

Journal of Inequalities in Pure and Applied Mathematics

ON THE PRODUCT OF DIVISORS OF n AND OF $\sigma(n)$

FLORIAN LUCA

Mathematical Institute, UNAM
Ap. Postal 61-3 (Xangari), CP 58 089
Morelia, Michoacán, Mexico.
E-Mail: fluca@matmor.unam.mx

©2000 Victoria University
ISSN (electronic): 1443-5756
021-03



volume 4, issue 2, article 46,
2003.

*Received 15 January, 2003;
accepted 21 February, 2003.*

Communicated by: J. Sandor

[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

Let $n \geq 1$ be a positive integer. In [7], Sándor introduced the function $T(n) := \prod_{d|n} d$ as the multiplicative analog of $\sigma(n)$, which is the sum of all the positive divisors of n , and studied some of its properties. In particular, he proved several results pertaining to *multiplicative perfect numbers*, which, by analogy, are numbers n for which the relation $T(n) = n^k$ holds with some positive integer k .

In this paper, we compare $T(n)$ with $T(\sigma(n))$. Our first result is:

Theorem 1. *The inequality $T(\sigma(n)) > T(n)$ holds for almost all positive integers n .*

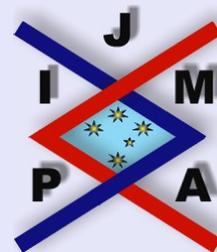
In light of Theorem 1, one can ask whether or not there exist infinitely many n for which $T(\sigma(n)) \leq T(n)$ holds. The fact that this is indeed so is contained in the following more precise statement.

Theorem 2. *Each one of the divisibility relations $T(n) \mid T(\sigma(n))$ and $T(\sigma(n)) \mid T(n)$ holds for an infinite set of positive integers n .*

Finally, we ask whether there exist positive integers $n > 1$ so that $T(n) = T(\sigma(n))$. The answer is no.

Theorem 3. *The equation $T(n) = T(\sigma(n))$ has no positive integer solution $n > 1$.*

Throughout this paper, for a positive real number x and a positive integer k we write $\log_k x$ for the recursively defined function given by $\log_k x := \max\{\log \log_{k-1} x, 1\}$, where \log stands for the natural logarithm function. When $k = 1$, we simply write $\log x$, and we understand that this number is always



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 2 of 15

greater than or equal to 1. For a positive real number x we use $[x]$ for the integer part of x , i.e., the largest integer k so that $k \leq x$. We use the Vinogradov symbols \gg and \ll as well as the Landau symbols O and o with their regular meanings. For a positive integer n , we write $\tau(n)$, and $\omega(n)$ for the number of divisors of n , and the number of distinct prime divisors of n , respectively.

Proof of Theorem 1. Let x be a large positive real number, and let n be a positive integer in the interval $I := (x/\log x, x)$. Since

$$\frac{1}{x} \cdot \sum_{n < x} \tau(n) = O(\log x),$$

it follows that the inequality

$$(1) \quad \tau(n) < \log^2 x$$

holds for all $n \in I$, except for a subset of such n of cardinality $O(x/\log x) = o(x)$.

A straightforward adaptation of the arguments from [4, p. 349] show that the inequality

$$(2) \quad \omega(\sigma(n)) > \frac{1}{3} \cdot \log_2^2 x$$

holds for all $n \in I$, except, eventually, for a subset of such n of cardinality $o(x)$. So, we can say that for most $n \in I$ both inequalities (1) and (2) hold. For such n , we have

$$(3) \quad T(n) = n^{\frac{\tau(n)}{2}} = \exp\left(\frac{\tau(n) \log n}{2}\right) < \exp\left(\frac{\log^3 x}{2}\right),$$



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 3 of 15

while

$$\begin{aligned}
 (4) \quad T(\sigma(n)) &= (\sigma(n))^{\frac{\tau(\sigma(n))}{2}} \\
 &> n^{\frac{\tau(\sigma(n))}{2}} \\
 &> \exp\left(\frac{\tau(\sigma(n)) \log n}{2}\right) \\
 &> \exp\left(\frac{2^{\omega(\sigma(n))} \log n}{2}\right) \\
 &> \exp\left(\frac{2^{\frac{\log^2 x}{3}}}{2} \cdot \log\left(\frac{x}{\log x}\right)\right),
 \end{aligned}$$

