Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 2, Article 27, 2003

INEQUALITIES ARISING OUT OF THE VALUE DISTRIBUTION OF A DIFFERENTIAL MONOMIAL

INDRAJIT LAHIRI AND SHYAMALI DEWAN

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF KALYANI,
WEST BENGAL 741235, INDIA.
indrajit@cal2.vsnl.net.in

Received 4 June, 2002; accepted 23 January, 2003
Communicated by A. Sofo

ABSTRACT. In the paper we derive two inequalities that describe the value distribution of a differential monomial generated by a transcendental meromorphic function and which improve some earlier results.

Key words and phrases: Meromorphic function, Monomial, Value distribution, Inequality.

2000 *Mathematics Subject Classification.* 30D35, 30A10.

1. INTRODUCTION AND DEFINITIONS

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. We do not explain the standard definitions and notations of the value distribution theory as these are available in [3].

Definition 1.1. A meromorphic function $\alpha \equiv \alpha(z)$ defined in $\mathbb{C}$ is called a small function of f if $T(r, \alpha) = S(r, f)$.

Hiong [5] proved the following inequality.

Theorem A. If a, b, c are three finite complex numbers such that $b \neq 0$, $c \neq 0$ and $b \neq c$ then

$$T(r, f) \leq N(r, a; f) + N(r, b; f^{(k)}) + N(r, c; f^{(k)}) - N(r, 0; f^{(k+1)}) + S(r, f).$$

Improving Theorem A, K.W.Yu [7] proved the following result.

ISSN (electronic): 1443-5756

The second author is thankful to C.S.I.R. for awarding her a junior research fellowship.

The authors are thankful to the referee for valuable comments towards the improvement of the paper.

066-02

Theorem B. Let $\alpha(\not\equiv 0,\infty)$ be a small function of f, then for any finite non-zero distinct complex numbers b and c and any positive integer k for which $\alpha f^{(k)}$ is non-constant, we obtain

$$T(r,f) \le N(r,0;f) + N(r,b;\alpha f^{(k)}) + N(r,c;\alpha f^{(k)}) - N(r,\infty;f) - N(r,0;\left(\alpha f^{(k)}\right)') + S(r,f).$$

Recently K.W.Yu [8] has further improved Theorem B and has proved the following result.

Theorem C. Let $\alpha(\not\equiv 0,\infty)$ be a small function of f. Suppose that b and c are any two finite non-zero distinct complex numbers and $k(\ge 1)$, $n(\ge 0)$ are integers. If $n=0$ or $n \ge 2+k$ then

$$(1+n)T(r,f) \le (1+n)N(r,0;f) + N\left(r,b;\alpha(f)^n f^{(k)}\right) + N\left(r,c;\alpha(f)^n f^{(k)}\right) - N(r,\infty;f) - N\left(r,0;\left(\alpha(f)^n f^{(k)}\right)'\right) + S(r,f). \tag{1.1}$$

If, in particular, f is entire, then (1.1) is true for all non-negative integers $n(\ne 1)$.

Yu [8] also remarked that inequality (1.1) might be valid even for $n=1$ if f is entire.

In this paper we first show that inequality (1.1) is valid for all integers $n(\ge 0)$ and $k(\ge 1)$ even if f is meromorphic.

Next we prove that the following inequality of Q.D. Zhang [9] can be extended to a differential monomial of the form $\alpha(f)^n(f^{(k)})^p$, where $\alpha(\not\equiv 0,\infty)$ is a small function of f and $n(\ge 0)$, $p(\ge 1)$, $k(\ge 1)$ are integers.

Theorem D. [9] Let $\alpha(\not\equiv 0,\infty)$ be a small function of f, then

$$2T(r,f) \le \overline{N}(r,\infty;f) + 2\overline{N}(r,0;f) + \overline{N}(r,1;\alpha f f') + S(r,f).$$

Definition 1.2. For a positive integer k we denote by $N_k(r,0;f)$ the counting function of zeros of f, where a zero with multiplicity q is counted q times if $q \le k$ and is counted k times if $q > k$.

2. LEMMAS

In this section we discuss some lemmas which will be needed in the sequel.

