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SOME SPECIAL SUBCLASSES OF CARATHÉODORY'S OR STARLIKE FUNCTIONS AND RELATED COEFFICIENT PROBLEMS

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ABSTRACT. Let $\mathcal{P}$ be the class of analytic functions in the unit disk $U=\{|z|<1\}$ with $p(0)=0$ and $\Re p(z)>0$ in U. Let also $\mathcal{S}^*$, $\mathcal{K}$ be the well known classes of normalized univalent starlike and convex functions respectively. For $\Re\alpha>0$ we introduce the classes $\mathcal{P}_{[\alpha]}$, $\mathcal{S}^*_{[\alpha]}$ and $\mathcal{K}_{[\alpha]}$ which are subclasses of $\mathcal{P}$, $\mathcal{S}^*$ and $\mathcal{K}$ respectively, being defined as follows: $p\in\mathcal{P}_{[\alpha]}$ iff $p\in\mathcal{P}\ with\ p(z)\neq\alpha\,\forall\, z\in U$, $f\in\mathcal{S}^*_{[\alpha]}$ iff $\frac{zf'}{f}\in\mathcal{P}_{[\alpha]}$ and $f\in\mathcal{K}_{[\alpha]}$ iff $1+\frac{zf''(z)}{f'(z)}\in\mathcal{P}_{[\alpha]}$. In this paper we study different kind of coefficient problems for the above mentioned classes $\mathcal{P}_{[\alpha]}$, $\mathcal{S}^*_{[\alpha]}$ and $\mathcal{P}_{[\alpha]}$. All the estimations obtained are the best possible.

Key words and phrases: Coefficient problem; Carathéodory's functions; Starlike functions; Convex functions.

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Let $\mathcal{H}$ be the class of holomorphic functions in the unit disk $U=\{z:|z|<1\}$ and $\mathcal{A}$ the class of functions $f\in\mathcal{H}(U)$ with $f(0)=f'(0)-1=0$. Let also

$$\mathcal{W}=\{w\in\mathcal{H}(U):w(0)=0\ \text{ and }\ |w(z)|<1\ \text{ in }\ U\}\ \text{ and}$$
$$\mathcal{P}=\{p\in\mathcal{H}(U):p(0)=1\ \text{ and }\ \Re p(z)>0\ \text{ in }\ U\}\equiv\{(1+w)(1-w)^{-1}:w\in\mathcal{W}\}.$$

Classes $\mathcal{P}$ (also known as Carathéodory's class) and $\mathcal{W}$ (also known as class of Schwarz functions) are fundamental in Geometric Function Theory, while a large number of papers written during the last century involve them. Apart from the independent interest revealed by their study, these two classes are significantly useful, since many results related to them prove to be essential while working with other classes of equal importance. Furthermore, several classes' coefficients can be formed as expressions of the relative coefficients of classes $\mathcal{P}$ and $\mathcal{W}$.

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Two fundamental examples are given by the classes $\mathcal{S}^*$ and $\mathcal{K}$, consisting of the univalent functions $f \in \mathcal{A}$ which are starlike and convex respectively. It is known that:

$$f \in \mathcal{S}^* \text{ iff } f \in \mathcal{A} \text{ and } \frac{zf'(z)}{f(z)} \in \mathcal{P} \tag{1}$$

and

$$f \in \mathcal{K} \text{ iff } f \in \mathcal{A} \text{ and } 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}. \tag{2}$$

The most known subclasses of class $\mathcal{P}$ have been initially introduced by the exclusion from each function's domain, of an entire surface of the right half-plain $\{z : \Re z > 0\}$. More specifically, we derive the classes $\mathcal{P}_a$ and $\mathcal{P}_{(a)}$ being defined as follows:

$$f \in \mathcal{P}_a \text{ iff } f \in \mathcal{P} \text{ and } \Re f(z) > a \text{ in } U, \ (a > 0)$$

$$\text{and} \qquad f \in \mathcal{P}_{(a)} \text{ iff } f \in \mathcal{P} \text{ and } |\arg f(z)| < \frac{a\pi}{2} \text{ in } U, \ (1 > a > 0).$$

If in relation (1) we replace class $\mathcal{P}$ by $\mathcal{P}_a$ or $\mathcal{P}_{(a)}$, we obtain the classes $\mathcal{S}^*_a$ and $\mathcal{S}^*_{(a)}$ respectively, known as starlike of order a and strongly starlike of order a. In a similar way, classes $\mathcal{K}_a$ and $\mathcal{K}_{(a)}$ known as convex of order a and strongly convex of order a are obtained, applying the same substitutions in relation (2).

Our idea is to study coefficient problems, about classes which are "very close" to initial classes $\mathcal{P}$, $\mathcal{S}^*$ and $\mathcal{K}$, which are obtained excluding from their domains a single point belonging to the right half-plain $\{z : \Re z > 0\}$. More specifically if $\Re\alpha > 0$, we introduce classes $\mathcal{P}_{[\alpha]}$, $\mathcal{S}^*_{[\alpha]}$ and $\mathcal{K}_{[\alpha]}$ as follows:

$$\mathcal{P}_{[\alpha]} = \{f \in \mathcal{P} : f(z) \neq \alpha, \forall\, z \in U\}, \ \ f \in \mathcal{S}^*_{[\alpha]} \text{ iff } \frac{zf'}{f} \in \mathcal{P}_{[\alpha]}$$

$$\text{and } f \in \mathcal{K}_{[\alpha]} \text{ iff } 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}_{[\alpha]}.$$

Similar problems, involving class $\mathcal{B}_{[a]} = \{\varphi \in \mathcal{H}(U) : |\varphi(z)| \leq 1, \ \varphi(z) \neq a, \ \forall z \in U\}$, $(|a| < 1)$, were studied by Ruud Ermers (see [2]). For $\mathcal{B}_{[a]}$ the author gives the best possible upper bound for the first and second Taylor coefficient, generalizing the relative results provided by Krzyż (see [3]) for the well known class $\mathcal{B}_{[0]}$.

