_n^1) + \liminf_{n\to\infty} b_n(v_n^1) \leq a(v) + \liminf_{n\to\infty} b_n\left(v_n^1\right).$$

Using the epi-regularity of g at $\overline{x}$,

$$\liminf_{n\to\infty} b_n(v_n^1) = \liminf_{n\to\infty} b_n(v_n^2) = \liminf_{n\to\infty} b_n(v).$$

Then

$$\liminf_{n\to\infty} c_n(v_n^1) \leq a(v) + b(v).$$

The conclusion is thus immediate, that is, $f+g$ is epi-differentiable at $\overline{x}$. □

4. OPTIMALITY CONDITIONS

Fix $y^* \in Y^*$ and let $C_F(y^*,x) := \inf_{y\in F(x)} \langle y^*, y\rangle$,

$$N_{F(\overline{x})}(\overline{y}) = \{y^* \in Y^* : \langle y^*, \overline{y}\rangle = C_F(y^*,\overline{x})\}$$

and

$$Y_F^* = \{y^* \in Y^* : C_F(y^*,\overline{x}) > -\infty\}.$$

Let $I(x) := \{y^* \in Y_F^* \text{ such that } C_F(y^*,x) = d_F(x)\}$. In particular, if $\overline{x} \in C$ we have $I(\overline{x}) := \{y^* \in Y_F^* \text{ such that } C_F(y^*,\overline{x}) = 0\}$. Consider, when $C_F(y^*,\cdot)$ is epi-differentiable at $\overline{x}$

$$D_F(\overline{x}) := \{d \in X : \forall y^* \in I(\overline{x}),\ \forall x^* \in \partial_e C_F(y^*,\cdot)(\overline{x}) \qquad \langle x^*, d\rangle \leq 0\}$$

and

$$H_F(\overline{x}) := \left\{d \in X : \forall y^* \in I(\overline{x}) \qquad C'_F(y^*,\cdot)_{\bar{x}}(d) \leq 0\right\}.$$

Definition 4.1. $\overline{x}$ is said to be regular if there exist two reals $\lambda, r > 0$ such that

$$d(0, F(x)) \geq \lambda\, d(x, F^{-}(0))$$

for all $x \in \overline{x} + r\mathbb{B}_X$.

For every $x \in X$, let $d_F(x) := d(0, F(x))$.

Proposition 4.1. *The following inclusions are always true*

$$K(F^{-}(0), \overline{x}) \subset H_F(\overline{x}) \subset D_F(\overline{x}).$$

Proof. Let $d \in K(F^{-}(0), \overline{x})$. There exists $(t_n, d_n) \to (0^{+}, d)$ such that $\overline{x} + t_n d_n \in F^{-}(0)$. Consider $y^* \in I(\overline{x})$ and $x^* \in X^*$ such that $C_F^{'}(y^*, \cdot)_{\bar{x}}\,(v) \geq \langle x^*, v\rangle$ for all $v \in X$. All we have to show is that $C_F^{'}(y^*, \cdot)_{\bar{x}}\,(d) \leq 0$. Indeed, since $0 \in F(\overline{x}+t_n d_n)$ we have $C_F(y^*, \overline{x}+t_n d_n) \leq 0$. But $C_F(y^*, \overline{x}) = 0$, consequently $C_F^{'}(y^*, \cdot)_{\bar{x}}\,(d) \leq \liminf_{n\to\infty} t_n^{-1}[C_F(y^*, \overline{x}+t_n d_n) - C_F(y^*, \overline{x})] \leq 0$. Then $C_F^{'}(y^*, \cdot)_{\bar{x}}\,(d) \leq 0$ and $\langle x^*, d\rangle \leq 0$. The proof is thus complete. □

The following lemma will play a very crucial role in the remainder of the paper.

