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MONOTONICITY RESULTS FOR THE GAMMA FUNCTION

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Abstract

The function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1,\infty)$, the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2,\infty)$, and the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing on $[1,\infty)$, respectively. From these, some inequalities, for example, the Minc-Sathre inequality, are deduced, and two open problems posed by the second author are solved partially.

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Contents

1. Introduction

In [14], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{\frac{1}{r}}$, then

$$1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r}, \tag{1.1}$$

which can be rearranged as

$$[\Gamma(1+r)]^{\frac{1}{r}} < [\Gamma(2+r)]^{\frac{1}{r+1}} \tag{1.2}$$

and

$$\frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r} > \frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}. \tag{1.3}$$

In [1, 13], H. Alzer and J.S. Martins refined the right inequality in (1.1) and showed that, if n is a positive integer, then, for all positive real numbers r, we have

$$\frac{n}{n+1} < \left(\frac{1}{n}\sum_{i=1}^{n} i^r \Big/ \frac{1}{n+1}\sum_{i=1}^{n+1} i^r\right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1.4}$$

Both bounds in (1.4) are the best possible.

There have been many extensions and generalizations of inequalities in (1.4), please refer to [3, 4, 12, 15, 16, 22, 23, 28] and references therein.

The inequalities in (1.1) were refined and generalized in [17, 8, 24, 25, 26] and the following inequalities were obtained:

$$\text{(1.5)} \qquad \frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i\right)^{\frac{1}{n}} \Bigg/ \left(\prod_{i=k+1}^{n+m+k} i\right)^{\frac{1}{(n+m)}} \le \sqrt{\frac{n+k}{n+m+k}},$$

where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (1.5) is valid.

In [18], inequalities in (1.5) were generalized and Qi obtained the following inequalities on the ratio for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer k and natural numbers n and m:

$$\text{(1.6)} \qquad \frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{\frac{1}{(n+m)}}} \le \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}},$$

where $\alpha \in [0,1]$ is a constant. For $n = m = 1$, the equality in (1.6) is valid.

Furthermore, for nonnegative integer k and natural numbers n and m, we have

$$\text{(1.7)} \qquad \frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \le \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}},$$

where a is a positive constant and b a nonnegative integer. For $n = m = 1$, the equality in (1.7) is valid. See [9].

It is clear that inequalities in (1.7) extend those in (1.6).

In [10], the following monotonicity results for the Gamma function were established. The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x > 0$, which recover the inequalities in (1.1) which refer to integer values of r. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0,\infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})^x]$ decreases for $0 < x < 1$, where $\gamma = 0.57721566\cdots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1,\infty)$.

In [8], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{1.8}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}}. \tag{1.9}$$

Inequality (1.9) extends and generalizes inequality (1.5), since $\Gamma(n+1) = n!$.

In an unpublished paper drafted by the second author, the following related results were obtained: Let f be a positive function such that $x\big[f(x+1)/f(x) - 1\big]$ is increasing on $[1,\infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^{n} f(i)} \Big/ f(n+1) \right\}_{n=1}^{\infty}$

is decreasing. If f is a logarithmically concave and positive function defined on $[1,\infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^{n} f(i)} \Big/ \sqrt{f(n)} \right\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} \Big/ \sqrt[n+m]{\prod_{i=k+1}^{n+k+m} f(i)}$ of the geometric mean sequence $\left\{ \sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} \right\}_{n=1}^{\infty}$ are obtained, where k is a nonnegative integer and m a natural number.

In [9, 8], the second author, F. Qi, posed the following.

Open Problem 1. *For positive real numbers x and y, we have*

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \leq \sqrt{\frac{x+y}{x+y+1}}, \tag{1.10}$$

where Γ denotes the Gamma function.

Open Problem 2. *For any positive real number z, define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z-1]$, and $[z]$ denotes Gauss function whose value is the largest integer not more than z. Let $x > 0$ and $y \geq 0$ be real numbers, then*

$$\frac{x+1}{x+y+1} \leq \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \leq \sqrt{\frac{x}{x+y}}. \tag{1.11}$$

Hence inequalities in (1.10) and (1.11) are equivalent to the following monotonicity results in some sense for $x \geq 1$, which are the main results of this paper.