We also denote

$$\mathcal{W}_{[a]} = \mathcal{W} \cap \mathcal{B}_{[a]}.$$

If $\alpha \equiv \frac{1+a}{1-a}$ $(|a| < 1)$ (we will retain this symbolism throughout the paper), then it is easy to see that

$$w \in \mathcal{W}_{[a]} \iff \frac{1+w}{1-w} \in \mathcal{P}_{[\alpha]}.$$

In order to obtain the Taylor expansions mentioned we will use the forms $f(z) = f_0 + f_1 z + f_2 z^2 + \dots$ and $f^{-1}(w) = F_0 + F_1 w + F_2 w^2 + \dots$.

In this paper we give the following results:

(i) For the class $\mathcal{P}_{[\alpha]}$ we calculate the quantities $\max\limits_f |f_n|$, $n = 1, 2$ for $\Re\alpha > 0$ and $\max\limits_f |f_3|$ for $\alpha > 0$.

(ii) For the classes $\mathcal{S}^*_{[\alpha]}$ and $\mathcal{K}_{[\alpha]}$:

- (α) we calculate $\max\limits_f |f_2|$ and $\max\limits_f |F_2|$ for $\Re\alpha > 0$,
- (β) we solve the Fekete–Szegö problem for every $\mu \in \mathbb{C}$, determining the quantities $\max\limits_f |f_3 - \mu f_2^2|$ and $\max\limits_f |F_3 - \mu F_2^2|$,

(γ) we calculate $\max_f |f_4|$ and $\max_f |F_4|$ for $\alpha > 0$.

The following three lemmas will be very useful in order to prove the theorems where our main results are stated. First we present the Szynal–Prokhorov lemma (see [4]) which is crucial for the estimation of our results. Through this lemma the value

$$\Phi(x_1, x_2) = \max_{f \in \mathcal{W}} |w_3 + x_1 w_1 w_2 + x_2 w_1^3|$$

for $x_1,\ x_2 \in \mathbb{R}$ is obtained. For the formulation of the lemma we will need the following denotations:

$$\begin{aligned}
S_1(x_1, x_2) &= \frac{1}{2} - |x_1|, \\
S_2(x_1, x_2) &= 2 - x_1, \\
S_3(x_1, x_2) &= 4 - x_1, \\
S_4(x_1, x_2) &= x_2 + 1, \\
S_5(x_1, x_2) &= 1 - x_2, \\
S_6(x_1, x_2) &= x_2 - \left(\frac{4}{27}(|x_1| + 1)^3 - (|x_1| + 1) \right), \\
S_7(x_1, x_2) &= -\frac{2}{3}(|x_1| + 1) - x_2, \\
S_8(x_1, x_2) &= x_2 - \frac{1}{12}(x_1^2 + 8), \\
S_9(x_1, x_2) &= x_2 - \frac{2}{3}(|x_1| - 1), \\
S_{10}(x_1, x_2) &= \frac{2|x_1|(|x_1| + 1)}{x_1^2 + 2|x_1| + 4} - x_2, \\
S_{11}(x_1, x_2) &= \frac{2|x_1|(|x_1| - 1)}{x_1^2 - 2|x_1| + 4} - x_2,
\end{aligned}$$

$$\begin{aligned}
D_1 &= \big\{(x_1, x_2) : S_1(x_1, x_2) \ge 0,\ S_4(x_1, x_2) \ge 0,\ S_5(x_1, x_2) \ge 0\big\}, \\
D_2 &= \big\{(x_1, x_2) : -S_1(x_1, x_2) \ge 0,\ S_2(x_1, x_2) \ge 0,\ S_5(x_1, x_2) \ge 0,\ S_6(x_1, x_2) \ge 0\big\}, \\
D_3 &= \big\{(x_1, x_2) : S_1(x_1, x_2) \ge 0,\ -S_4(x_1, x_2) \ge 0\big\}, \\
D_4 &= \big\{(x_1, x_2) : -S_1(x_1, x_2) \ge 0,\ S_7(x_1, x_2) \ge 0\big\}, \\
D_5 &= \big\{(x_1, x_2) : S_2(x_1, x_2) \ge 0,\ -S_7(x_1, x_2) \ge 0\big\}, \\
D_6 &= \big\{(x_1, x_2) : -S_2(x_1, x_2) \ge 0,\ S_3(x_1, x_2) \ge 0,\ S_8(x_1, x_2) \ge 0\big\}, \\
D_7 &= \big\{(x_1, x_2) : -S_3(x_1, x_2) \ge 0,\ S_9(x_1, x_2) \ge 0\big\}, \\
D_8 &= \big\{(x_1, x_2) : -S_1(x_1, x_2) \ge 0,\ S_2(x_1, x_2) \ge 0,\ -S_7(x_1, x_2) \ge 0,\ -S_6(x_1, x_2) \ge 0\big\}, \\
D_9 &= \big\{(x_1, x_2) : -S_2(x_1, x_2) \ge 0,\ -S_7(x_1, x_2) \ge 0,\ S_{10}(x_1, x_2) \ge 0\big\}, \\
D_{10} &= \big\{(x_1, x_2) : -S_2(x_1, x_2) \ge 0,\ S_3(x_1, x_2) \ge 0,\ -S_{10}(x_1, x_2) \ge 0,\ -S_8(x_1, x_2) \ge 0\big\}, \\
D_{11} &= \big\{(x_1, x_2) : -S_3(x_1, x_2) \ge 0,\ -S_{10}(x_1, x_2) \ge 0,\ S_{11}(x_1, x_2) \ge 0\big\} \ \textit{and} \\
D_{12} &= \big\{(x_1, x_2) : -S_3(x_1, x_2) \ge 0,\ -S_{11}(x_1, x_2) \ge 0,\ -S_9(x_1, x_2) \ge 0\big\}.
\end{aligned}$$