Lemma 4.2. *Let ∂d_F be the Clarke subdifferential. We assume F to have the following properties.*

- **i.** *F is Lipschitz at $\overline{x}$,*
- **ii.** *$\partial_e C_F(y^*, \cdot)\,(x)$ is upper semicontinuous (at (y^*, x)) when X^*, Y^* are endowed with the weak-star topology and X with the strong topology, that is, if $x_n^* \in \partial_e C_F(y_n^*, \cdot)\,(x_n)$ where $x_n^* \overset{w^*}{\to} x^*$ in X^*, $y_n^* \overset{w^*}{\to} y^*$ in Y^* and $x_n \to x$ in X, then $x^* \in \partial_e C_F(y^*, \cdot)\,(x)$.*

Then

$$\partial d_F(\overline{x}) \subset co\,\{\partial_e C_F(y^*, \overline{x}) : y^* \in I(\overline{x}) \cap \mathbb{B}_Y^*\}\,.$$

Proof. To prove the lemma, we need the following result of Thibault [11].

Let $h : X \to \mathbb{R}$ be a locally Lipschitzian function, H a subset of X such that X/H is Haar-nul set and at every $x \in H$, the function h is Gateaux differentiable and has Gateaux differential $\nabla h\,(x)$. Then we have

(1) $h\,(x, v) = \max\{\langle x^*, v\rangle : x^* \in L_H\,(h, x)\}$ for every $v \in X$,
(2) $\partial h\,(x) = \{\overline{co} L_H\,(h, x)\}\,,$

where $L_H\,(h, x) = \left\{\limsup_{n\to\infty} \nabla h\,(x_n) : x_n \in H,\ x_n \to x\right\}$ and the "limit " of $\{\nabla h\,(x_n)\}$ is in the weak* topology.

Now, from Christensen's Theorem [3] applied to the locally Lipschitzian function d_F it follows that there exists a subset $M \subset X$ such that d_F is Gateaux differentiable on M and X/M is a Haar-nul set.

For every $x_n \in M$, $y_n^* \in I\,(x_n) \cap \mathbb{B}_Y^*$, $v \in X$ we have

$$\begin{aligned}\langle \nabla d_F\,(x_n)\,, v\rangle &= \lim_{\varepsilon\to 0} \frac{d_F\,(x_n + \varepsilon v) - d_F\,(x_n)}{\varepsilon}\\ &\leq \lim_{\varepsilon\to 0} \frac{C_F(y_n^*, x_n + \varepsilon v) - C_F(y_n^*, x_n)}{\varepsilon}\\ &\leq \limsup_{k\to\infty} \frac{C_F(y_n^*, x_n + \varepsilon_k v) - C_F(y_n^*, x_n)}{\varepsilon_k}.\end{aligned}$$

Since C_F is epi-derivable, there exists $v_k \to v$ such that

$$C_F^{\prime}(y_n^*, \cdot)_{x_n}\,(v) \geq \limsup_{k\to\infty} \frac{C_F(y_n^*, x_n + \varepsilon_k v_k) - C_F(y_n^*, x_n)}{\varepsilon_k}.$$

On the other hand, C_F is $\alpha \|y_n^*\|$-Lipschitz. Consequently

$$\begin{aligned} \langle \nabla d_F(x_n), v\rangle &\leq \limsup_{k\to\infty} \frac{C_F(y_n^*, x_n + \varepsilon_k v_k) - C_F(y_n^*, x_n) + \alpha\varepsilon_k \|y_n^*\| \|v_k - v\|}{\varepsilon_k} \\ &\leq C_F'(y_n^*, \cdot)_{x_n}(v) + \limsup_{k\to\infty} \frac{\alpha\varepsilon_k \|y_n^*\| \|v_k - v\|}{\varepsilon_k} \\ &\leq C_F'(y_n^*, \cdot)_{x_n}(v). \end{aligned}$$

Hence

$$\nabla d_F(x_n) \in \partial_e C_F(y_n^*, \cdot)_{x_n}. \tag{4.1}$$

It is easily seen that the set-valued map $x \mapsto I(x) \cap \mathbb{B}_Y^*$ is upper semi-continuous. Moreover, since $\partial_e C_F(y^*, \cdot)(x)$ is upper semicontinuous, the set valued-map $x \mapsto G(x)$ defined by

$$G(x) := \{x^* : x^* \in \partial_e C_F(y^*, x) \text{ and } y^* \in I(x) \cap \mathbb{B}_Y^*\}$$

is upper semi-continuous as well.