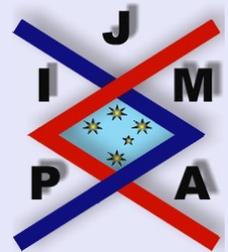
and it is easy to see that for large values of x the function appearing in the right hand side of (4) is larger than the function appearing on the right hand side of (3). This completes the proof of Theorem 1. \square

Proof of Theorem 2. We first construct infinitely many n such that $T(n) \mid T(\sigma(n))$. Let λ be an odd number to be chosen later and put $n := 2^\lambda \cdot 3$. Then, $\tau(n) = 2(\lambda + 1)$, therefore

$$(5) \quad T(n) = (2^\lambda \cdot 3)^{\frac{\tau(n)}{2}} \mid 6^{(\lambda+1)^2}.$$

Now $\sigma(n) = 4 \cdot (2^{\lambda+1} - 1)$ is a multiple of 6 because $\lambda + 1$ is even, and so $2^{\lambda+1} - 1$ is a multiple of 3. Thus, $T(\sigma(n))$ is a multiple of

$$6^{\lfloor \frac{\tau(4(2^{\lambda+1}-1))}{2} \rfloor} = 6^{\lfloor \frac{3\tau(2^{\lambda+1}-1)}{2} \rfloor},$$



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 4 of 15

and since the inequality $\lfloor 3k/2 \rfloor \geq k$ holds for all positive integers k , it follows that $T(\sigma(n))$ is a multiple of $6^{\tau(2^{\lambda+1}-1)}$.

It suffices therefore to see that we can choose infinitely many such odd λ so that $\tau(2^{\lambda+1} - 1) > (\lambda + 1)^2$. Since $\tau(2^{\lambda+1} - 1) \geq 2^{\omega(2^{\lambda+1}-1)}$, it follows that it suffices to show that we can choose infinitely many odd λ so that

$$2^{\omega(2^{\lambda+1}-1)} > (\lambda + 1)^2,$$

which is equivalent to

$$\omega(2^{\lambda+1} - 1) > \frac{2}{\log 2} \cdot \log(\lambda + 1).$$

Since $2/\log 2 < 3$, it suffices to show that the inequality

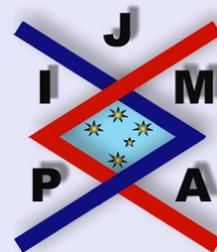
$$(6) \quad \omega(2^{\lambda+1} - 1) > 3 \log(\lambda + 1)$$

holds for infinitely many odd positive integers λ .

Let $(u_k)_{k \geq 1}$ be the *Lucas sequence* of general term $u_k := 2^k - 1$ for $k = 1, 2, \dots$. The primitive divisor theorem (see [1], [2]), says that for all $d \mid k$, $d \neq 1, 6$, there exists a prime number $p \mid u_d$ (hence, $p \mid u_k$ as well), so that $p \nmid u_m$ for any $1 \leq m < d$. In particular, the inequality $\omega(2^k - 1) \geq \tau(k) - 2$ holds for all positive integers k . Thus, in order to prove that (6) holds for infinitely many odd positive integers λ , it suffices to show that the inequality

$$\tau(\lambda + 1) \geq 2 + 3 \log(\lambda + 1)$$

holds for infinitely many odd positive integers λ .



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 5 of 15

Choose a large real number y and put

$$(7) \quad \lambda + 1 := \prod_{p < y} p.$$

Clearly, $\lambda + 1$ is even, therefore λ is odd. With the prime number theorem, we have that

$$\lambda + 1 = \exp(1 + o(1))y$$

holds for large y , and therefore the inequality

$$\lambda + 1 < \exp(2y)$$

holds for large values of y . In particular,

$$(8) \quad 2 + 3 \log(\lambda + 1) < 2 + 6y$$

holds for large y . However,

$$\tau(\lambda + 1) \geq 2^{\omega(\lambda+1)} = 2^{\pi(y)},$$

where we write $\pi(y)$ for the number of prime numbers $p < y$. Since $\pi(y) \geq y / \log y$ holds for all $y > 17$ (see [6]), it follows that for y sufficiently large we have

$$(9) \quad \tau(\lambda + 1) \geq 2^{\frac{y}{\log y}}.$$

It is now clear that the right hand side of (9) is larger than the right hand side of (8) for sufficiently large values of y , and therefore the numbers λ shown at (7) do fulfill inequality (6) for large values of y .