Lemma 1. *(See* [4]*)*

$$\Phi(x_1,x_2)=\begin{cases} 1 & \textit{if } (x_1,x_2)\in D_1\cup D_2,\\ |x_2| & \textit{if } (x_1,x_2)\in \bigcup_{k=3}^{7} D_k,\\ \Phi_1(x_1,x_2) & \textit{if } (x_1,x_2)\in D_8\cup D_9,\\ \Phi_2(x_1,x_2) & \textit{if } (x_1,x_2)\in D_{10}\cup D_{11},\\ \Phi_3(x_1,x_2) & \textit{if } (x_1,x_2)\in D_{12}, \end{cases}$$

where:

$$\Phi_1(x_1,x_2)=\frac{2}{3}(|x_1|+1)\left(\frac{|x_1|+1}{3(|x_1|+1+x_2)}\right)^{\frac{1}{2}},$$
$$\Phi_2(x_1,x_2)=\frac{1}{3}x_2\left(\frac{x_1^2-4}{x_1^2-4x_2}\right)\left(\frac{x_1^2-4}{3(x_2-1)}\right)^{\frac{1}{2}} \quad \textit{and}$$
$$\Phi_3(x_1,x_2)=\frac{2}{3}\left(|x_1|-1\right)\left(\frac{|x_1|-1}{3(|x_1|-1-x_2)}\right)^{\frac{1}{2}}.$$

Lemma 2. *For every* $(x_1,x_2)\in\mathbb{C}^2$ *it holds that* $\max\limits_{w\in\mathcal{W}}|x_1w_1^2+x_2w_2|=\max\{|x_1|,|x_2|\}$.

Proof. Let $w_1,w_2\in\mathbb{C}$. Applying the Carathéodory–Toeplitz (C–T) Theorem (see [1]) in the class $\mathcal{W}$, there exists a $w\in\mathcal{W}$ with $w'(0)=w_1$ and $w''(0)=2w_2$ if and only if

(3) $$|w_1|\le 1 \text{ and } |w_2|\le 1-|w_1|^2,$$

or equivalently there exist $(r_1,r_2)\in[0,1]^2$ and $|z_1|=|z_2|=1$ such that

(4) $$w_1=r_1z_1 \text{ and } w_2=(1-r_1^2)r_2z_2.$$

Using (4) we derive that

$$\begin{aligned}\max_{w\in\mathcal{W}}&|x_1w_1^2+x_2w_2|\\ &=\max\left\{|x_1r_1^2z_1^2+x_2(1-r_1^2)r_2z_2| : 0\le r_1\le 1,\ 0\le r_2\le 1,\ |z_1|=|z_2|=1\right\}\\ &=\max\left\{|x_1|r_1^2+|x_2|(1-r_1^2) : 0\le r_1\le 1\right\}\\ &=\max\left\{r_1^2\left(|x_1|-|x_2|\right)+|x_2| : 0\le r_1\le 1\right\}\\ &=\max\left\{|x_1|,|x_2|\right\}.\end{aligned}$$

□

Lemma 3. (α) *If* $\Re\alpha>0$ *then* $f\in\mathcal{P}_{[\alpha]}$ *if and only if it has the form*

$$\frac{1-|a|^{(p(z)+1)}+a(1-|a|^{(p(z)-1)})}{1-|a|^{(p(z)+1)}-a(1-|a|^{(p(z)-1)})}$$

with $p\in\mathcal{P}$.

(β) *For* $(a_1,a_2,a_3)\subset\mathbb{C}^3$ *the following propositions are equivalent:*

(i) *There is a function* $f\in\mathcal{P}_{[\alpha]}$ *such that* $f_1=a_1$, $f_2=a_2$ *and* $f_3=a_3$.

(ii) *There is a function* $w = w_1 z + w_2 z^2 + \ldots \in \mathcal{W}$ *such that:*

$$a_1 = \frac{4a\log|a|}{-1+|a|^2}\, w_1,$$
$$a_2 = \frac{4a\log|a|}{-1+|a|^2}\left(\frac{(-1+|a|^2+(-1+2a-|a|^2)\log|a|)}{-1+|a|^2} w_1^2 + w_2\right),$$
$$a_3 = \frac{4a\log|a|}{-1+|a|^2}\Bigg(w_3 + \frac{2\,(-1+|a|^2-(1-2a+|a|^2)\log|a|)}{-1+|a|^2}\, w_1 w_2 + \frac{1}{3(-1+|a|^2)^2}\big(3(-1+|a|^2)^2 + 2\log|a|(3-3|a|^4+6a(-1+|a|^2) + (1+6a^2+4|a|^2+|a|^4-6a(1+|a|^2))\log|a|)\big)\, w_1^3\Bigg).$$

Proof. For the proof of this lemma we consider the following propositions:

(i) $f \in \mathcal{P}_{[\alpha]}$ if and only if f has the form $f = \frac{1+w}{1-w}$ with $w \in \mathcal{W}_{[a]}$.

(ii) $f \in \mathcal{W}_{[a]}$ if and only if w has the form

$$w = \frac{\alpha - w_1}{1-\bar{\alpha} w_1}$$

with $w_1 \in \mathcal{B}_{[0]}$ and $w_1(0) = a$.