Lemma 2.1. [4] *Let $A > 1$, then there exists a set $M(A)$ of upper logarithmic density at most $\delta(A) = \min\{(2e^{A-1}-1)^{-1}, 1+e(A-1)\exp(e(1-A))\}$ such that for $k=1,2,3,\dots$*

$$\limsup_{r\longrightarrow\infty, r\notin M(A)} \frac{T(r,f)}{T(r,f^{(k)})} \le 3eA.$$

Lemma 2.2. *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)$ be a small function of f, then $\psi = \alpha(f)^n\left(f^{(k)}\right)^p$ is non-constant, where $n(\ge 0)$, $p(\ge 1)$ and $k(\ge 1)$ are integers.*

Proof. We consider the following two cases.

Case 1. Let $n=0$.
If possible suppose that ψ is a constant, then we get

$$T(r,\left(f^{(k)}\right)^p) \le T(r,\alpha) + O(1) = S(r,f)$$

i.e.,

$$T(r,f^{(k)}) = S(r,f),$$

which is impossible by *Lemma 2.1*. Hence ψ is non-constant in this case.

Case 2. Let $n \geq 1$.
Since

$$\left(\frac{1}{f}\right)^{p+n} = \alpha\left(\frac{f^{(k)}}{f}\right)^{p}\frac{1}{\psi},$$

it follows, by the first fundamental theorem and the Milloux theorem ([3, p.55]), that

$$\begin{aligned}(p+n)T(r,f) &\leq T(r,\alpha) + pT\left(r,\frac{f^{(k)}}{f}\right) + T(r,\psi) + O(1) \\ &= pN\left(r,\frac{f^{(k)}}{f}\right) + T(r,\psi) + S(r,f) \\ &\leq pk\{\overline{N}(r,0;f) + \overline{N}(r,\infty;f)\} + T(r,\psi) + S(r,f).\end{aligned} \tag{2.1}$$

We note that if all the zeros (poles) of $(f)^n(f^{(k)})^p$ are poles (zeros) of α in the same multiplicities then

$$\overline{N}(r,0;f) \leq N(r,0;(f)^n(f^{(k)})^p) = N(r,\infty;\alpha) = S(r,f)$$

and

$$\overline{N}(r,\infty;f) \leq N(r,\infty;(f)^n(f^{(k)})^p) = N(r,0;\alpha) = S(r,f),$$

because $n \geq 1$. Since $n \geq 1$, it follows that

$$\overline{N}(r,0;f) \leq N(r,0;\psi) + S(r,f) \text{ and } \overline{N}(r,\infty;f) \leq N(r,\infty;\psi) + S(r,f).$$

Hence, from (2.1), we get

$$\begin{aligned}(p+n)T(r,f) &\leq pk\{N(r,0;\psi) + N(r,\infty;\psi)\} + T(r,\psi) + S(r,f) \\ &\leq (2pk+1)T(r,\psi) + S(r,f),\end{aligned}$$

which shows that ψ is non-constant. This proves the lemma. □

Lemma 2.3. [1] *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)$ be a small function of f. If $\psi = \alpha(f)^n\left(f^{(k)}\right)^p$, where $n(\geq 0)$, $p(\geq 1)$ and $k(\geq 1)$ are integers, then*

$$T(r,\psi) \leq \{n + (1+k)p\}T(r,f) + S(r,f).$$

3. THEOREMS

In this section we prove the main results of the paper.

Theorem 3.1. *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)$ be a small function of f. Suppose that b and c are any two finite non-zero distinct complex numbers. If $\psi = \alpha(f)^n\left(f^{(k)}\right)^p$, where $n(\geq 0)$, $p(\geq 1)$ and $k(\geq 1)$ are integers, then*

$$\begin{aligned}(p+n)T(r,f) \leq (p+n)N(r,0;f) &+ N(r,b;\psi) + N(r,c;\psi) \\ &- N(r,\infty;f) - N(r,0;\psi') + S(r,f).\end{aligned}$$

