Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 2, Article 44, 2003

MONOTONICITY RESULTS FOR THE GAMMA FUNCTION

CHAO-PING CHEN AND FENG QI

DEPARTMENT OF APPLIED MATHEMATICS AND INFORMATICS,
JIAOZUO INSTITUTE OF TECHNOLOGY,
JIAOZUO CITY, HENAN 454000,
THE PEOPLE'S REPUBLIC OF CHINA.
qifeng@jzit.edu.cn
URL: http://rgmia.vu.edu.au/qi.html

Received 01 June, 2002; accepted 01 May, 2003
Communicated by A. Laforgia

ABSTRACT. The function $\frac{[\Gamma(x+1)]^{1/x}}{x+1}$ is strictly decreasing on $[1,\infty)$, the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ is strictly increasing on $[2,\infty)$, and the function $\frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ is strictly increasing on $[1,\infty)$, respectively. From these, some inequalities, for example, the Minc-Sathre inequality, are deduced, and two open problems posed by the second author are solved partially.

Key words and phrases: Gamma function, Monotonicity, Inequality.

2000 *Mathematics Subject Classification.* Primary 33B15; Secondary 26D07.

1. INTRODUCTION

In [14], H. Minc and L. Sathre proved that, if r is a positive integer and $\phi(r) = (r!)^{\frac{1}{r}}$, then

$$1 < \frac{\phi(r+1)}{\phi(r)} < \frac{r+1}{r}, \tag{1.1}$$

which can be rearranged as

$$[\Gamma(1+r)]^{\frac{1}{r}} < [\Gamma(2+r)]^{\frac{1}{r+1}} \tag{1.2}$$

and

$$\frac{[\Gamma(1+r)]^{\frac{1}{r}}}{r} > \frac{[\Gamma(2+r)]^{\frac{1}{r+1}}}{r+1}. \tag{1.3}$$

ISSN (electronic): 1443-5756

The authors were supported in part by NNSF (#10001016) of China, SF for the Prominent Youth of Henan Province (#0112000200), SF of Henan Innovation Talents at Universities, NSF of Henan Province (#004051800), SF for Pure Research of Natural Science of the Education Department of Henan Province (#1999110004), Doctor Fund of Jiaozuo Institute of Technology, China.

The authors would also like to express many thanks to the anonymous referee and the Editor, Professor A. Laforgia, for their thoughful comments.

065-02

In [1, 13], H. Alzer and J.S. Martins refined the right inequality in (1.1) and showed that, if n is a positive integer, then, for all positive real numbers r, we have

$$\frac{n}{n+1} < \left(\frac{1}{n} \sum_{i=1}^{n} i^r \bigg/ \frac{1}{n+1} \sum_{i=1}^{n+1} i^r \right)^{\frac{1}{r}} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}}. \tag{1.4}$$

Both bounds in (1.4) are the best possible.

There have been many extensions and generalizations of inequalities in (1.4), please refer to [3, 4, 12, 15, 16, 22, 23, 28] and references therein.

The inequalities in (1.1) were refined and generalized in [17, 8, 24, 25, 26] and the following inequalities were obtained:

$$\frac{n+k+1}{n+m+k+1} < \left(\prod_{i=k+1}^{n+k} i \right)^{\frac{1}{n}} \bigg/ \left(\prod_{i=k+1}^{n+m+k} i \right)^{\frac{1}{(n+m)}} \leq \sqrt{\frac{n+k}{n+m+k}}, \tag{1.5}$$

where k is a nonnegative integer, n and m are natural numbers. For $n = m = 1$, the equality in (1.5) is valid.

In [18], inequalities in (1.5) were generalized and Qi obtained the following inequalities on the ratio for the geometric means of a positive arithmetic sequence with unit difference for any nonnegative integer k and natural numbers n and m:

$$\frac{n+k+1+\alpha}{n+m+k+1+\alpha} < \frac{\left[\prod_{i=k+1}^{n+k}(i+\alpha)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(i+\alpha)\right]^{\frac{1}{(n+m)}}} \leq \sqrt{\frac{n+k+\alpha}{n+m+k+\alpha}}, \tag{1.6}$$

where $\alpha \in [0, 1]$ is a constant. For $n = m = 1$, the equality in (1.6) is valid.