**On the Product of Divisors of n
and of $\sigma(n)$**

Florian Luca

Title Page

Contents

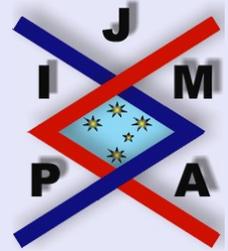


Go Back

Close

Quit

Page 6 of 15



**On the Product of Divisors of n
and of $\sigma(n)$**

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 7 of 15

We now construct infinitely many n such that $T(\sigma(n)) \mid T(n)$. For coprime integers a and d with d positive and a large positive real number x let $\pi(x; d, a)$ be the number of primes $p < x$ with $p \equiv a \pmod{d}$. For positive real numbers $y < x$ let $\pi(x; y)$ stand for the number of primes $p < x$ so that $p + 1$ is free of primes $q \geq y$. Let \mathcal{E} denote the set of all real numbers E in the range $0 < E < 1$ so that there exists a positive constant $\gamma(E)$ and a real number $x_1(E)$ such that the inequality

$$(10) \quad \pi(x; x^{1-E}) > \gamma(E)\pi(x)$$

holds for all $x > x_1(E)$. Thus, \mathcal{E} is the set of all real numbers E in the interval $0 < E < 1$ such that for large x a positive proportion (depending on E) of all the prime numbers p up to x have $p + 1$ free of primes $q \geq x^{1-E}$. Erdős (see [3]) showed that \mathcal{E} is nonempty. In fact, he did not exactly treat this question, but the analogous question for the primes $p < x$ such that $p - 1$ is free of primes larger than x^{1-E} , but his argument can be adapted to the situation in which $p - 1$ is replaced by $p + 1$, which is our instance. The best result known about \mathcal{E} is due to Friedlander [5], who showed that every positive number E smaller than $1 - (2\sqrt{e})^{-1}$ belongs to \mathcal{E} . Erdős has conjectured that \mathcal{E} is the full interval $(0, 1)$.

Let E be some number in \mathcal{E} . Let $x > x_1(E)$ be a large real number. Let $\mathcal{P}_E(x)$ be the set of all the primes $p < x$ counted by $\pi(x; x^{1-E})$. Note that all the primes $p < x^{1-E}$ are already in $\mathcal{P}_E(x)$. Put

$$(11) \quad n := \prod_{p \in \mathcal{P}_E(x)} p.$$

Clearly,

$$(12) \quad T(n) = n^{\frac{\tau(n)}{2}},$$

and

$$(13) \quad \frac{\tau(n)}{2} = 2^{\#\mathcal{P}_E(x)-1} = 2^{\pi(x; x^{1-E})-1} > 2^{c\pi(x)} > 2^{\frac{cx}{\log x}},$$

where one can take $c := \gamma(E)/2$, and inequality (13) holds for sufficiently large values of x . In particular, $T(n)$ is divisible by all primes $q < x^{1-E}$, and each one of them appears at the power at least $2^{\frac{cx}{\log x}}$.