(iii) $w_1 \in \mathcal{B}_{[0]}$ with $w_1(0) = a$ if and only if w_1 has the form $w_1 = a|a|^{p-1}$ with $p \in \mathcal{P}$.

We now observe that the proof of propositions (i) and (ii) is rather simple. The proof of proposition (iii) is obtained by relation $w_1 \in \mathcal{B}_{[0]}$ if and only if w_1 gets the form $w_1 = \lambda e^{-tp}$ with $|\lambda| = 1$, $t > 0$ and $p \in \mathcal{P}$. Setting $w_1(0) = \lambda e^{-t} = a$ we obtain the result of the proposition.

Thus combining the results of propositions (i), (ii) and (iii) we get the result of the lemma. □

Our results are stated in the following theorems.

Theorem 4. (α) *For* $\Re\alpha > 0$ *it holds that:*

(i)

$$\max_{f\in\mathcal{P}_{[\alpha]}} |f_1| = -\frac{4|a|\log|a|}{1-|a^2|}$$

and

(ii)

$$\max_{f\in\mathcal{P}_{[\alpha]}} |f_2| = \max\left\{\left|\frac{4a\log|a|(-1-|a|^2(-1+\log|a|)+(-1+2a)\log|a|)}{(-1+|a|^2)^2}\right|, \left|\frac{4a\log|a|}{-1+|a|^2}\right|\right\}.$$

(β) *For $\alpha > 0$ it holds that:*

$$\max_{f\in\mathcal{P}_{[\alpha]}}|f_3|$$
$$= \begin{cases} |x_{31}(a)|\ \Phi_1(x_{11}(a), x_{21}(a)) & \textit{for} \quad \alpha \in (0, 0.76227] \cup [1.05537, 1.39636], \\ |x_{31}(a)|\ \Phi_2(x_{11}(a), x_{21}(a)) & \textit{for} \quad \alpha \in [0.76227, 0.883736] \cup [1.04583, 1.05537], \\ |x_{31}(a)|\ \Phi_3(x_{11}(a), x_{21}(a)) & \textit{for} \quad \alpha \in [0.883736, 0.914114] \cup [1.03238, 1.04583], \\ |x_{31}(a)|\ |x_{21}(a)| & \textit{for} \quad \alpha \in [0.914114, 1.03238], \\ |x_{31}(a)| & \textit{for} \quad \alpha \in [1.39636, \infty), \end{cases}$$

with:

$$x_{11}(a) = \frac{2\big(-1 + a^2 - (-1+a)^2 \log|a|\big)}{-1+a^2},$$
$$x_{21}(a) = \frac{(-1+a)^2\big(-3(-3+a)(1+a)\log|a| + 2(1-4a+a^2)(\log|a|)^2\big)}{3(-1+a^2)^2}$$

and

$$x_{31}(a) = \frac{4a\log|a|}{-1+a^2}.$$

Theorem 5. (α) *For $\Re\alpha > 0$ and $\mu \in \mathbb{C}$ it holds that:*

(i)
$$\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_2| = -\frac{4|a|\log|a|}{1-|a^2|},$$

(ii)
$$\max_{f\in\mathcal{S}^*_{[\alpha]}}|F_2| = \max_{f\in\mathcal{S}^*_{[\alpha]}}|f_2|,$$

(iii)
$$\max_{f\in\mathcal{K}_{[\alpha]}}|f_2| = \max_{f\in\mathcal{K}_{[\alpha]}}|F_2| = \frac{1}{2}\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_2|,$$

(iv)
$$\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_3 - \mu f_2^2|$$
$$= \max\left\{\left|\frac{2a\log|a|(1-|a|^2 + (1+|a|^2+2a(-3+4\mu))\log|a|)}{(-1+|a|^2)^2}\right|, \left|\frac{2a\log|a|}{-1+|a|^2}\right|\right\},$$

(v)
$$\max_{f\in\mathcal{S}^*_{[\alpha]}}|F_3 - \mu F_2^2| = \max_{f\in\mathcal{S}^*_{[\alpha]}}|f_3 + (\mu-2)f_2^2|,$$

(vi)
$$\max_{f\in\mathcal{K}_{[\alpha]}}|f_3 - \mu f_2^2| = \frac{1}{3}\max_{f\in\mathcal{S}^*_{[\alpha]}}\left|f_3 - \frac{3}{4}\,\mu\, f_2^2\right|$$

and

(vii)
$$\max_{f\in\mathcal{K}_{[\alpha]}}|F_3 - \mu F_2^2| = \frac{1}{3}\max_{f\in\mathcal{S}^*_{[\alpha]}}\left|f_3 + (\mu-2)\frac{3}{4}f_2^2\right|.$$

(β) *For $\alpha > 0$ it holds that:*

(i) $\max_{f\in\mathcal{S}^*_{[\alpha]}} |f_4|$

$$= \begin{cases} |x_{32}(a)|\,|x_{22}(a)| & \textit{for}\ \ \alpha \in (0, 1.02357] \cup [1.14133, 1.33331] \\ & \qquad\qquad \cup [1.76736, \infty), \\ |x_{32}(a)|\,\Phi_3(x_{12}(a), x_{22}(a)) & \textit{for}\ \ \alpha \in [1.02357, 1.03283], \\ |x_{32}(a)|\,\Phi_2(x_{12}(a), x_{22}(a)) & \textit{for}\ \ \alpha \in [1.03283, 1.0378] \cup [1.73905, 1.76736], \\ |x_{32}(a)|\,\Phi_1(x_{12}(a), x_{22}(a)) & \textit{for}\ \ \alpha \in [1.0378, 1.14133] \cup [1.33331, 1.73905], \end{cases}$$

with:

$$x_{12}(a) = -\frac{2(1 - a^2 + (1 - 5a + a^2)\log|a|)}{-1 + a^2},$$

$$x_{22}(a) = 1 - \frac{6(-1 + 5a - 5a^3 + a^4)\log|a|}{3(-1 + a^2)^2} + \frac{2(1 + a(-15 + a(40 + (-15 + a)a)))(\log|a|)^2}{3(-1 + a^2)^2}$$

and

$$x_{32}(a) = \frac{4a\log|a|}{3(-1 + a^2)},$$

(ii)

$$\max_{f\in\mathcal{K}_{[\alpha]}} |f_4| = \frac{1}{4} \max_{f\in\mathcal{S}^*_{[\alpha]}} |f_4|,$$

