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EXTENSIONS OF POPOVICIU'S INEQUALITY USING A GENERAL METHOD

This note is dedicated to my wife Mari – a very brave lady.

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Abstract

A lemma of considerable generality is proved from which one can obtain inequalities of Popoviciu's type involving norms in a Banach space and Gram determinants.

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1. Introduction

Let x_k and y_k $(1 \leq k \leq n)$ be non-negative real numbers. Let $\frac{1}{p} + \frac{1}{q} = 1,\ p, q > 1$, and suppose that $\sum x_k^p \leq 1$ and $\sum y_k^q \leq 1$. Then an inequality due to Popoviciu reads:

$$\left(1 - \sum x_k y_k\right) \geq \left(1 - \sum x_k^p\right)^{\frac{1}{p}} \left(1 - \sum y_k^q\right)^{\frac{1}{q}}. \tag{1.1}$$

When we make the substitutions

$$x_k^p \to w_k \left(\frac{a_k}{a}\right)^p \quad \text{and} \quad y_k^q \to w_k \left(\frac{b_k}{b}\right)^q \tag{1.2}$$

in (1.1) and then multiply throughout by ab we get the more usual, but no more general, form of the inequality; (see [1, p.118], or [2, p.58], for example). The case $p = q = 2$ is called Aczèl's Inequality [1, p.117] or [2, p.57].

Our purpose here is to present a general inequality, (see lemma below), whose proof is short but which yields many generalizations of (1.1).

We shall present all our results in a 'reduced form' like (1.1) but it would be a simple matter to rescale them in the spirit of (1.2) to obtain inequalities of more apparent generality.

2. The Basic Result

Lemma 2.1. *Let f be a real-valued function defined and continuous on $[0,1]$ that is positive, strictly decreasing and strictly log-concave on the open interval $(0,1)$. Let $x,y,z \in [0,1]$ and $z \le xy$ there. Then with p and q defined as above we have:*

$$f(z) \ge f(xy) \ge [f(x^p)]^{\frac{1}{p}}[f(y^q)]^{\frac{1}{q}}. \tag{2.1}$$

Proof. Write $L(x) = \log f(x)$. The properties of f imply that L is a strictly decreasing and strictly concave function on $(0,1)$. Hence if $x,y,z \in (0,1)$ and $z \le xy$ we have

$$L(z) \ge L(xy) \ge L\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right) \ge \frac{1}{p}L(x^p) + \frac{1}{q}L(y^q). \tag{2.2}$$

The second step here uses the arithmetic mean-geometric mean inequality and the third uses the strict concavity of L; these inequalities are strict if $x \ne y$. Taking exponentials we get

$$f(z) \ge f(xy) \ge [f(x^p)]^{\frac{1}{p}}[f(y^q)]^{\frac{1}{q}}. \tag{2.3}$$

Appealing to the continuity of f we now extend this to the case in which $x,y,z \in [0,1]$ and $z \le xy$ there. This completes the proof of the lemma. □

Note: The conditions in Lemma 2.1 are satisfied if f is twice differentiable on $(0,1)$ and on that open interval $f > 0$, $f' < 0$, $ff'' - (f')^2 < 0$. Our reason for working in $(0,1)$ and then proceeding to $[0,1]$ via continuity is because our main specialization below will be $f(x) = 1 - x$.

3. Specializations

We now state some inequalities which result by specializing (2.1).

(1) Suppose that B is a Banach space whose dual is B^*. Let $g \in B$ and $F \in B^*$. Recalling that $|F(g)| \leq \|F\| \|g\|$ we read this as $z \leq xy$ and then (2.1) reads:

$$f(|F(g)|) \geq f(\|F\| \|g\|) \geq [f(\|F\|^p)]^{\frac{1}{p}} [f(\|g\|^q)]^{\frac{1}{q}}, \tag{3.1}$$

provided that $\|F\|, \|g\| \leq 1$.

(2) Taking $f(t) = (1-t)$ specializes (3.1) further to:

$$(1 - |F(g)|) \geq (1 - \|F\| \|g\|) \geq (1 - \|F\|^p)^{\frac{1}{p}} (1 - \|g\|^q)^{\frac{1}{q}}, \tag{3.2}$$

provided that $\|F\|, \|g\| \leq 1$.

(3) Examples of (3.1) and (3.2) are afforded by taking B to be the sequence space $B = l_q^{(n)}$ in which case B^* is the space $l_p^{(n)}$. Then the outer inequalities of (3.1) and (3.2) yield:

$$f\left(\left|\sum x_k y_k\right|\right) \geq \left[f\left(\sum |x_k^p|\right)\right]^{\frac{1}{p}} \left[f\left(\sum |y_k^q|\right)\right]^{\frac{1}{q}} \tag{3.3}$$

and

$$\left(1 - \left|\sum x_k y_k\right|\right) \geq \left(1 - \sum |x_k^p|\right)^{\frac{1}{p}} \left(1 - \sum |y_k^q|\right)^{\frac{1}{q}}, \tag{3.4}$$

provided that, in each case, $\left(\sum |x_k^p|\right)^{\frac{1}{p}} \leq 1$ and $\left(\sum |y_k^q|\right)^{\frac{1}{q}} \leq 1$.

The inequality (3.4) is a slightly stronger form of (1.1).