Furthermore, for nonnegative integer k and natural numbers n and m, we have

$$\frac{a(n+k+1)+b}{a(n+m+k+1)+b} < \frac{\left[\prod_{i=k+1}^{n+k}(ai+b)\right]^{\frac{1}{n}}}{\left[\prod_{i=k+1}^{n+m+k}(ai+b)\right]^{\frac{1}{n+m}}} \leq \sqrt{\frac{a(n+k)+b}{a(n+m+k)+b}}, \tag{1.7}$$

where a is a positive constant and b a nonnegative integer. For $n = m = 1$, the equality in (1.7) is valid. See [9].

It is clear that inequalities in (1.7) extend those in (1.6).

In [10], the following monotonicity results for the Gamma function were established. The function $[\Gamma(1+\frac{1}{x})]^x$ decreases with $x > 0$ and $x[\Gamma(1+\frac{1}{x})]^x$ increases with $x > 0$, which recover the inequalities in (1.1) which refer to integer values of r. These are equivalent to the function $[\Gamma(1+x)]^{\frac{1}{x}}$ being increasing and $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x}$ being decreasing on $(0, \infty)$, respectively. In addition, it was proved that the function $x^{1-\gamma}[\Gamma(1+\frac{1}{x})^x]$ decreases for $0 < x < 1$, where $\gamma = 0.57721566\cdots$ denotes the Euler's constant, which is equivalent to $\frac{[\Gamma(1+x)]^{\frac{1}{x}}}{x^{1-\gamma}}$ being increasing on $(1, \infty)$.

In [8], the following monotonicity result was obtained: The function

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{x+y+1} \tag{1.8}$$

is decreasing in $x \geq 1$ for fixed $y \geq 0$. Then, for positive real numbers x and y, we have

$$\frac{x+y+1}{x+y+2} \leq \frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{\frac{1}{x}}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{\frac{1}{x+1}}}. \tag{1.9}$$

Inequality (1.9) extends and generalizes inequality (1.5), since $\Gamma(n+1) = n!$.

In an unpublished paper drafted by the second author, the following related results were obtained: Let f be a positive function such that $x\big[f(x+1)/f(x) - 1\big]$ is increasing on $[1,\infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^{n} f(i)} \Big/ f(n+1) \right\}_{n=1}^{\infty}$ is decreasing. If f is a logarithmically concave and positive function defined on $[1,\infty)$, then the sequence $\left\{ \sqrt[n]{\prod_{i=1}^{n} f(i)} \Big/ \sqrt{f(n)} \right\}_{n=1}^{\infty}$ is increasing. As consequences of these monotonicities, the lower and upper bounds for the ratio $\sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} \Big/ \sqrt[n+m]{\prod_{i=k+1}^{n+k+m} f(i)}$ of the geometric mean sequence $\left\{ \sqrt[n]{\prod_{i=k+1}^{n+k} f(i)} \right\}_{n=1}^{\infty}$ are obtained, where k is a nonnegative integer and m a natural number.

In [9, 8], the second author, F. Qi, posed the following.

Open Problem 1. For positive real numbers x and y, we have

$$\frac{[\Gamma(x+y+1)/\Gamma(y+1)]^{1/x}}{[\Gamma(x+y+2)/\Gamma(y+1)]^{1/(x+1)}} \le \sqrt{\frac{x+y}{x+y+1}}, \tag{1.10}$$

where Γ denotes the Gamma function.

Open Problem 2. For any positive real number z, define $z! = z(z-1)\cdots\{z\}$, where $\{z\} = z - [z-1]$, and $[z]$ denotes Gauss function whose value is the largest integer not more than z. Let $x > 0$ and $y \ge 0$ be real numbers, then

$$\frac{x+1}{x+y+1} \le \frac{\sqrt[x]{x!}}{\sqrt[x+y]{(x+y)!}} \le \sqrt{\frac{x}{x+y}}. \tag{1.11}$$

Hence inequalities in (1.10) and (1.11) are equivalent to the following monotonicity results in some sense for $x \ge 1$, which are the main results of this paper.

