

# Journal of Inequalities in Pure and Applied Mathematics

## EXTENSIONS OF POPOVICIU'S INEQUALITY USING A GENERAL METHOD

*This note is dedicated to my wife Mari – a very brave lady.*

A. McD. MERCER

Department of Mathematics and Statistics  
University of Guelph  
Guelph, Ontario, N1G 1J4, Canada;

Box 12, RR 7. Belleville,  
Ontario, K8N 4Z7, Canada.  
EMail: [amercer@reach.net](mailto:amercer@reach.net)

©2000 Victoria University  
ISSN (electronic): 1443-5756  
132-02



---

volume 4, issue 1, article 11,  
2003.

*Received 23 July 2002;  
accepted 22 November 2002.*

*Communicated by: P.S. Bullen*

---

Abstract

Contents



Home Page

Go Back

Close

Quit

## Abstract

A lemma of considerable generality is proved from which one can obtain inequalities of Popoviciu's type involving norms in a Banach space and Gram determinants.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Aczel's inequality, Popoviciu's inequality, Inequalities for Grammians.

## Contents

1	Introduction .....	3
2	The Basic Result .....	4
3	Specializations .....	5
4	Inequalities for Grammians .....	7
5	Some Final Remarks .....	8
	References	



---

### Extensions of Popoviciu's Inequality Using a General Method

A. McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 10

# 1. Introduction

Let  $x_k$  and  $y_k$  ( $1 \leq k \leq n$ ) be non-negative real numbers. Let  $\frac{1}{p} + \frac{1}{q} = 1$ ,  $p, q > 1$ , and suppose that  $\sum x_k^p \leq 1$  and  $\sum y_k^q \leq 1$ . Then an inequality due to Popoviciu reads:

$$(1.1) \quad \left(1 - \sum x_k y_k\right) \geq \left(1 - \sum x_k^p\right)^{\frac{1}{p}} \left(1 - \sum y_k^q\right)^{\frac{1}{q}}.$$

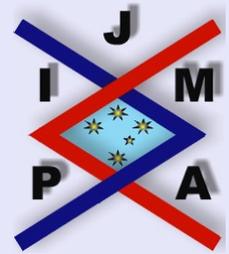
When we make the substitutions

$$(1.2) \quad x_k^p \rightarrow w_k \left(\frac{a_k}{a}\right)^p \quad \text{and} \quad y_k^q \rightarrow w_k \left(\frac{b_k}{b}\right)^q$$

in (1.1) and then multiply throughout by  $ab$  we get the more usual, but no more general, form of the inequality; (see [1, p.118], or [2, p.58], for example). The case  $p = q = 2$  is called Aczèl's Inequality [1, p.117] or [2, p.57].

Our purpose here is to present a general inequality, (see lemma below), whose proof is short but which yields many generalizations of (1.1).

We shall present all our results in a 'reduced form' like (1.1) but it would be a simple matter to rescale them in the spirit of (1.2) to obtain inequalities of more apparent generality.



Extensions of Popoviciu's  
Inequality Using a General  
Method

A. McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 3 of 10

## 2. The Basic Result

**Lemma 2.1.** *Let  $f$  be a real-valued function defined and continuous on  $[0, 1]$  that is positive, strictly decreasing and strictly log-concave on the open interval  $(0, 1)$ . Let  $x, y, z \in [0, 1]$  and  $z \leq xy$  there. Then with  $p$  and  $q$  defined as above we have:*

$$(2.1) \quad f(z) \geq f(xy) \geq [f(x^p)]^{\frac{1}{p}} [f(y^q)]^{\frac{1}{q}}.$$

*Proof.* Write  $L(x) = \log f(x)$ . The properties of  $f$  imply that  $L$  is a strictly decreasing and strictly concave function on  $(0, 1)$ . Hence if  $x, y, z \in (0, 1)$  and  $z \leq xy$  we have

$$(2.2) \quad L(z) \geq L(xy) \geq L\left(\frac{1}{p}x^p + \frac{1}{q}y^q\right) \geq \frac{1}{p}L(x^p) + \frac{1}{q}L(y^q).$$

The second step here uses the arithmetic mean-geometric mean inequality and the third uses the strict concavity of  $L$ ; these inequalities are strict if  $x \neq y$ .