Taking other interpretations of B and B^* we give two more examples of the outer inequalities of (3.2) as follows:

(4)

$$\left(1-\left|\int_E uv\right|\right) \geq \left(1-\int_E |u|^p\right)^{\frac{1}{p}} \left(1-\int_E |v|^q\right)^{\frac{1}{q}},$$

provided that $\left(\int_E |u|^p\right)^{\frac{1}{p}} \leq 1$ and $\left(\int_E |v|^q\right)^{\frac{1}{q}} \leq 1$. The integrals are Lebesgue integrals and E is a bounded measurable subset of the real numbers.

(5) When we take $B \equiv C[0,1]$ and $B^* \equiv BV[0,1]$ in (3.2) we get the somewhat exotic result:

$$\left(1-\left|\int_0^1 h(t)d\alpha(t)\right|\right) \geq [1-(\mathrm{Max}\,|h|)^p]^{\frac{1}{p}}[1-(\mathrm{Var}(\alpha))^q]^{\frac{1}{q}},$$

where the maximum and total variation are taken over $[0,1]$ and each is less than or equal to 1.

4. Inequalities for Grammians

Let $\Gamma(\mathbf{x},\mathbf{y})$, $\Gamma(\mathbf{x})$ and $\Gamma(\mathbf{y})$ denote the determinants of size n whose (i,j)th elements are respectively the inner products $(\mathbf{x}_i,\mathbf{y}_j)$, $(\mathbf{x}_i,\mathbf{x}_j)$ and $(\mathbf{y}_i,\mathbf{y}_j)$ where the $\mathbf{x}$'s and $\mathbf{y}$'s are vectors in a Hilbert space. Then it is known, see [1, p.599], that

$$[\Gamma(\mathbf{x},\mathbf{y})]^2 \le \Gamma(\mathbf{x})\Gamma(\mathbf{y}). \tag{4.1}$$

It is also well-known that the two factors on the right side of (3.3) are each non-negative so that we have

$$|\Gamma(\mathbf{x},\mathbf{y})|^{2\alpha} \le [\Gamma(\mathbf{x})]^{\alpha}\,[\Gamma(\mathbf{y})]^{\alpha} \quad \text{if} \quad \alpha > 0. \tag{4.2}$$

If we read this as $z \le xy$ and we take $f(t) = 1 - t$ again then (2.1) gives

$$(1 - |\Gamma(\mathbf{x},\mathbf{y})|^{2\alpha}) \ge [1 - (\Gamma(\mathbf{x}))^{p\alpha}]^{\frac{1}{p}}[1 - (\Gamma(\mathbf{y}))^{q\alpha}]^{\frac{1}{q}},$$

provided that $\Gamma(\mathbf{x}) \le 1$ and $\Gamma(\mathbf{y}) \le 1$.

When $p = q = 2$ and $\alpha = \frac{1}{2}$ this is a result due to J. Pečarić, [4], and when $p = q = 2$ and $\alpha = \frac{1}{4}$ we get a result due to S.S. Dragomir and B. Mond, [1, Theorem 2].

5. Some Final Remarks

In giving examples of the use of (2.1) we have used the function $f(t) = 1 - t$ since that is the source of Popoviciu's result. But interesting inequalities arise also from other suitable choices of f. For example, taking

$$f(t) = \frac{(\alpha - t)}{(\beta - t)} \text{ in } [0,1] \quad (1 < \alpha < \beta)$$

we are led to the result

$$\left[\frac{\alpha - \left|\sum x_k y_k\right|}{\beta - \left|\sum x_k y_k\right|}\right] \geq \left[\frac{\alpha - \sum |x_k^p|}{\beta - \sum |x_k^p|}\right]^{\frac{1}{p}} \left[\frac{\alpha - \sum |y_k^q|}{\beta - \sum |y_k^q|}\right]^{\frac{1}{q}}$$

provided that $\left(\sum |x_k^p|\right)^{\frac{1}{p}} \leq 1$, $\left(\sum |y_k^q|\right)^{\frac{1}{q}} \leq 1$. This reduces to Popoviciu's inequality (3.2) if we multiply throughout by β, let $\beta \to \infty$ and $\alpha \to 1$.

Next let f possess the same properties as in Lemma 2.1 above but now take $x, y, z, w \in [0,1]$ with $w \leq xyz$. Then one finds that

$$f(w) \geq f(xyz) \geq [f(x^p)]^{\frac{1}{p}} [f(y^q)]^{\frac{1}{q}} [f(z^r)]^{\frac{1}{r}},$$

where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (p, q, r < 1).$$

Specializing this by again taking $f(t) = 1 - t$ and reading the extended Hölder inequality

$$\left|\sum x_k y_k z_k\right| \leq \left[\sum |x_k|^p\right]^{\frac{1}{p}} \left[\sum |y_k|^q\right]^{\frac{1}{q}} \left[\sum |z_k|^r\right]^{\frac{1}{r}}$$

as $w \leq xyz$ we get the Popoviciu-type inequality:

$$\left(1 - \left|\sum x_k y_k z_k\right|\right) \geq \left(1 - \sum |x_k^p|\right)^{\frac{1}{p}} \left(1 - \sum |y_k^q|\right)^{\frac{1}{q}} \left(1 - \sum |z_k^r|\right)^{\frac{1}{r}},$$

provided $(\sum |x_k^p|)^{\frac{1}{p}} \leq 1$, $(\sum |y_k^q|)^{\frac{1}{q}} \leq 1$, $(\sum |z_k^r|)^{\frac{1}{r}} \leq 1$.

One can also construct inequalities which involve products of four or more factors, in the same way.

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