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Abstract

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Abstract

The function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1, \infty)$, the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2, \infty)$, and the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing on $[1, \infty)$, respectively. From these, some inequalities, for example, the Minc-Sathre inequality, are deduced, and two open problems posed by the second author are solved partially.

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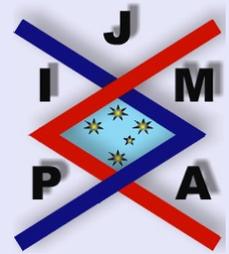
Key words: Gamma function, Monotonicity, Inequality

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1. Introduction

In [14], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{\frac{1}{r}}$, then

$$(1.1) \quad 1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r},$$

which can be rearranged as

$$(1.2) \quad [\Gamma(1+r)]^{\frac{1}{r}} < [\Gamma(2+r)]^{\frac{1}{r+1}}$$

and

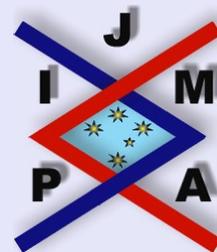
$$(1.3) \quad \frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r} > \frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}.$$

In [1, 13], H. Alzer and J.S. Martins refined the right inequality in (1.1) and showed that, if n is a positive integer, then, for all positive real numbers r , we have

$$(1.4) \quad \frac{n}{n+1} < \left(\frac{1}{n} \sum_{i=1}^n i^r \bigg/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{n+1 \sqrt[n+1]{(n+1)!}}.$$

Both bounds in (1.4) are the best possible.

There have been many extensions and generalizations of inequalities in (1.4), please refer to [3, 4, 12, 15, 16, 22, 23, 28] and references therein.



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The inequalities in (1.1) were refined and generalized in [17, 8, 24, 25, 26] and the following inequalities were obtained:

$$(1.5) \quad \frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{\frac{1}{n}} / \left(\prod_{i=k+1}^{n+m+k} i \right)^{\frac{1}{(n+m)}} \leq \sqrt{\frac{n+k}{n+m+k}},$$

where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (1.5) is valid.

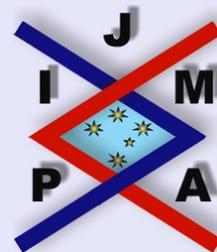
In [18], inequalities in (1.5) were generalized and Qi obtained the following inequalities on the ratio for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer k and natural numbers n and m :

$$(1.6) \quad \frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left[\prod_{i=k+1}^{n+k} (i+\alpha) \right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k} (i+\alpha) \right]^{\frac{1}{(n+m)}}} \leq \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}},$$

where $\alpha \in [0, 1]$ is a constant. For $n = m = 1$, the equality in (1.6) is valid.

Furthermore, for nonnegative integer k and natural numbers n and m , we have

$$(1.7) \quad \frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k} (ai+b) \right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k} (ai+b) \right]^{\frac{1}{n+m}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}},$$



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where a is a positive constant and b a nonnegative integer. For $n = m = 1$, the equality in (1.7) is valid. See [9].

It is clear that inequalities in (1.7) extend those in (1.6).

In [10], the following monotonicity results for the Gamma function were established. The function $[\Gamma(1 + \frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1 + \frac{1}{x})]^x$ increases with $x > 0$, which recover the inequalities in (1.1) which refer to integer values of r . These are equivalent to the function $[\Gamma(1 + x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0, \infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1 + \frac{1}{x})]^x$ decreases for $0 < x < 1$, where $\gamma = 0.57721566 \dots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1, \infty)$.

In [8], the following monotonicity result was obtained: The function

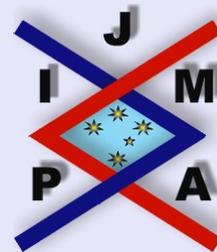
$$(1.8) \quad \frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{\frac{1}{x}}}{x + y + 1}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y , we have

$$(1.9) \quad \frac{x + y + 1}{x + y + 2} \leq \frac{[\Gamma(x + y + 1)/\Gamma(y + 1)]^{\frac{1}{x}}}{[\Gamma(x + y + 2)/\Gamma(y + 1)]^{\frac{1}{x+1}}}.$$

Inequality (1.9) extends and generalizes inequality (1.5), since $\Gamma(n + 1) = n!$.