Taking exponentials we get

$$(2.3) \quad f(z) \geq f(xy) \geq [f(x^p)]^{\frac{1}{p}} [f(y^q)]^{\frac{1}{q}}.$$

Appealing to the continuity of  $f$  we now extend this to the case in which  $x, y, z \in [0, 1]$  and  $z \leq xy$  there. This completes the proof of the lemma.  $\square$

**Note:** The conditions in Lemma 2.1 are satisfied if  $f$  is twice differentiable on  $(0, 1)$  and on that open interval  $f > 0$ ,  $f' < 0$ ,  $f f'' - (f')^2 < 0$ . Our reason for working in  $(0, 1)$  and then proceeding to  $[0, 1]$  via continuity is because our main specialization below will be  $f(x) = 1 - x$ .



Extensions of Popoviciu's  
Inequality Using a General  
Method

A. McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 4 of 10

### 3. Specializations

We now state some inequalities which result by specializing (2.1).

- (1) Suppose that  $B$  is a Banach space whose dual is  $B^*$ . Let  $g \in B$  and  $F \in B^*$ . Recalling that  $|F(g)| \leq \|F\| \|g\|$  we read this as  $z \leq xy$  and then (2.1) reads:

$$(3.1) \quad f(|F(g)|) \geq f(\|F\| \|g\|) \geq [f(\|F\|^p)]^{\frac{1}{p}} [f(\|g\|^q)]^{\frac{1}{q}},$$

provided that  $\|F\|, \|g\| \leq 1$ .

- (2) Taking  $f(t) = (1 - t)$  specializes (3.1) further to:

$$(3.2) \quad (1 - |F(g)|) \geq (1 - \|F\| \|g\|) \geq (1 - \|F\|^p)^{\frac{1}{p}} (1 - \|g\|^q)^{\frac{1}{q}},$$

provided that  $\|F\|, \|g\| \leq 1$ .

- (3) Examples of (3.1) and (3.2) are afforded by taking  $B$  to be the sequence space  $B = l_q^{(n)}$  in which case  $B^*$  is the space  $l_p^{(n)}$ . Then the outer inequalities of (3.1) and (3.2) yield:

$$(3.3) \quad f\left(\left|\sum x_k y_k\right|\right) \geq \left[f\left(\sum |x_k^p|\right)\right]^{\frac{1}{p}} \left[f\left(\sum |y_k^q|\right)\right]^{\frac{1}{q}}$$

and

$$(3.4) \quad \left(1 - \left|\sum x_k y_k\right|\right) \geq \left(1 - \sum |x_k^p|\right)^{\frac{1}{p}} \left(1 - \sum |y_k^q|\right)^{\frac{1}{q}},$$

provided that, in each case,  $(\sum |x_k^p|)^{\frac{1}{p}} \leq 1$  and  $(\sum |y_k^q|)^{\frac{1}{q}} \leq 1$ .



Extensions of Popoviciu's  
Inequality Using a General  
Method

A. McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 5 of 10

The inequality (3.4) is a slightly stronger form of (1.1).

Taking other interpretations of  $B$  and  $B^*$  we give two more examples of the outer inequalities of (3.2) as follows:

(4)

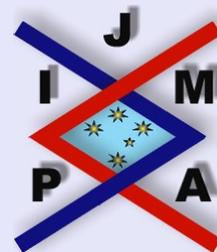
$$\left(1 - \left| \int_E uv \right| \right) \geq \left(1 - \int_E |u|^p\right)^{\frac{1}{p}} \left(1 - \int_E |v|^q\right)^{\frac{1}{q}},$$

provided that  $\left(\int_E |u|^p\right)^{\frac{1}{p}} \leq 1$  and  $\left(\int_E |v|^q\right)^{\frac{1}{q}} \leq 1$ . The integrals are Lebesgue integrals and  $E$  is a bounded measurable subset of the real numbers.