(iii) $\max_{f\in\mathcal{S}^*_{[\alpha]}} |F_4|$

$$= \begin{cases} |x_{33}(a)|\,|x_{23}(a)| & \textit{for}\ \ \alpha \in (0, 0.711625] \cup [0.824185, 0.936408] \\ & \qquad\qquad \cup [0.983596, \infty), \\ |x_{33}(a)|\,\Phi_3(x_{13}(a), x_{23}(a)) & \textit{for}\ \ \alpha \in [0.711625, 0.71718] \cup [0.977731, 0.983596], \\ |x_{33}(a)|\,\Phi_2(x_{13}(a), x_{23}(a)) & \textit{for}\ \ \alpha \in [0.71718, 0.732352] \cup [0.975309, 0.977731], \\ |x_{33}(a)|\,\Phi_1(x_{13}(a), x_{23}(a)) & \textit{for}\ \ \alpha \in [0.732352, 0, 824185] \cup [0.936408, 0.975309], \end{cases}$$

with:

$$x_{13}(a) = -\frac{2\big(1 - a^2 + (1 + 10a + a^2)\log|a|\big)}{-1 + a^2},$$

$$x_{23}(a) = 1 - \frac{6(-1 + a)(1 + a)(1 + a(10 + a))\log|a|}{3(-1 + a^2)^2} + \frac{2(1 + a(30 + a(130 + (30 + a)a)))(\log|a|)^2}{3(-1 + a^2)^2}$$

and

$$x_{33}(a) = \frac{4a\log|a|}{3(-1 + a^2)}$$

and

(iv) $\max_{f\in\mathcal{K}_{[\alpha]}} |F_4|$

$$= \begin{cases} |x_{34}(a)|\,|x_{24}(a)| & \textit{for } \alpha\in(0,0.565815]\cup[0.750011,0.876173] \\ & \qquad\cup[0.976968,\infty), \\ |x_{34}(a)|\,\Phi_2(x_{14}(a),x_{24}(a)) & \textit{for } \alpha\in[0.565815,0.575026]\cup[0.963576,0.968213], \\ |x_{34}(a)|\,\Phi_1(x_{14}(a),x_{24}(a)) & \textit{for } \alpha\in[0.575026,0.750011]\cup[0.876173,0.963576], \\ |x_{34}(a)|\,\Phi_3(x_{14}(a),x_{24}(a)) & \textit{for } \alpha\in[0.968213,0.976968], \end{cases}$$

with:

$$x_{14}(a) = -\frac{2(1-a^2+(1+5a+a^2)\log|a|)}{-1+a^2},$$
$$x_{24}(a) = 1-\frac{6(-1-5a+5a^3+a^4)\log|a|}{3(-1+a^2)^2} + \frac{2(1+a(15+a(40+(15+a)a)))(\log|a|)^2}{3(-1+a^2)^2}$$

and

$$x_{34}(a) = \frac{a\log|a|}{-3(-1+a^2)}.$$

Proof. (i) For every $f\in\mathcal{P}_{[\alpha]}$, we set the coefficients f_1, f_2 and f_3 in the form of Lemma 3 (β). Using the relation $\max_{w\in\mathcal{W}}|w_1| = 1$, we find that $\max_{f\in\mathcal{P}_{[\alpha]}}|f_1|$ coincides with the form given in Theorem 4.
In a similar way, $\max_{f\in\mathcal{P}_{[\alpha]}}|f_2|$ presented in Theorem 4 follows using Lemma 2.

(ii) Using Lemma 3 (β), after the calculations we obtain that

$$\max_{f\in\mathcal{P}_{[\alpha]}}|f_3| = |x_{31}(a)|\;\;\Phi\big(x_{11}(a),x_{21}(a)\big).$$

In order to find for any $a\in(-1,1)$ the corresponding branch of Φ, we proceed finding all the roots of each equation $S_i\big(x_1(a),x_2(a)\big)=0$ $(i=1,\dots,11)$, with respect to a, that belong in $(-1,1)$. The procedure by which these calculations are obtained will be described later.

In the next step, we consider the partition of the interval $(-1,1)$ formed by the above roots, into successive subintervals, being defined $\forall i$ respectively. Checking in each subinterval the sign of all $S_i\big(x_1(a),x_2(a)\big)$, through Lemma 1, we select the corresponding branch of Φ to this subinterval. More specifically we verify that the roots of all quantities $S_i\big(x_1(a),x_2(a)\big)$ belong to the set:

$$A = \{-0.1349,-0.0761,-0.06172,-0.04487,0.01593,0.0224,\\ 0.02694,0.09078,0.1654,0.17104,0.2707\}.$$

Checking the signs of functions $S_i\big(x_1(a), x_2(a)\big)$ in the twelve subintervals defined, we obtain the formulation of the following inequalities:

$$\begin{aligned}
&S_1 \geq 0 && \text{iff } a \in [0.17104, 0.27070],\\
&S_2 \geq 0 && \text{iff } a \in [0.09078, 1),\\
&S_3 \geq 0 && \text{iff } a \in (-1, -0.07610] \cup [0.03902, 1),\\
&S_4 \geq 0 && \text{iff } a \in (-1, 0.10256] \cup [0.16413, 1),\\
&S_5 \geq 0 && \text{iff } a \in [0.03220, 1),\\
&S_6 \geq 0 && \text{iff } a \in [0.16540, 1),\\
&S_7 \geq 0 && \text{iff } a \in [0.17550, 0.22472],\\
&S_8 \geq 0 && \text{iff } a \in [-0.04027, 0.01306],\\
&S_9 \geq 0 && \text{iff } a \in (-1, -0.25267] \cup [-0.04487, 0.01593] \cup [0.44174, 1),\\
&S_{10} \geq 0 && \text{iff } a \in (-1, -0.13490] \cup [0.02694, 1) \quad \text{and}\\
&S_{11} \geq 0 && \text{iff } a \in (-1, -0.06172] \cup [0.02240, 1).
\end{aligned}$$

In this way we also get for every function S_i, the set of $a \in (-1, 1)$ with $S_i\,(x_1(a), x_2(a)) < 0$. Therefore by Lemma 1 we obtain the result given in part (ii) of Theorem 4, which completes the proof of the theorem.

It only remains to solve the equations $S_i\,(x_1(a), x_2(a)) = 0$. In the following paragraphs, we outline our methodology for the solution of these equations. But first let us stress that although we will use a numeric computation program like Mathematica, its use will be restricted *only* to the following cases:

(i) once we have rigorously proved that a given function of a single real variable has a *unique* root in a given closed interval then, we will compute this root by Mathematica,

(ii) given a polynomial of a single variable, we will use Mathematica to compute the k roots of it and

(iii) we will use Mathematica to perform both numeric and symbolic algebraic calculations that can, in principle, be performed by hand.

Replacing in the initial equation all the expressions of the form $|\Pi|$ with $\pm\Pi$, we form all possible combinations, deriving some equations of the form

$$Q(t) = Q_0(t) + Q_1(t)\log|t| + \cdots + Q_k(t)\,(\log|t|)^k = 0,$$

with Q_i $i = 0, \ldots, k$ rational functions of a single real variable t. Therefore, it suffices to solve the new equations and then check which of their roots are also roots of the original one (we call k the *logarithmic degree* of $Q(t)$ – in our case $k = 3$).

By dividing with $Q_k(t)$, we may further suppose that the maximum logarithmic degree coefficient $Q_k(t)$ is constantly 1 (this step requires a check whether any root of Q_k is also a root of the whole equation). Therefore, we now have to deal with an equation of the form

$$Q(t) = Q_0(t) + Q_1(t)\log t + \cdots + Q_{k-1}(t)\big(\log t\big)^{k-1} + \big(\log t\big)^k = 0.$$

Our crucial observation now is that if we differentiate $Q(t)$, we obtain a function of the same form, but with its logarithmic degree decremented by one (and possibly with a non-constant coefficient for $\big(\log t\big)^{k-1}$). Assume that we have computed the roots of the latter equation of logarithmic degree $k-1$. Then we can locate the (closed) intervals where the original $Q(t)$ is either strictly increasing or strictly decreasing. In such an interval, $Q(t)$ can have at most one root. We can easily determine if at this interval $Q(t)$ has a unique root or no root at all. In case it has a unique root, we compute it using Mathematica.

These remarks lead to the following recursive computation of the roots of $Q(t)$. Differentiate repeatedly and between any two differentiations divide with the maximum degree coefficient, until a function with no logarithms (of logarithmic degree 0) is obtained. This obviously is a rational function. Find its roots by the use of Mathematica. Then backtrack step by step to the original function using the strict monotonicity intervals at each step to locate the intervals where the function of the previous step has a unique solution. Then find the roots of the previous step by Mathematica and proceed further back to larger logarithmic degree, until the original function is reached and all of its roots are computed. This ends the description of our methodology for the solution of $S_i\,(x_1(a), x_2(a)) = 0$ and completes the proof of the theorem. □

Remark 6. For $a \in (-1, 1)$ we remark that:

(i) For $a \to 1$ (equivalently $\alpha \to 0$) then $\max_{f\in\mathcal{P}_{[a]}} |f_k| \to 2$, $k = 1, 2, 3$.
(ii) For $a \to -1$ (equivalently $\alpha \to \infty$) then $\max_{f\in\mathcal{P}_{[a]}} |f_k| \to 2$, $k = 1, 2, 3$.

These remarks are making obvious that our theorem consists an extension of Carathé -odory's inequality in the cases $i = 1, 2, 3$. In Figures 1 – 3 we give the graphical representations of $\max_{f\in\mathcal{P}_{[\alpha]}} |f_k|$ as functions of a, for $k = 1, 2, 3$ respectively.

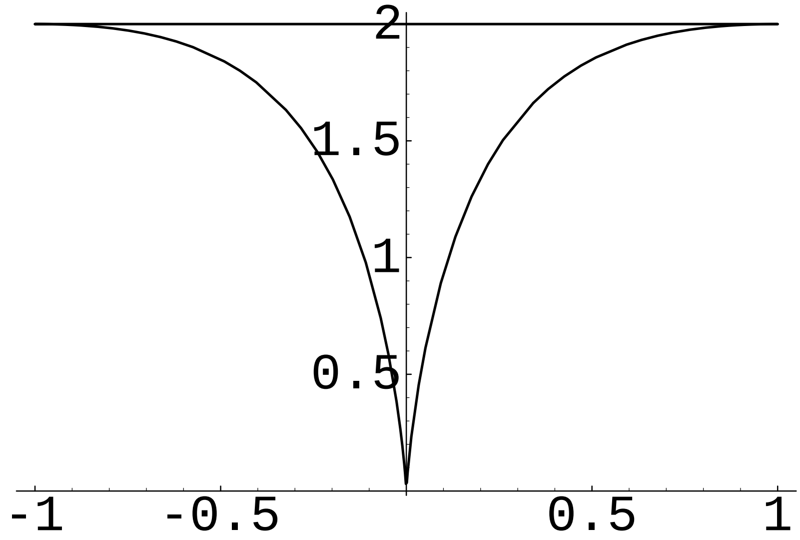

Figure 1:

We also remark that:

(iii) For $a \to \pm 1$ then $\max_{f\in\mathcal{S}^*_{[a]}} |f_n| \to n$, $n = 2, 3, 4$.
(iv) For $a \to \pm 1$ then $\max_{f\in\mathcal{K}_{[a]}} |f_n| \to 1$, $n = 2, 3, 4$.

Remarks (iii) and (iv) are pointing out that our theorem consists as well an extension of Rogosinski's inequality (see [5]), in the cases $i = 2, 3, 4$, for starlike or convex functions respectively. The graphical representations of $\max_{f\in\mathcal{S}^*_{[\alpha]}} |f_n|$, as functions of a for $n = 2, 3, 4$, are given in Figures 4 – 6 respectively.

Proof of Theorem 5. A function $f \in \mathcal{S}^*_{[\alpha]} \iff \frac{zf'}{f} \equiv q \in \mathcal{P}_{[\alpha]}$ while $f := z + f_2z^2 + f_3z^3 + \dots$, and $q := 1 + q_1z + q_2z^2 + \dots$. Since

$$f_2 = q_1,\ f_3 = \frac{1}{2}(q_2 + q_1^2) \text{ and } f_4 = \frac{1}{6}(q_1^3 + 3q_1q_2 + 2q_3),$$

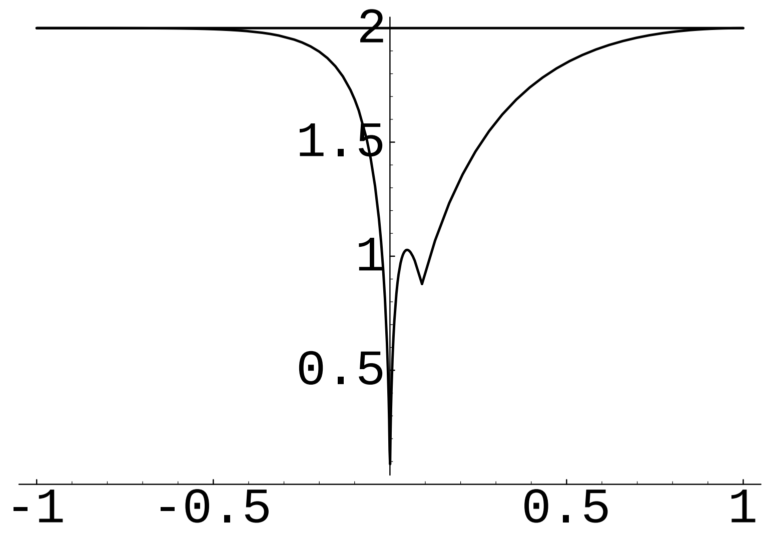

Figure 2:

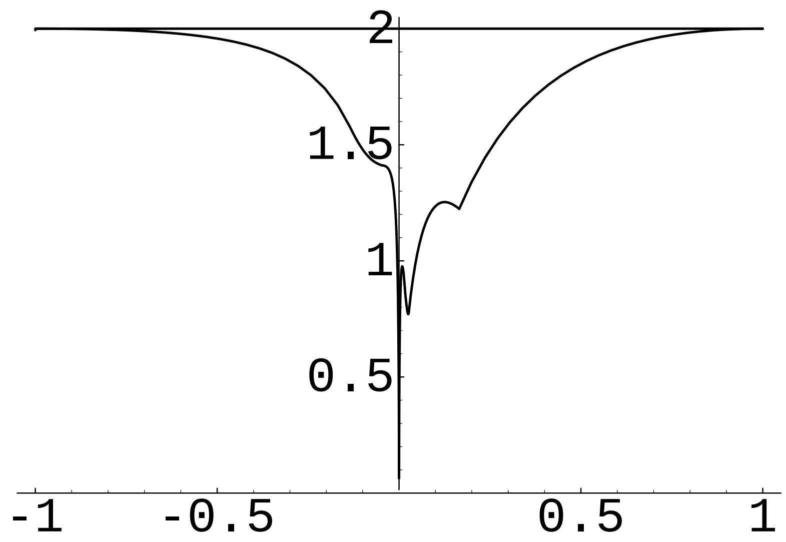

Figure 3:

expanding f to Taylor series and substituting quantities q_1, q_2 and q_3 with their form described in Lemma 3, we obtain that:

$$f_2 = \frac{4a \log |a|}{-1 + |a|^2}\, w_1,$$

$$f_3 = \frac{2a \log |a|}{(-1 + |a^2|)^2} \left((-1 + |a^2|)w_2 + \left(-1 + |a^2| + (-1 + 6a - |a^2|) \log |a|\right) w_1^2\right)$$