Proof. By *Lemma 2.2* we see that ψ is non-constant. We now get

$$m\left(r,\frac{1}{\alpha(f)^{p+n}}\right) \leq m(r,0;\psi) + m\left(r,\left(\frac{f^{(k)}}{f}\right)^{p}\right) + O(1),$$

$$m\left(r,\frac{1}{\alpha(f)^{p+n}}\right) = T(r,\alpha(f)^{p+n}) - N(r,0;\alpha(f)^{p+n}) + O(1)$$

and

$$m(r,0;\psi) = T(r,\psi) - N(r,0;\psi) + O(1).$$

Hence we obtain

$$T(r,\alpha(f)^{p+n}) \le N(r,0;\alpha(f)^{p+n}) + T(r,\psi) - N(r,0;\psi) + m\left(r,\left(\frac{f^{(k)}}{f}\right)^{p}\right) + O(1) = N(r,0;\alpha(f)^{p+n}) + T(r,\psi) - N(r,0;\psi) + S(r,f). \tag{3.1}$$

By the second fundamental theorem we get

$$T(r,\psi) \le N(r,0;\psi) + N(r,b;\psi) + N(r,c;\psi) - N_1(r,\psi) + S(r,\psi), \tag{3.2}$$

where $N_1(r,\psi) = 2N(r,\infty;\psi) - N(r,\infty;\psi') + N(r,0;\psi')$.

Let z_0 be a pole of f with multiplicity $q(\ge 1)$. ψ and ψ' have a pole with multiplicities $nq + (q+k)p + t$ and $nq + (q+k)p + 1 + t$ respectively, where $t = 0$ if z_0 is neither a pole nor a zero of α, $t = s$ if z_0 is a pole of α with multiplicity s and $t = -s$ if z_0 is a zero of α with multiplicity s, where s is a positive integer.

Thus,

$$\begin{aligned} 2\{nq + (q+k)p + t\} - \{nq + (q+k)p + 1 + t\} &= nq + (q+k)p + t - 1 \\ &= q + t + nq + (q+k)p - q - 1 \\ &\ge q + t \end{aligned}$$

because

$$nq + (q+k)p - q - 1 \ge k - 1 \ge 0.$$

Since $T(r,\alpha) = S(r,f)$, it follows that

$$N_1(r,\psi) \ge N(r,\infty;f) + N(r,0;\psi') + S(r,f). \tag{3.3}$$

Now, we get from (3.1), (3.2) and (3.3) in view of *Lemma 2.3*

$$T(r,\alpha(f)^{p+n}) \le N(r,0;\alpha(f)^{p+n}) + N(r,b;\psi) + N(r,c;\psi) - N(r,\infty;f) - N(r,0;\psi') + S(r,f)$$

i.e.,

$$(p+n)T(r,f) \le (p+n)N(r,0;f) + N(r,b;\psi) + N(r,c;\psi) - N(r,\infty;f) - N(r,0;\psi') + S(r,f).$$

This proves the theorem. □

Theorem 3.2. *Let f be a transcendental meromorphic function and $\alpha(\not\equiv 0,\infty)$ be a small function of f. If $\psi = \alpha(f)^n\left(f^{(k)}\right)^p$, where $n(\ge 0)$, $p(\ge 1)$, $k(\ge 1)$ are integers, then for any small function $a(\not\equiv 0,\infty)$ of ψ,*

$$(p+n)T(r,f) \le \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + pN_k(r,0;f) + \overline{N}(r,a;\psi) + S(r,f).$$

Proof. Since by *Lemma 2.2* ψ is non-constant, by Nevanlinna's three small functions theorem ([3, p. 47]) we get

$$T(r,\psi) \le \overline{N}(r,0;\psi) + \overline{N}(r,\infty;\psi) + \overline{N}(r,a;\psi) + S(r,\psi).$$

So from (3.1) we obtain

$$T(r,\alpha(f)^{p+n}) \le N(r,0;\alpha(f)^{p+n}) + \overline{N}(r,0;\psi) + \overline{N}(r,\infty;\psi) + \overline{N}(r,a;\psi) - N(r,0;\psi) + S(r,\psi).$$