(5) When we take  $B \equiv C[0, 1]$  and  $B^* \equiv BV[0, 1]$  in (3.2) we get the somewhat exotic result:

$$\left(1 - \left| \int_0^1 h(t) d\alpha(t) \right| \right) \geq [1 - (\text{Max } |h|)^p]^{\frac{1}{p}} [1 - (\text{Var}(\alpha))^q]^{\frac{1}{q}},$$

where the maximum and total variation are taken over  $[0, 1]$  and each is less than or equal to 1.



**Extensions of Popoviciu's  
Inequality Using a General  
Method**

A. McD. Mercer

Title Page

Contents



Go Back

Close

Quit

Page 6 of 10

## 4. Inequalities for Grammians

Let  $\Gamma(\mathbf{x}, \mathbf{y})$ ,  $\Gamma(\mathbf{x})$  and  $\Gamma(\mathbf{y})$  denote the determinants of size  $n$  whose  $(i, j)$ th elements are respectively the inner products  $(\mathbf{x}_i, \mathbf{y}_j)$ ,  $(\mathbf{x}_i, \mathbf{x}_j)$  and  $(\mathbf{y}_i, \mathbf{y}_j)$  where the  $\mathbf{x}$ 's and  $\mathbf{y}$ 's are vectors in a Hilbert space. Then it is known, see [1, p.599], that

$$(4.1) \quad [\Gamma(\mathbf{x}, \mathbf{y})]^2 \leq \Gamma(\mathbf{x})\Gamma(\mathbf{y}).$$

It is also well-known that the two factors on the right side of (3.3) are each non-negative so that we have

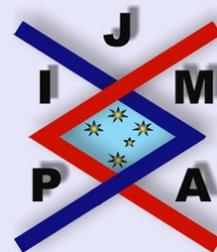
$$(4.2) \quad |\Gamma(\mathbf{x}, \mathbf{y})|^{2\alpha} \leq [\Gamma(\mathbf{x})]^\alpha [\Gamma(\mathbf{y})]^\alpha \quad \text{if } \alpha > 0.$$

If we read this as  $z \leq xy$  and we take  $f(t) = 1 - t$  again then (2.1) gives

$$(1 - |\Gamma(\mathbf{x}, \mathbf{y})|^{2\alpha}) \geq [1 - (\Gamma(\mathbf{x}))^{p\alpha}]^{\frac{1}{p}} [1 - (\Gamma(\mathbf{y}))^{q\alpha}]^{\frac{1}{q}},$$

provided that  $\Gamma(\mathbf{x}) \leq 1$  and  $\Gamma(\mathbf{y}) \leq 1$ .

When  $p = q = 2$  and  $\alpha = \frac{1}{2}$  this is a result due to J. Pečarić, [4], and when  $p = q = 2$  and  $\alpha = \frac{1}{4}$  we get a result due to S.S. Dragomir and B. Mond, [1, Theorem 2].



---

Extensions of Popoviciu's  
Inequality Using a General  
Method

A. McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 7 of 10

## 5. Some Final Remarks

In giving examples of the use of (2.1) we have used the function  $f(t) = 1 - t$  since that is the source of Popoviciu's result. But interesting inequalities arise also from other suitable choices of  $f$ . For example, taking

$$f(t) = \frac{(\alpha - t)}{(\beta - t)} \text{ in } [0, 1] \quad (1 < \alpha < \beta)$$