In an unpublished paper drafted by the second author, the following related results were obtained: Let f be a positive function such that $x[f(x + 1)/f(x) - 1]$ is increasing on $[1, \infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^n f(i)} / f(n + 1) \right\}_{n=1}^{\infty}$



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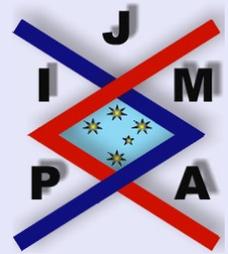


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is decreasing. If f is a logarithmically concave and positive function defined on $[1, \infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^n f(i)} / \sqrt{f(n)} \right\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} / \sqrt[n+m]{\prod_{i=k+1}^{n+k+m} f(i)}$ of the geometric mean sequence $\left\{ \sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} \right\}_{n=1}^{\infty}$ are obtained, where k is a nonnegative integer and m a natural number.

In [9, 8], the second author, F. Qi, posed the following.

Open Problem 1. For positive real numbers x and y , we have

$$(1.10) \quad \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \leq \sqrt{\frac{x+y}{x+y+1}},$$

where Γ denotes the Gamma function.

Open Problem 2. For any positive real number z , define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z - 1]$, and $[z]$ denotes Gauss function whose value is the largest integer not more than z . Let $x > 0$ and $y \geq 0$ be real numbers, then

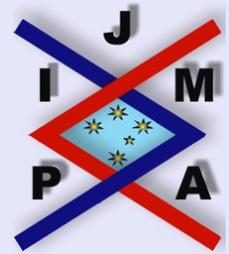
$$(1.11) \quad \frac{x+1}{x+y+1} \leq \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \leq \sqrt{\frac{x}{x+y}}.$$

Hence inequalities in (1.10) and (1.11) are equivalent to the following monotonicity results in some sense for $x \geq 1$, which are the main results of this paper.

Theorem 1.1. *The function $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1, \infty)$, the function $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2, \infty)$, and the function $h(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing on $[1, \infty)$, respectively.*

Remark 1.1. *Note that the function $f(x)$ is a special case of the function (1.8). In this paper, we will give a new and simple proof for the monotonicity of $f(x)$. Theorem 1.1 partially solves the two open problems above.*

Remark 1.2. *In recent years, many monotonicity results and inequalities involving the Gamma and incomplete Gamma functions have been established, please refer to [5, 6, 7, 19, 20, 21, 25, 27] and some references therein.*



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2. Proof of Theorem 1.1

For $x > 1$, the following double inequalities are stated in [11, p. 431]:

$$(2.1) \quad 0 < \ln \Gamma(x) - \left[\left(x - \frac{1}{2} \right) \ln x - x + \frac{1}{2} \ln(2\pi) \right] < \frac{1}{x},$$

$$(2.2) \quad \frac{1}{2x} < \ln x - \frac{\Gamma'(x)}{\Gamma(x)} < \frac{1}{x},$$

$$(2.3) \quad \frac{1}{x} < \frac{d^2}{dx^2} \ln \Gamma(x) < \frac{1}{x-1}.$$

In [29, pp. 103–105], the following formula was given:

$$(2.4) \quad \frac{\Gamma'(z)}{\Gamma(z)} + \gamma = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt = \int_0^1 \frac{1 - t^{z-1}}{1 - t} dt,$$

where γ denotes the Euler constant and $\gamma = 0.57721566490153286060651 \dots$.

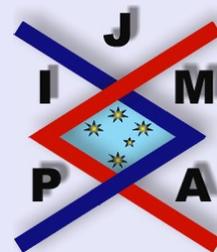
See [29, p. 94]. Formula (2.4) can be used to calculate $\Gamma'(k)$ for $k \in \mathbb{N}$. We call

$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [2, p. 71].