From (4.1), we have

$$L_M(d_F, x) \subset G(x).$$

The lemma now follows from the second part of the mentioned result of Thibault and the compactness of the set $G(x)$ (in the weak topology). □

In the following, we give necessary optimality conditions of type Fermat. Throughout the reminder of the paper, we assume that the function f is epi-differentiable, the support function $C_F(z^*, \cdot)$ of the set valued-map F is epi-differentiable and that $\partial_e C_F(y^*, \cdot)(x)$ is upper semicontinuous.

Theorem 4.3. *Consider problem (P). Suppose also that $\overline{x}$ is regular.*

If $\bar{x}$ is a solution of (P) then

$$f_{\overline{x}}'(v) \leq 0 \tag{4.2}$$

for all $v \in D_F(\overline{x})$.

We begin by giving an important lemma which we shall use later on.

Lemma 4.4. *If $\overline{x}$ is regular then*

$$H_F(\overline{x}) = D_F(\overline{x}) = K(F^-(0), \bar{x}).$$

Proof. All we have to show is that $D_F(\overline{x}) \subset K(F^-(0), \bar{x})$. Let $d \in D_F(\overline{x})$ with $d \neq 0$. Without loss of generality we can assume that $\|d\| = 1$. As $\overline{x}$ is regular, we can fix $r > 0$ and $\lambda > 0$ such that

$$d_F(x) \geq \lambda d(x, F^-(0)) \tag{4.3}$$

for all $x \in \overline{x} + r\mathbb{B}_X$.

Let $t_n \searrow 0$. Since $\partial d_F(\cdot)$ is upper semicontinuous, there exists $r_n > 0$ such that

$$\partial d_F(x) \subset \partial d_F(\overline{x}) + t_n \lambda \mathbb{B} \tag{4.4}$$

for all $x \in \overline{x} + r_n \mathbb{B}_X$.

Setting $\mu_n := \min(r, r_n, t_n)$, and by Lebourg's Mean value Theorem [7], we can assert that for any $n \in \mathbb{N}$ there exist $x_n \in [\overline{x}, \overline{x} + \mu_n d]$ and $x_n^* \in \partial d_F(x_n)$ such that

$$d_F(\overline{x} + \mu_n d) - d_F(\overline{x}) = \langle x_n^*, \mu_n d\rangle \leq \sup_{y_n^* \in \partial d_F(x_n)} \langle y_n^*, \mu_n d\rangle = \mu_n \sup_{y_n^* \in \partial d_F(x_n)} \langle y_n^*, d\rangle.$$

Observe that $x_n \in [\overline{x}, \overline{x} + \mu_n d] \subset \overline{x} + r_n\mathbb{B}$. From (4.4) we get

$$\sup_{y_n^* \in \partial d_F(x_n)} \langle y_n^*, d\rangle \leq \sup_{y_n^* \in \partial d_F(\overline{x})} \langle y_n^*, d\rangle + t_n\lambda.$$

By virtue of Lemma 4.2,

$$\partial d_F(\overline{x}) \subset co\{\partial_e C_F(y^*, \overline{x}) : y^* \in I(\overline{x}) \cap \mathbb{B}_Y^*\}$$

and consequently $d_F(\overline{x}+\mu_n d) \leq \mu_n t_n \lambda$. Taking account of (4.3), we deduce $d(\overline{x}+\mu_n d, F^-(0)) \leq \mu_n t_n < 2\mu_n t_n$.

This implies the existence of a sequence (v_n) such that for n large enough, $\overline{x}+\mu_n v_n \in F^-(0)$ and $\|\overline{x} + \mu_n d - (\overline{x} + \mu_n v_n)\| = \mu_n \|v_n - d\| < 2\mu_n t_n$. Hence $d \in K(F^-(0), \overline{x})$. This ends the proof of the lemma. □

Proof of Theorem 4.3. Let $v \in D_F(\overline{x})$. By virtue of Lemma 4.4, $v \in K(F^-(0), \bar{x})$, hence there exist $(t_n) \searrow 0$ and $v_n \to v$ such that $\overline{x} + t_n v_n \in F^-(0)$. Since $\overline{x}$ is a solution of (P), there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ $f(\overline{x} + t_n v_n) \leq f(\overline{x})$. From the epi-differentiability of f at $\overline{x}$, one gets $f'_{\overline{x}}(v) \leq \liminf_{n\to\infty}(\Delta_{t_n} f)_{\overline{x}}(v_n) \leq 0$. The proof is thus complete. □