We now look at $T(\sigma(n))$. We have

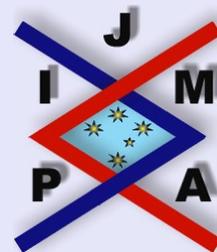
$$(14) \quad T(\sigma(n)) = \left(\prod_{p \in \mathcal{P}_E(x)} (p+1) \right)^{\frac{\tau(\sigma(n))}{2}}.$$

From the definition of $\mathcal{P}_E(x)$, we know that the only primes that can divide $T(\sigma(n))$ are the primes $q < x^{1-E}$. Thus, to conclude, it suffices to show that the exponent at which each one of these primes $q < x^{1-E}$ appears in the prime factorization of $T(\sigma(n))$ is smaller than $2^{\frac{cx}{\log x}}$. Let q be such a prime, and let α_q be so that $q^{\alpha_q} \parallel \sigma(n)$. It is easy to see that

$$(15) \quad \alpha_q \leq \pi(x, q, -1) + \pi(x, q^2, -1) + \cdots + \pi(x, q^j, -1) + \cdots.$$

Let $j \geq 1$. Then $\pi(x; q^j, -1)$ is the number of primes $p < x$ such that $q^j \mid p+1$. In particular, $\pi(x; q^j, -1)$ is at most the number of numbers $m < x+1$ which are multiples of q^j , and this number is $\left\lfloor \frac{x+1}{q^j} \right\rfloor \leq \frac{x+1}{q^j}$. Thus,

$$\alpha_q < (x+1) \sum_{j \geq 1} \frac{1}{q^j} = \frac{x+1}{q-1} \leq x+1.$$



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 8 of 15

Thus,

$$\alpha_q + 1 \leq x + 2 < 2x$$

holds for all $q < x^{1-E}$, and therefore

$$\tau(\sigma(n)) < (2x)^{\pi(x^{1-E})} = \exp(\pi(x^{1-E}) \cdot \log(2x)).$$

By the prime number theorem,

$$\pi(x^{1-E}) = (1 + o(1)) \cdot \frac{x^{1-E}}{\log(x^{1-E})},$$

and therefore the inequality

$$(16) \quad \pi(x^{1-E}) < \frac{2x^{1-E}}{\log(x^{1-E})} = \frac{2}{1-E} \cdot \frac{x^{1-E}}{\log x}$$

holds for large values of x . Thus,

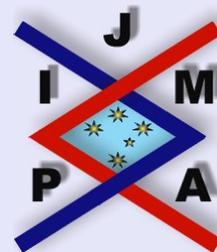
$$(17) \quad \tau(\sigma(n)) < \exp\left(\frac{2}{1-E} \cdot \frac{x^{1-E}}{\log x} \cdot \log(2x)\right) < \exp\left(\frac{3x^{1-E}}{1-E}\right),$$

holds for large values of x . In particular, the exponent at which a prime number $q < x^{1-E}$ can appear in the prime factorization of $T(\sigma(n))$ is at most

$$(18) \quad \alpha_q \cdot \frac{\tau(\sigma(n))}{2} < \tau(\sigma(n))^2 < \exp\left(\frac{6x^{1-E}}{1-E}\right).$$

Comparing (13) with (18), it follows that it suffices to show that the inequality

$$(19) \quad \exp\left(\frac{6x^{1-E}}{1-E}\right) < 2^{\frac{cx}{\log x}}$$



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 9 of 15

holds for large values of x , and taking logarithms in (19), we see that (19) is equivalent to

$$(20) \quad c_1 \log x < x^E,$$

where $c_1 := \frac{6}{c(1-E)\log 2}$, and it is clear that (20) holds for large values of x . Theorem 2 is therefore proved. \square

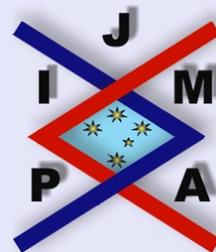
Proof of Theorem 3. Assume that $n > 1$ satisfies $T(n) = T(\sigma(n))$. Write $t := \omega(n)$. It is clear that $t > 1$, for otherwise the number n will be of the form $n = q^\alpha$ for some prime number q and some positive integer α , and the contradiction comes from the fact that $\sigma(q^\alpha)$ is coprime to q . We now note that it is not possible that the prime factors of n are in $\{2, 3\}$. Indeed, if this were so, then $n = 2^{\alpha_1} \cdot 3^{\alpha_2}$, and $\sigma(n) = (2^{\alpha_1+1} - 1)(3^{\alpha_2+1} - 1)$. Since the prime factors of $\sigma(n)$ are also in the set $\{2, 3\}$, we get the diophantine equations $2^{\alpha_1+1} - 1 = 3^{\beta_1}$ and $3^{\alpha_2+1} - 1 = 2^{\beta_2}$, and it is wellknown and very easy to prove that the only positive integer solution $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ of the above equations is $(1, 1, 1, 3)$. Thus, $n = 6$, and the contradiction comes from the fact that this number does not satisfy the equation $T(n) = T(\sigma(n))$.