Theorem 1.1. *The function* $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$ *is strictly decreasing on* $[1, \infty)$, *the function* $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ *is strictly increasing on* $[2, \infty)$, *and the function* $h(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ *is strictly incresing on* $[1, \infty)$, *respectively.*

Remark 1.1. *Note that the function* $f(x)$ *is a special case of the function* (1.8). *In this paper, we will give a new and simple proof for the monotonicity of* $f(x)$. *Theorem 1.1 partially solves the two open problems above.*

Remark 1.2. *In recent years, many monotonicity results and inequalities involving the Gamma and incomplete Gamma functions have been established, please refer to [5, 6, 7, 19, 20, 21, 25, 27] and some references therein.*

2. Proof of Theorem 1.1

For $x > 1$, the following double inequalities are stated in [11, p. 431]:

$$0 < \ln \Gamma(x) - \left[\left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi)\right] < \frac{1}{x}, \tag{2.1}$$

$$\frac{1}{2x} < \ln x - \frac{\Gamma'(x)}{\Gamma(x)} < \frac{1}{x}, \tag{2.2}$$

$$\frac{1}{x} < \frac{d^2}{dx^2}\ln \Gamma(x) < \frac{1}{x-1}. \tag{2.3}$$

In [29, pp. 103–105], the following formula was given:

$$\frac{\Gamma'(z)}{\Gamma(z)} + \gamma = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}dt = \int_0^1 \frac{1 - t^{z-1}}{1 - t}dt, \tag{2.4}$$

where γ denotes the Euler constant and $\gamma = 0.577215664901532860651\cdots$. See [29, p. 94]. Formula (2.4) can be used to calculate $\Gamma'(k)$ for $k \in \mathbb{N}$. We call $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [2, p. 71].

Taking the logarithm yields

$$\ln f(x) = \frac{1}{x}\ln \Gamma(x+1) - \ln(x+1). \tag{2.5}$$

Differentiating with x on both sides of (2.5) and using double inequalities (2.1) and (2.2) gives us

$$x^2\frac{f'(x)}{f(x)} = -\ln\Gamma(x+1) + x\frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{x^2}{x+1}$$

$$(2.6)\qquad \begin{aligned} &< -\left[\left(x+\frac{1}{2}\right)\ln(x+1) - (x+1) + \frac{1}{2}\ln(2\pi)\right] \\ &\quad + x\left[\ln(x+1) - \frac{1}{2(x+1)}\right] - \frac{x^2}{x+1} \\ &= -\frac{1}{2}\ln(x+1) - \frac{1}{2(x+1)} + \frac{1}{2}[3-\ln(2\pi)] \\ &\triangleq \phi(x), \end{aligned}$$

By direct computation, we have

$$\phi'(x) = -\frac{x}{2(x+1)^2} < 0.$$

Thus, the function $\phi(x)$ is strictly decreasing, and then $\phi(x) \le \phi(1) = \frac{5}{4} - \frac{1}{2}\ln(4\pi) < 0$. Therefore $f'(x) < 0$ and $f(x)$ is strictly decreasing on $[1,\infty)$.

Straightforward calculating and using inequalities in (2.3) for $x > 1$ produces

$$(2.7)\qquad \ln g(x) = \frac{1}{x}\ln\Gamma(x+1) - \frac{1}{2}\ln x,$$

$$(2.8)\qquad x^2\frac{g'(x)}{g(x)} = -\ln\Gamma(x+1) + x\frac{d}{dx}\ln\Gamma(x+1) - \frac{1}{2}x \triangleq \varphi(x),$$

$$\varphi'(x) = x\frac{d^2}{dx^2}\ln\Gamma(x+1) - \frac{1}{2} \tag{2.9}$$
$$> \frac{x}{x+1} - \frac{1}{2} = \frac{x-1}{2(x+1)} > 0.$$

Therefore, function $\varphi(x)$ is strictly increasing, and $\varphi(x) \geq \varphi(2) = \Gamma'(3) - 1 - \ln 2 > 0$ by (2.4). Thus $g'(x) > 0$ and then $g(x)$ is strictly increasing on $[2, \infty)$.

Direct computing and using inequalities in (2.3) for $x > 1$ produces

$$\ln h(x) = \frac{1}{x}\ln\Gamma(x+1) - \frac{1}{2}\ln(x+1), \tag{2.10}$$

$$x^2\frac{h'(x)}{h(x)} = -\ln\Gamma(x+1) + x\frac{d}{dx}\ln\Gamma(x+1) - \frac{x^2}{2(x+1)} \triangleq \tau(x), \tag{2.11}$$

$$\tau'(x) = x\frac{d^2}{dx^2}\ln\Gamma(x+1) - \frac{x(2+x)}{2(1+x)^2} \tag{2.12}$$
$$> \frac{x}{x+1} - \frac{x(2+x)}{2(1+x)^2} = \frac{x^2}{2(x+1)^2} > 0.$$

Therefore, function $\tau(x)$ is strictly increasing, and $\tau(x) \geq \tau(1) = \Gamma'(2) - \frac{1}{4} > 0$. Thus $h'(x) > 0$ and then $h(x)$ is strictly increasing on $[1, \infty)$.

The proof is complete.

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