Theorem 1.1. *The function* $f(x) = \frac{[\Gamma(x+1)]^{1/x}}{x+1}$ *is strictly decreasing on* $[1,\infty)$*, the function* $g(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x}}$ *is strictly increasing on* $[2,\infty)$*, and the function* $h(x) = \frac{[\Gamma(x+1)]^{1/x}}{\sqrt{x+1}}$ *is strictly incresing on* $[1,\infty)$*, respectively.*

Remark 1.2. Note that the function $f(x)$ is a special case of the function (1.8). In this paper, we will give a new and simple proof for the monotonicity of $f(x)$. Theorem 1.1 partially solves the two open problems above.

Remark 1.3. In recent years, many monotonicity results and inequalities involving the Gamma and incomplete Gamma functions have been established, please refer to [5, 6, 7, 19, 20, 21, 25, 27] and some references therein.

2. PROOF OF THEOREM 1.1

For $x > 1$, the following double inequalities are stated in [11, p. 431]:

$$0 < \ln\Gamma(x) - \left[\left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi)\right] < \frac{1}{x}, \tag{2.1}$$

$$\frac{1}{2x} < \ln x - \frac{\Gamma'(x)}{\Gamma(x)} < \frac{1}{x}, \tag{2.2}$$

$$\frac{1}{x} < \frac{d^2}{dx^2}\ln\Gamma(x) < \frac{1}{x-1}. \tag{2.3}$$

In [29, pp. 103–105], the following formula was given:

$$\frac{\Gamma'(z)}{\Gamma(z)} + \gamma = \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt = \int_0^1 \frac{1 - t^{z-1}}{1 - t} dt, \tag{2.4}$$

where γ denotes the Euler constant and $\gamma = 0.577215664901532860606512 \cdots$. See [29, p. 94]. Formula (2.4) can be used to calculate $\Gamma'(k)$ for $k \in \mathbb{N}$. We call $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ the digamma or psi function. See [2, p. 71].

Taking the logarithm yields

$$\ln f(x) = \frac{1}{x} \ln \Gamma(x+1) - \ln(x+1). \tag{2.5}$$

Differentiating with x on both sides of (2.5) and using double inequalities (2.1) and (2.2) gives us

$$\begin{aligned} x^2 \frac{f'(x)}{f(x)} &= -\ln \Gamma(x+1) + x \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \frac{x^2}{x+1} \\ &< -\left[\left(x + \frac{1}{2} \right) \ln(x+1) - (x+1) + \frac{1}{2} \ln(2\pi) \right] \\ &\quad + x \left[\ln(x+1) - \frac{1}{2(x+1)} \right] - \frac{x^2}{x+1} \\ &= -\frac{1}{2} \ln(x+1) - \frac{1}{2(x+1)} + \frac{1}{2}[3 - \ln(2\pi)] \\ &\triangleq \phi(x), \end{aligned} \tag{2.6}$$

By direct computation, we have

$$\phi'(x) = -\frac{x}{2(x+1)^2} < 0.$$

Thus, the function $\phi(x)$ is strictly decreasing, and then $\phi(x) \leq \phi(1) = \frac{5}{4} - \frac{1}{2} \ln(4\pi) < 0$. Therefore $f'(x) < 0$ and $f(x)$ is strictly decreasing on $[1, \infty)$.

Straightforward calculating and using inequalities in (2.3) for $x > 1$ produces

$$\ln g(x) = \frac{1}{x} \ln \Gamma(x+1) - \frac{1}{2} \ln x, \tag{2.7}$$

$$x^2 \frac{g'(x)}{g(x)} = -\ln \Gamma(x+1) + x \frac{d}{dx} \ln \Gamma(x+1) - \frac{1}{2} x \triangleq \varphi(x), \tag{2.8}$$

$$\begin{aligned} \varphi'(x) &= x \frac{d^2}{dx^2} \ln \Gamma(x+1) - \frac{1}{2} \\ &> \frac{x}{x+1} - \frac{1}{2} = \frac{x-1}{2(x+1)} > 0. \end{aligned} \tag{2.9}$$

Therefore, function $\varphi(x)$ is strictly increasing, and $\varphi(x) \geq \varphi(2) = \Gamma'(3) - 1 - \ln 2 > 0$ by (2.4). Thus $g'(x) > 0$ and then $g(x)$ is strictly increasing on $[2, \infty)$.