we are led to the result

$$\left[ \frac{\alpha - |\sum x_k y_k|}{\beta - |\sum x_k y_k|} \right] \geq \left[ \frac{\alpha - \sum |x_k^p|}{\beta - \sum |x_k^p|} \right]^{\frac{1}{p}} \left[ \frac{\alpha - \sum |y_k^q|}{\beta - \sum |y_k^q|} \right]^{\frac{1}{q}}$$

provided that  $(\sum |x_k^p|)^{\frac{1}{p}} \leq 1$ ,  $(\sum |y_k^q|)^{\frac{1}{q}} \leq 1$ . This reduces to Popoviciu's inequality (3.2) if we multiply throughout by  $\beta$ , let  $\beta \rightarrow \infty$  and  $\alpha \rightarrow 1$ .

Next let  $f$  possess the same properties as in Lemma 2.1 above but now take  $x, y, z, w \in [0, 1]$  with  $w \leq xyz$ . Then one finds that

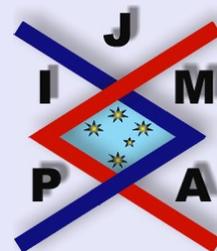
$$f(w) \geq f(xyz) \geq [f(x^p)]^{\frac{1}{p}} [f(y^q)]^{\frac{1}{q}} [f(z^r)]^{\frac{1}{r}},$$

where

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \quad (p, q, r < 1).$$

Specializing this by again taking  $f(t) = 1 - t$  and reading the extended Hölder inequality

$$\left| \sum x_k y_k z_k \right| \leq \left[ \sum |x_k|^p \right]^{\frac{1}{p}} \left[ \sum |y_k|^q \right]^{\frac{1}{q}} \left[ \sum |z_k|^r \right]^{\frac{1}{r}}$$



Extensions of Popoviciu's  
Inequality Using a General  
Method

A. McD. Mercer

Title Page

Contents



Go Back

Close

Quit

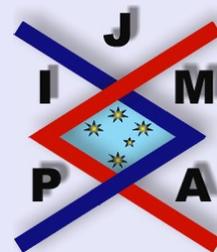
Page 8 of 10

as  $w \leq xyz$  we get the Popoviciu-type inequality:

$$\left(1 - \left| \sum x_k y_k z_k \right| \right) \geq \left(1 - \sum |x_k^p|\right)^{\frac{1}{p}} \left(1 - \sum |y_k^q|\right)^{\frac{1}{q}} \left(1 - \sum |z_k^r|\right)^{\frac{1}{r}},$$

provided  $(\sum |x_k^p|)^{\frac{1}{p}} \leq 1$ ,  $(\sum |y_k^q|)^{\frac{1}{q}} \leq 1$ ,  $(\sum |z_k^r|)^{\frac{1}{r}} \leq 1$ .

One can also construct inequalities which involve products of four or more factors, in the same way.



---

**Extensions of Popoviciu's  
Inequality Using a General  
Method**

A. McD. Mercer

---

Title Page

Contents



Go Back

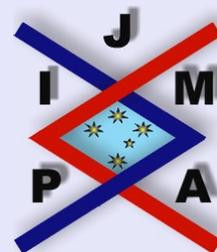
Close

Quit

Page 9 of 10

## References

- [1] S.S. DRAGOMIR AND B. MOND, Some inequalities of Aczèl type for Grammians in inner product spaces, *RGMIA. Res. Rep. Coll.*, **2**(2) (1999), Article 10. [ONLINE: <http://rgmia.vu.edu.au/v2n2.html>]
- [2] D.S. MITRINOVIĆ, *Analytic Inequalities*,(1970), Springer-Verlag.
- [3] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.
- [4] J.E. PEČARIĆ, On some classical inequalities in unitary spaces, *Mat. Bilten. (Skopje)*, **16** (1992), 63–72.



---

Extensions of Popoviciu's  
Inequality Using a General  
Method

A. McD. Mercer

---

Title Page

Contents



Go Back

Close

Quit

Page 10 of 10