Taking the logarithm yields

$$(2.5) \quad \ln f(x) = \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1).$$

Differentiating with x on both sides of (2.5) and using double inequalities (2.1)



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and (2.2) gives us

$$\begin{aligned}
 (2.6) \quad \frac{x^2 f'(x)}{f(x)} &= -\ln \Gamma(x+1) + x \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{x^2}{x+1} \\
 &< -\left[\left(x + \frac{1}{2}\right) \ln(x+1) - (x+1) + \frac{1}{2} \ln(2\pi) \right] \\
 &\quad + x \left[\ln(x+1) - \frac{1}{2(x+1)} \right] - \frac{x^2}{x+1} \\
 &= -\frac{1}{2} \ln(x+1) - \frac{1}{2(x+1)} + \frac{1}{2} [3 - \ln(2\pi)] \\
 &\triangleq \phi(x),
 \end{aligned}$$

By direct computation, we have

$$\phi'(x) = -\frac{x}{2(x+1)^2} < 0.$$

Thus, the function $\phi(x)$ is strictly decreasing, and then $\phi(x) \leq \phi(1) = \frac{5}{4} - \frac{1}{2} \ln(4\pi) < 0$. Therefore $f'(x) < 0$ and $f(x)$ is strictly decreasing on $[1, \infty)$.

Straightforward calculating and using inequalities in (2.3) for $x > 1$ produces

$$(2.7) \quad \ln g(x) = \frac{1}{x} \ln \Gamma(x+1) - \frac{1}{2} \ln x,$$

$$(2.8) \quad x^2 \frac{g'(x)}{g(x)} = -\ln \Gamma(x+1) + x \frac{d}{dx} \ln \Gamma(x+1) - \frac{1}{2} x \triangleq \varphi(x),$$



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$$(2.9) \quad \begin{aligned} \varphi'(x) &= x \frac{d^2}{dx^2} \ln \Gamma(x+1) - \frac{1}{2} \\ &> \frac{x}{x+1} - \frac{1}{2} = \frac{x-1}{2(x+1)} > 0. \end{aligned}$$

Therefore, function $\varphi(x)$ is strictly increasing, and $\varphi(x) \geq \varphi(2) = \Gamma'(3) - 1 - \ln 2 > 0$ by (2.4). Thus $g'(x) > 0$ and then $g(x)$ is strictly increasing on $[2, \infty)$.

Direct computing and using inequalities in (2.3) for $x > 1$ produces

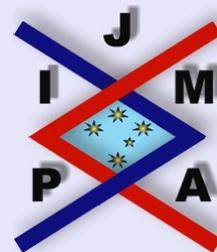
$$(2.10) \quad \ln h(x) = \frac{1}{x} \ln \Gamma(x+1) - \frac{1}{2} \ln(x+1),$$

$$(2.11) \quad x^2 \frac{h'(x)}{h(x)} = -\ln \Gamma(x+1) + x \frac{d}{dx} \ln \Gamma(x+1) - \frac{x^2}{2(x+1)} \triangleq \tau(x),$$

$$(2.12) \quad \begin{aligned} \tau'(x) &= x \frac{d^2}{dx^2} \ln \Gamma(x+1) - \frac{x(2+x)}{2(1+x)^2} \\ &> \frac{x}{x+1} - \frac{x(2+x)}{2(1+x)^2} = \frac{x^2}{2(x+1)^2} > 0. \end{aligned}$$

Therefore, function $\tau(x)$ is strictly increasing, and $\tau(x) \geq \tau(1) = \Gamma'(2) - \frac{1}{4} > 0$. Thus $h'(x) > 0$ and then $h(x)$ is strictly increasing on $[1, \infty)$.

The proof is complete.



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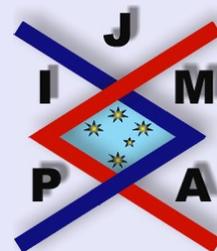
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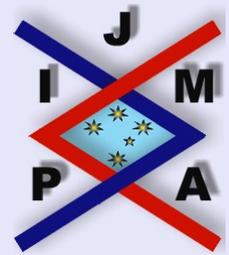
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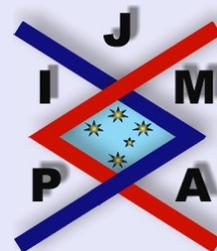
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