Direct computing and using inequalities in (2.3) for $x > 1$ produces

$$\ln h(x) = \frac{1}{x} \ln \Gamma(x+1) - \frac{1}{2} \ln(x+1), \tag{2.10}$$

$$x^2 \frac{h'(x)}{h(x)} = -\ln \Gamma(x+1) + x \frac{d}{dx} \ln \Gamma(x+1) - \frac{x^2}{2(x+1)} \triangleq \tau(x), \tag{2.11}$$

$$\tau'(x) = x\frac{d^2}{dx^2}\ln\Gamma(x+1) - \frac{x(2+x)}{2(1+x)^2} > \frac{x}{x+1} - \frac{x(2+x)}{2(1+x)^2} = \frac{x^2}{2(x+1)^2} > 0. \tag{2.12}$$

Therefore, function $\tau(x)$ is strictly increasing, and $\tau(x) \geq \tau(1) = \Gamma'(2) - \frac{1}{4} > 0$. Thus $h'(x) > 0$ and then $h(x)$ is strictly increasing on $[1, \infty)$.

The proof is complete.

REFERENCES

[1] H. ALZER, On an inequality of H. Minc and L. Sathre, *J. Math. Anal. Appl.*, **179** (1993), 396–402.

[2] P.S. BULLEN, *A Dictionary of Inequalities*, Pitman Monographs and Surveys in Pure and Applied Mathematics 97, Addison Wesley Longman Limited, 1998.

[3] T.H. CHAN, P. GAO AND F. QI, On a generalization of Martins' inequality, *Monatsh. Math.*, **138**(3) (2003), 179–187. *RGMIA Res. Rep. Coll.*, **4**(1) (2001), Art. 12, 93–101. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

[4] Ch.-P. CHEN, F. QI, P. CERONE AND S.S. DRAGOMIR, Monotonicity of sequences involving convex and concave functions, *Math. Inequal. Appl.*, **6** (2003), accepted. *RGMIA Res. Rep. Coll.*, **5**(1) (2002), Art. 1, 3–13. Available online at `http://rgmia.vu.edu.au/v5n1.html`.

[5] S.S. DRAGOMIR, R.P. AGARWAL, AND N.S. BARNETT, Inequalities for Beta and Gamma functions via some classical and new integral inequalities, *J. Inequal. Appl.*, **5** (2000), 103–165.

[6] Á. ELBERT AND A. LAFORGIA, An inequality for the product of two integrals relating to the incomplete gamma function, *J. Inequal. Appl.*, **5** (2000), 39–51.

[7] N. ELEZOVIĆ, C. GIORDANO AND J. PEČARIĆ, The best bounds in Gautschi's inequality, *Math. Inequal. Appl.*, **3**(2) (2000), 239–252.

[8] B.-N. GUO AND F. QI, Inequalities and monotonicity for the ratio of gamma functions, *Taiwanese J. Math.*, **7**(2) (2003), 239–247.

[9] B.-N. GUO AND F. QI, Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with arbitrary difference, *Tamkang. J. Math.*, **34**(3) (2003), accepted.

[10] D. KERSHAW AND A. LAFORGIA, Monotonicity results for the gamma function, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.*, **119** (1985), 127–133.

[11] J.-Ch. KUANG, Chángyòng Bùděngshì (Applied Inequalities), 2nd ed., Hunan Education Press, Changsha City, China, 1993. (Chinese)

[12] J.-Ch. KUANG, Some extensions and refinements of Minc-Sathre inequality, *Math. Gaz.*, **83** (1999), 123–127.

[13] J.S. MARTINS, Arithmetic and geometric means, an applications to Lorentz sequence spaces, *Math Nachr.*, **139** (1988), 281–288.

[14] H. MINC AND L. SATHRE, Some inequalities involving $(r!)^{1/r}$, *Proc. Edinburgh Math. Soc.*, **14**(2) (1964/65), 41–46.