Since by *Lemma 2.3* we can replace $S(r,\psi)$ by $S(r,f)$ and $\overline{N}(r,\infty;\psi)=\overline{N}(r,\infty;f)+S(r,f)$, we get

$$(3.4)\quad (p+n)T(r,f)\le N(r,0;(f)^{p+n})+\overline{N}(r,0;\psi)+\overline{N}(r,\infty;f) + \overline{N}(r,a;\psi)-N(r,0;\psi)+S(r,f).$$

Let z_0 be a zero of f with multiplicity $q(\ge 1)$. It follows that z_0 is a zero of ψ with multiplicity $nq+t$ if $q\le k$ and $nq+(q-k)p+t$ if $q\ge 1+k$, where $t=0$ if z_0 is neither a pole nor a zero of α, $t=s$ if z_0 is a zero of α with multiplicity s and $t=-s$ if z_0 is a pole of α with multiplicity s, where s is a positive integer.

Hence $(p+n)q+1-nq-t=pq+1-t$ if $q\le k$ and $(p+n)q+1-nq-(q-k)p-t=pk+1-t$ if $q\ge 1+k$. Since $T(r,\alpha)=S(r,f)$, we get

$$(3.5)\quad N(r,0;\alpha(f)^{p+n})+\overline{N}(r,0;\psi)-N(r,0;\psi)\le\overline{N}(r,0;f)+pN_k(r,0;f)+S(r,f).$$

Now the theorem follows from (3.4) and (3.5). This proves the theorem. □

Hayman [2] proved that if f is a transcendental meromorphic function and $n(\ge 3)$ is an integer then $(f)^n f'$ assumes all finite values, except possibly zero, infinitely often.

In the following corollary of *Theorem 3.2* we improve this result.

Corollary 3.3. *Let f be a transcendental meromorphic function and $\psi=\alpha(f)^n(f^{(k)})^p$, where $n(\ge 3)$, $k(\ge 1)$, $p(\ge 1)$ are integers and $\alpha(\not\equiv 0,\infty)$ is a small function of f, then*

$$\Theta(a;\psi)\le\frac{(1+k)p+2}{(1+k)p+n}$$

for any small function $a(\not\equiv 0,\infty)$ of f.

Proof. Since for $n\ge 1$,

$$(3.6)\quad T(r,f)\le BT(r,\psi)$$

holds except possibly for a set of r of finite linear measure, where B is a constant (see [6]), it follows that if $a(\not\equiv 0,\infty)$ is a small function of f, then it is also a small function of ψ.

Hence by *Theorem 3.2* we get

$$(n-2)T(r,f)\le\overline{N}(r,a;\psi)+S(r,f),$$

and so by *Lemma 2.3* and (3.6) we obtain

$$\frac{n-2}{(1+k)p+n}T(r,\psi)\le\overline{N}(r,a;\psi)+S(r,\psi),$$

from which the corollary follows. This proves the corollary. □

REFERENCES

[1] W. DOERINGER, Exceptional values of differential polynomials, *Pacific J. Math.*, **98**(1) (1982), 55–62.

[2] W.K. HAYMAN, Picard values of meromorphic functions and their derivatives, *Ann. Math.*, **70** (1959), 9–42.

[3] W.K. HAYMAN, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[4] W.K. HAYMAN AND J. MILES, On the growth of a meromorphic function and its derivatives, *Complex Variables*, **12** (1989), 245–260.

[5] K.L. HIONG, Sur la limitation de $T(r,f)$ sans intervention des pôles, *Bull. Sci. Math.*, **80** (1956), 175–190.

[6] L.R. SONS, Deficiencies of monomials, *Math. Z.*, **111** (1969), 53–68.

[7] K.W. YU, A note on the product of meromorphic functions and its derivatives, *Kodai Math. J.*, **24**(3) (2001), 339–343.

[8] K.W. YU, On the value distribution of $\phi(z)[f(z)]^{n-1}f^{(k)}(z)$, *J. Inequal. Pure Appl. Math.*, **3**(1) (2002), Article 8. [ONLINE http://jipam.vu.edu.au/v3n1/037_01.html]

[9] Q.D. ZHANG, On the value distribution of $\phi(z)f(z)f'(z)$ (in Chinese), *Acta Math. Sinica*, **37** (1994), 91–97.