Remark 4.5. Without the regularity of $\overline{x}$, the proof of Theorem 4.3 permit to get

$$f'_{\overline{x}}(v) \leq 0 \leq C'_F(y^*, \cdot)_{\bar{x}}(v) \qquad \text{for every } v \in K(F^-(0), \bar{x}) \text{ and every } y^* \in I(\overline{x}).$$

By virtue of the complexity of $F^-(0)$, we were forced to adopt $D_F(\overline{x})$ instead of $K(F^-(0), \bar{x})$ in Theorem 4.3.

Theorem 4.6. *Consider problem (P) and let us assume that* $\dim(X) < +\infty$. *Suppose that* $\overline{x}$ *is regular and that* f *is Lipschitz at* $\overline{x}$.

Then $\bar{x}$ *is a solution of (P) whenever* $f'_{\overline{x}}(v) < 0$ *for any* $v \in D_F(\overline{x})\setminus\{0\}$.

Proof. Assume the contrary, that is, that the statement of Theorem 4.6 is not true. Then there exists a sequence $(x_n) \subset F^-(0)$ satisfying $x_n \to \overline{x}$ and $f(x_n) > f(\overline{x})$ $\forall n$. Let $t_n := \|x_n - \overline{x}\|$ and $v_n := \dfrac{x_n - \overline{x}}{t_n}$. Since $\dim(X)$ is finite, there exists $\overline{v} \in X$ with $\|\overline{v}\| = 1$ and a subsequence noted another time (v_n) such that $v_n \to \overline{v}$. From the epi-differentiability of f at $\overline{x}$, there exists $\widetilde{v}_n \to \overline{v}$ such that

$$f'_{\overline{x}}(\overline{v}) \geq \limsup_{n\to+\infty} \frac{f(\overline{x} + t_n\widetilde{v}_n) - f(\overline{x})}{t_n}.$$

Setting $a_n := t_n^{-1}[f(\overline{x} + t_n\widetilde{v}_n) - f(\overline{x})]$, $b_n := t_n^{-1}[f(\overline{x} + t_n v_n) - f(\overline{x})]$ and $c_n := b_n - a_n$. We have that f is Lipschitz at $\overline{x}$, $\limsup_{n\to+\infty} c_n = 0$ and that $b_n \geq 0$; consequently $f'_{\overline{x}}(\overline{v}) \geq \limsup_{n\to+\infty} a_n \geq \limsup_{n\to+\infty} b_n - \limsup_{n\to+\infty} c_n \geq 0$. This conflicts with $\overline{v} \in D_F(\overline{x})\setminus\{0\}$ and the theorem follows. □

f is said to be hypo-differentiable at $\overline{x}$, with $f'_{h,\overline{x}}$ as its hypo-derivative, if $-f$ is epi-differentiable at $\overline{x}$. In this case $f'_{h,\overline{x}} = -(-f)'_{\overline{x}}$.

Theorem 4.7. *Consider problem (P) and let us assume that* $\dim(X) < +\infty$. *Suppose that* $\overline{x}$ *is regular and that* f *is hypo-differentiable at* $\overline{x}$.

Then $\bar{x}$ *is a solution of (P) whenever* $f'_{h,\overline{x}}(v) < 0$ *for any* $v \in D_F(\overline{x})\setminus\{0\}$.