Write

$$(21) \quad n := q_1^{\alpha_1} \cdots q_t^{\alpha_t}$$

where $q_1 < q_2 < \cdots < q_t$ are prime numbers and α_i are positive integers for $i = 1, \dots, t$. We claim that

$$(22) \quad q_1 \cdots q_t > e^t.$$



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 10 of 15

This is clearly so if $t = 2$, because in this case $q_1 q_2 \geq 2 \cdot 5 > e^2$. For $t \geq 3$, one proves by induction that the inequality

$$p_1 \cdots p_t > e^t$$

holds, where p_i is the i th prime number. This takes care of (22).

We now claim that

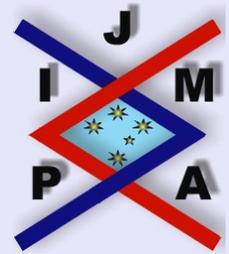
$$(23) \quad \frac{\sigma(n)}{n} < \exp(1 + \log t).$$

Indeed,

$$(24) \quad \begin{aligned} \frac{\sigma(n)}{n} &= \prod_{i=1}^t \left(1 + \frac{1}{q_i} + \cdots + \frac{1}{q_i^{\alpha_i}} \right) \\ &< \exp \left(\sum_{i=1}^t \sum_{\beta \geq 1} \frac{1}{p_i^\beta} \right) \\ &< \exp \left(\sum_{i=1}^t \frac{1}{p_i - 1} \right), \end{aligned}$$

and so, in order to prove (23), it suffices, via (24), to show that

$$(25) \quad \sum_{i=1}^t \frac{1}{p_i - 1} \leq 1 + \log t.$$



**On the Product of Divisors of n
and of $\sigma(n)$**

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 11 of 15

One checks that (25) holds at $t := 1$ and $t := 2$. Assume now that $t \geq 3$ and that (25) holds for $t - 1$. Then,

$$(26) \quad \sum_{i=1}^t \frac{1}{p_i - 1} = \frac{1}{p_t - 1} + \sum_{i=1}^{t-1} \frac{1}{p_i - 1} < 1 + \frac{1}{p_t - 1} + \log(t-1) < 1 + \log t,$$

where the last inequality in (26) above holds because it is equivalent to

$$\left(1 + \frac{1}{t-1}\right)^{p_t-1} > e,$$

which in turn holds because $p_t \geq t + 1$ holds for $t \geq 3$, and

$$\left(1 + \frac{1}{t-1}\right)^t > e$$

holds for all positive integers $t > 1$.

After these preliminaries, we complete the proof of Theorem 3. Write the relation $T(n) = T(\sigma(n))$ as

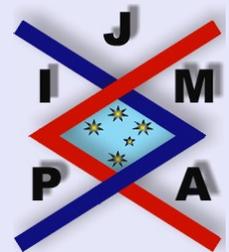
$$(27) \quad \sigma(n) = n^{\frac{\tau(n)}{\tau(\sigma(n))}} = n \cdot n^{\frac{\tau(n) - \tau(\sigma(n))}{\tau(\sigma(n))}}.$$

Since $\sigma(n) > n$, we get that $\tau(n) > \tau(\sigma(n))$. We now use (23) to say that

$$n^{\frac{\tau(n) - \tau(\sigma(n))}{\tau(\sigma(n))}} = \frac{\sigma(n)}{n} < \exp(1 + \log t),$$

therefore

$$(28) \quad \frac{\tau(n) - \tau(\sigma(n))}{\tau(\sigma(n))} < \frac{1 + \log t}{\log n}.$$