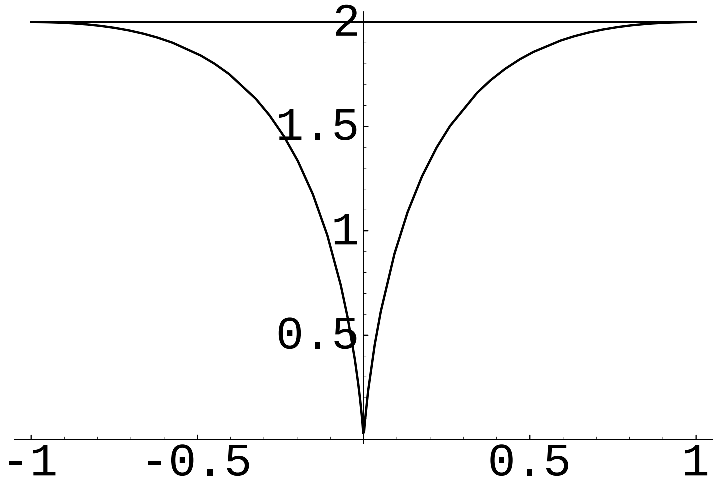

Figure 4:

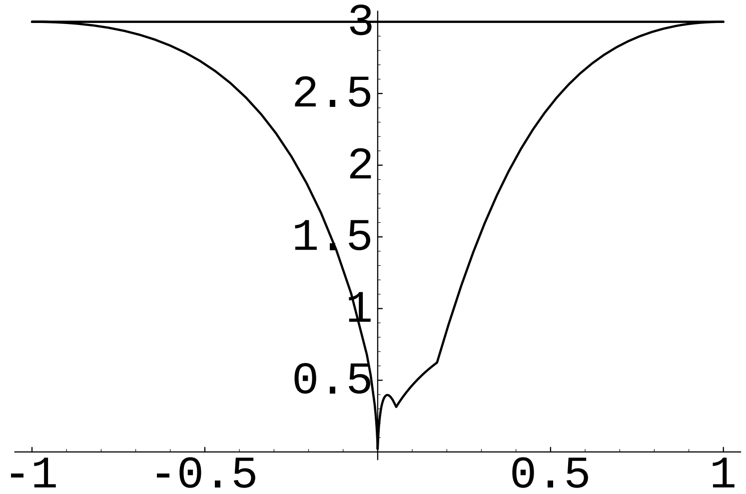

Figure 5:

and

$$f_4 = \frac{4a\log|a|}{9(-1+a^2)^3}\Big(3(-1+a^2)^2 w_3 - 6(-1+a^2)\left(1-a^2+(1-5a+a^2)\log|a|\right)w_1w_2$$
$$+(3(-1+a^2)^2 - 6(-1+5a-5a^3+a^4)\log|a| + 2(1-15a+40a^2-15a^3+a^4)(\log|a|)^2)w_1^3\Big).$$

Quantities $\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_2|$, $\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_3-\mu f_2^2|$ and $\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_4|$, are derived following the procedure of the previous theorem.

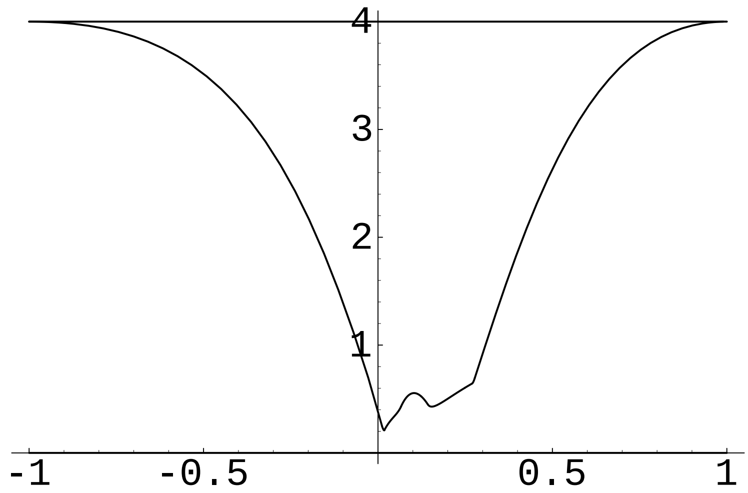

Figure 6:

For the inverse function $F \equiv f^{-1}$ it holds that:

$$F_2 = -f_2, \;\; F_3 = 2f_2^2 - f_3 \;\text{ and }\; F_4 = -5f_2^3 + 5f_2f_3 - f_4.$$

It is obvious that the most complicated case is the calculation of $\max_{f\in\mathcal{S}^*_{[\alpha]}}|F_4|$, since $\max_{f\in\mathcal{S}^*_{[\alpha]}}|F_2|$ and $\max_{f\in\mathcal{S}^*_{[\alpha]}}|F_3 - \mu F_2^2|$ are derived straightforwardly. Substituting f_2, f_3 and f_4 in F_4, we obtain a polynomial expression with respect to w_1, w_2 and w_3. We work in a similar way with the calculation of $\max_{f\in\mathcal{P}_{[\alpha]}}|f_3|$ to complete the proof.

Furthermore, due to the initial assumption of this paper, a function $g \in \mathcal{K}_{[\alpha]} \iff 1 + \frac{zg''}{g'} \equiv q \in \mathcal{P}_{[\alpha]}$ with $g := z + g_2z^2 + g_3z^3 + \ldots$ and $q := 1 + q_1z + q_2z^2 + \ldots$. Now let $\mathcal{S}^*_{[\alpha]} \ni f := z + f_2z^2 + f_3z^3 + \ldots$. It can be easily seen that:

$$g_2 = \frac{f_2}{2}, \quad g_3 = \frac{f_3}{3} \;\text{ and }\; g_4 = \frac{f_4}{4},$$

which renders the proof of (α) (iii), (vi), (vii) and (β) (ii) straightforward. In order to prove (β) (iv), we work in the same manner with the calculation of $\max_{f\in\mathcal{P}_{[\alpha]}}|f_3|$ and $\max_{f\in\mathcal{S}^*_{[\alpha]}}|f_4|$. $\square$

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