Proof. The argument is slightly similar to that used above, but we give it for the convenience of the reader. Assume the contrary, that is, that the statement of Theorem 4.6 is not true. Then there exists a sequence $(x_n) \subset F^-(0)$ satisfying $x_n \to \overline{x}$ and $f(x_n) > f(\overline{x})$ $\forall n$. Let $t_n := \|x_n - \overline{x}\|$

and $v_n := \dfrac{x_n - \overline{x}}{t_n}$. Since $\dim(X)$ is finite, there exists $\overline{v} \in X$ with $\|\overline{v}\| = 1$ and a subsequence noted another time (v_n) such that $v_n \to \overline{v}$. From the hypo–differentiability of f at $\overline{x}$,

$$(-f)'_{\overline{x}}(\overline{v}) \leq \liminf_{n \to +\infty} \frac{(-f)(\overline{x} + t_n v_n) - (-f)(\overline{x})}{t_n} \leq 0.$$

Hence $f'_{h,\overline{x}}(\overline{v}) \geq 0$. This is a contradiction since $\overline{v} \in D_F(\overline{x}) \setminus \{0\}$. □

As an application for the above results, we are concerned with the mathematical programming problem

$$(P^*) \qquad \begin{aligned} &\max \ f(x) \\ &\text{subject to } g_i(x) \leq 0 \text{ and } h_j(x) = 0 \\ &\text{for all } i \in \{1, 2, \ldots, m\} \text{ and all } j \in \{1, 2, \ldots, k\}. \end{aligned}$$

Let $C := \{x : g_i(x) \leq 0,\ h_j(x) = 0 \text{ for all } i,\ j\}$. Let $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))$ and $h(x) = (h_1(x), h_2(x), \ldots, h_k(x))$. The problem ($P^*$) reduces to ($P$), where the set-valued mapping $F : X \rightrightarrows Y = \mathbb{R}^m \times \mathbb{R}^k$ is defined by

$$F(x) := (g(x), h(x)) + \mathbb{R}^m_+ \times \{0_{\mathbb{R}^k}\}.$$

Obviously, in that case $Y^*_F = \mathbb{R}^m_+ \times \mathbb{R}^k$ and for any $y^* = (\lambda, \mu) \in Y^*_F$ we have

$$C_F(y^*, x) = \langle \lambda, g(x) \rangle + \langle \mu, h(x) \rangle.$$

It can be verified that $C_F(y^*, \overline{x}) = 0$ if and only if $\lambda_i g_i(\overline{x}) = 0$ for all $i \in \{1, 2, \ldots, m\}$.

Then $I(\overline{x}) = \left\{(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k : \lambda_i g_i(\overline{x}) = 0 \ \forall i = 1, \ldots, m\right\}$, and consequently

$$H_F(\overline{x}) := \left\{ v \in X : \forall (\lambda, \mu) \in I(\overline{x}) \quad \sum_{i=1}^{m} \lambda_i g'_{i\overline{x}}(v) + \sum_{j=1}^{k} \mu_j h'_{j\overline{x}}(v) \leq 0 \right\}.$$

We deduce from Theorem 4.3 and Theorem 4.6 the following optimality conditions for problem (P^*).

Theorem 4.8. *Let $\overline{x}$ be a solution of (P^*). Suppose that the functions f and h_j are epi-differentiable at $\overline{x}$, the functions g_i are epi-regular at $\overline{x}$ and there exist $r > 0$ and $\lambda > 0$ such that*

$$d(g(x), \mathbb{R}^m_-) \leq \lambda \, d(x, C) \tag{4.5}$$

for every $x \in \overline{x} + r\mathbb{B}_X$.

Then for any $v \in X$ such that $\forall (\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k$ satisfying $\lambda_i g_i(\overline{x}) = 0$ and $\sum_{i=1}^m \lambda_i g'_{i\overline{x}}(v) + \sum_{j=1}^k \mu_j h'_{j\overline{x}}(v) \leq 0$ we have

$$f'_{\overline{x}}(v) \leq 0.$$

Remark 4.9. The condition (4.5) ensures the regularity of $\overline{x}$.

Theorem 4.10. *Suppose that $\dim(X) < \infty$ and that f is Lipschitz at $\overline{x}$.*

$\overline{x}$ will be a solution of (P^*) if for any $v \in X \setminus \{0\}$ such that $\forall (\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^k$, verifying $\lambda_i g_i(\overline{x}) = 0$ and

$$\sum_{i=1}^{m} \lambda_i g'_{i\overline{x}}(v) + \sum_{j=1}^{k} \mu_j h'_{j\overline{x}}(v) \leq 0,$$

we have

$$f'_{\overline{x}}(v) < 0.$$

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