[15] F. QI, An algebraic inequality, *J. Inequal. Pure Appl. Math.*, **2**(1) (2001), Art. 13. Available online at `http://jipam.vu.edu.au/v2n1/006_00.html`. *RGMIA Res. Rep. Coll.*, **2**(1) (1999), Art. 8, 81–83. Available online at `http://rgmia.vu.edu.au/v2n1.html`.

[16] F. QI, Generalization of H. Alzer's inequality, *J. Math. Anal. Appl.*, **240** (1999), 294–297.

[17] F. QI, Inequalities and monotonicity of sequences involving $\sqrt[n]{(n+k)!/k!}$, *Soochow J. Math.*, **29** (2003), accepted. *RGMIA Res. Rep. Coll.*, **2**(5) (1999), Art. 8, 685–692. Available online at `http://rgmia.vu.edu.au/v2n5.html`.

[18] F. QI, Inequalities and monotonicity of the ratio for the geometric means of a positive arithmetic sequence with unit difference, *Internat. J. Math. Edu. Sci. Tech.*, **34** (2003), accepted.

[19] F. QI, Monotonicity results and inequalities for the gamma and incomplete gamma functions, *Math. Inequal. Appl.*, **5**(1) (2002), 61–67. RGMIA Res. Rep. Coll. 2 (1999), no. 7, Art. 7, 1027–1034. Available online at `http://rgmia.vu.edu.au/v2n7.html`.

[20] F. QI, The extended mean values: definition, properties, monotonicities, comparison, convexities, generalizations, and applications, *Cubo Mathematica Educational*, **4** (2002), in press. *RGMIA Res. Rep. Coll*. **5**(1) (2002), Art. 5, 57–80. Available online at `http://rgmia.vu.edu.au/v5n1.html`.

[21] F. QI, L.-H. CUI, AND S.-L. XU, Some inequalities constructed by Tchebysheff's integral inequality, *Math. Inequal. Appl.*, **2**(4) (1999), 517–528.

[22] F. QI AND B.-N. GUO, An inequality between ratio of the extended logarithmic means and ratio of the exponential means, *Taiwanese J. Math.*, **7**(2) (2003), 229–237. *RGMIA Res. Rep. Coll.*, **4**(1) (2001), Art. 8, 55–61. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

[23] F. QI AND B.-N. GUO, Monotonicity of sequences involving convex function and sequence, *RGMIA Res. Rep. Coll.*, **3**(2) (2000), Art. 14, 321–329. Available online at `http://rgmia.vu.edu.au/v3n2.html`.

[24] F. QI AND B.-N. GUO, Some inequalities involving the geometric mean of natural numbers and the ratio of gamma functions, *RGMIA Res. Rep. Coll.*, **4**(1) (2001), Art. 6, 41–48. Available online at `http://rgmia.vu.edu.au/v4n1.html`.

[25] F. QI AND S.-L. GUO, Inequalities for the incomplete gamma and related functions, *Math. Inequal. Appl.*, **2**(1) (1999), 47–53.

[26] F. QI AND Q.-M. LUO, Generalization of H. Minc and J. Sathre's inequality, *Tamkang J. Math.*, **31**(2) (2000), 145–148. *RGMIA Res. Rep. Coll.*, **2**(6) (1999), Art. 14, 909–912. Available online at `http://rgmia.vu.edu.au/v2n6.html`.

[27] F. QI AND J.-Q. MEI, Some inequalities for the incomplete gamma and related functions, *Z. Anal. Anwendungen*, **18**(3) (1999), 793–799.

[28] F. QI AND N. TOWGHI, An inequality for the ratios of the arithmetic means of functions with a positive parameter, *Nonlinear Funct. Anal. Appl.*, **8** (2003), accepted. *RGMIA Res. Rep. Coll.*, **4**(2) (2001), Art. 15, 305–309. Available online at `http://rgmia.vu.edu.au/v4n2.html`.

[29] Zh.-X. WANG AND D.-R. GUO, Tèshū Hánshù Gàilùn (*Introduction to Special Functions*), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)