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 12 of 15

Let $d := \gcd(\tau(n), \tau(\sigma(n))) = \gcd(\tau(n) - \tau(\sigma(n)), \tau(\sigma(n)))$. From (28), we get that

$$d < \left(\frac{1 + \log t}{\log n} \right) \cdot \tau(\sigma(n)).$$

Write

$$(29) \quad \frac{\tau(n) - \tau(\sigma(n))}{\tau(\sigma(n))} = \frac{\beta}{\gamma},$$

where β and γ are coprime positive integers. We have

$$(30) \quad \gamma = \frac{\tau(\sigma(n))}{d} > \frac{\log n}{1 + \log t}.$$

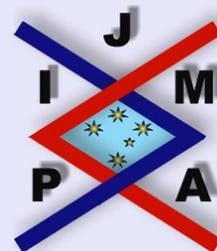
The number $n^{\frac{\beta}{\gamma}} = \sigma(n)/n$ is both a rational number and an algebraic integer, and is therefore an integer. Since β and γ are coprime, it follows, by unique factorization, that α_i is a multiple of γ for all $i = 1, \dots, t$. Thus, $\alpha_i \geq \gamma$ holds for $i = 1, \dots, t$, therefore

$$(31) \quad n \geq (q_1 \cdots q_t)^\gamma > e^{t\gamma} = \exp(t\gamma) > \exp\left(\frac{t \log n}{1 + \log t}\right) = n^{\frac{t}{1 + \log t}},$$

and now (31) implies that

$$1 + \log t > t,$$

which is impossible. Theorem 3 is therefore proved. \square



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



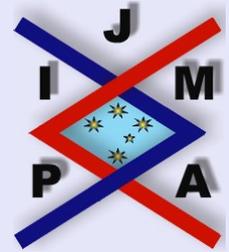
Go Back

Close

Quit

Page 13 of 15

Remark 0.1. We close by noting that if n is a multiply perfect number, then $T(n) \mid T(\sigma(n))$. Recall that a multiply perfect number n is a number so that $n \mid \sigma(n)$. If n has this property, then $\tau(\sigma(n)) > \tau(n)$, and now it is easy to see that $T(\sigma(n)) = \sigma(n)^{\frac{\tau(\sigma(n))}{2}}$ is a multiple of $n^{\frac{\tau(n)}{2}} = T(n)$. Unfortunately, we still do not know if the set of multiply perfect numbers is infinite.



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

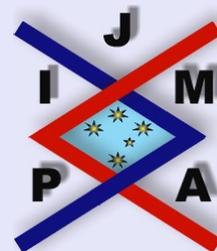
Close

Quit

Page 14 of 15

References

- [1] Y. BILU, G. HANROT AND P.M. VOUTIER, Existence of primitive divisors of Lucas and Lehmer numbers. With an appendix by M. Mignotte, *J. Reine Angew. Math.*, **539** (2001), 75–122.
- [2] R.D. CARMICHAEL, On the numerical factors of the arithmetic forms $\alpha^n \pm \beta^n$, *Ann. Math.*, **15**(2) (1913), 30–70.
- [3] P. ERDŐS, On the normal number of prime factors of $p - 1$ and some other related problems concerning Euler's ϕ -function, *Quart. J. of Math.* (Oxford Ser.), **6** (1935), 205–213.
- [4] P. ERDŐS AND C. POMERANCE, On the normal number of prime factors of $\phi(n)$, *Rocky Mtn. J. of Math.*, **15** (1985), 343–352.
- [5] J.B. FRIEDLANDER, Shifted primes without large prime factors, in *Number Theory and Applications*, (Ed. R.A. Mollin), (Kluwer, NATO ASI, 1989), 393–401.
- [6] J.B. ROSSER AND L. SHOENFELD, Approximate formulas for some functions of prime numbers, *Illinois J. of Math.*, **6** (1962), 64–94.
- [7] J. SÁNDOR, On multiplicatively perfect numbers, *J. Inequal. Pure Appl. Math.*, **2**(1) (2001), Article 3. [ONLINE: http://jipam.vu.edu.au/v2n1/019_99.html]



On the Product of Divisors of n
and of $\sigma(n)$

Florian Luca

Title Page

Contents



Go Back

Close

Quit

Page 15 of 15