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SECOND-ORDER DIFFERENTIAL PROXIMAL METHODS FOR EQUILIBRIUM PROBLEMS

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ABSTRACT. An approximate procedure for solving equilibrium problems is proposed and its convergence is established under natural conditions. The result obtained in this paper includes, as a special case, some known results in convex minimization and monotone inclusion fields.

Key words and phrases: Equilibrium, Proximal method, Minimization, Monotone inclusion.

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1. INTRODUCTION AND PRELIMINARIES

Equilibrium problems theory has emerged as an interesting branch of applicable mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, and operations research in a general and unified way. There are a substantial number of papers on existence results for solving equilibrium problems based on different relaxed monotonicity notions and various compactness assumptions. But up to now only few iterative methods to solve such problems have been done. Inspired by numerical methods developed by A. S. Antipin for optimization and monotone inclusion, and motivated by its research in the continuous case, we consider a class of equilibrium problems which includes variational inequalities as well as complementarity problems, convex optimisation, saddle point-problems, problems of finding a zero of a maximal monotone operator and Nash equilibria problems as special cases. Then, we propose and investigate iterative methods for solving such problems.

To begin with, let $\mathcal{H}$ be a real Hilbert space and $|\cdot|$ the norm generated by the scalar product $\langle\cdot,\cdot\rangle$. We will focus our attention on the following problem

$$(\mathcal{E}P) \qquad \text{find } \overline{x} \in C \quad \text{such that} \quad F(\overline{x}, x) \geq 0 \quad \forall x \in C,$$

where C is a nonempty, convex, and closed set of $\mathcal{H}$ and $F : C \times C \to \mathbb{R}$ is a given bifunction satisfying $F(x,x) = 0$ for all $x \in C$.

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This problem has potential and useful applications in nonlinear analysis and mathematical economics. For example, if we set $F(x,y)=\varphi(y)-\varphi(x) \quad \forall x,y \in C$, $\varphi : C \to \mathbb{R}$ a real-valued function, then ($\mathcal{E}P$) reduces to the following *minimization problem* subject to implicit constraints

($\mathcal{C}O$) $\qquad$ find $\overline{x} \in C$ such that $\varphi(\overline{x}) \le \varphi(x) \quad \forall x \in C.$

The basic case of *monotone inclusion* corresponds to $F(x,y) = \sup_{\zeta \in Bx} \langle \zeta, y-x\rangle$ with $B : C \rightrightarrows X$ a set-valued maximal monotone operator. Actually, the equilibrium problem ($\mathcal{E}P$) is nothing but

($\mathcal{M}I$) $\qquad$ find $\overline{x} \in C$ such that $0 \in B(\overline{x})$.

Moreover, if $B = T + N_C$, then inclusion ($\mathcal{M}I$) reduces to the classical *variational inequality*

($\mathcal{V}I$) $\qquad$ find $\overline{x} \in C$ such that $\langle T(\bar{x}), x-\bar{x}\rangle \ge 0 \quad \forall x \in C,$

T being a univoque operator and N_C standing for the normal cone to C.

In particular if C is a closed convex cone, then the inequality ($\mathcal{V}I$) can be written as

($\mathcal{C}P$) $\qquad$ find $\overline{x} \in C \quad T(\overline{x}) \in C^*$ and $\langle T(\overline{x}), \overline{x}\rangle = 0,$

where $C^* = \{x \in X;\ \langle x,y\rangle \ge 0\ \forall y \in C\}$ is the polar cone to C.

The problem of finding such a $\overline{x}$ is an important instance of the well-known *complementarity problem* of mathematical programming.

Now, let $P : C \to C$ be a given mapping, if we set $F(x,y) = \langle x - Px, y - x\rangle$, then ($\mathcal{E}P$) is nothing but the *problem of finding fixed points of* P. On the other hand, monotonicity of F is equivalent to saying $\langle Px - Py, x - y\rangle \le |x-y|$ which is clearly satisfied when P is nonexpansive.

Another example corresponds to *Nash equilibria* in noncooperative games. Let I (the set of players) be a finite index set. For every $i \in I$ let C_i (the strategy set of the i-th player) be a given set, f_i (the loss function of the i-th player, depending on the strategies of all players) : $C \to \mathbb{R}$ a given function with $C := \prod_{i\in I} C_i$. For $x = (x_i)_{i\in I} \in C$, we define $x^i := (x_j)_{j\in I,\, j\neq i}$. The point $\overline{x} = (\overline{x}_i)_{i\in I} \in C$ is called a Nash equilibrium if and only if for all $i \in I$ the following inequalities hold true:

($\mathcal{N}E$) $\qquad f_i(\overline{x}) \le f_i(\overline{x}^i, y_i)$ for all $y_i \in C_i,$

(i.e. no player can reduce his loss by varying his strategy alone).

Let us define $F : C \times C \to \mathbb{R}$ by

$$F(x,y) = \sum_{i\in I} \left(f_i(x^i, y_i) - f_i(x)\right).$$

Then $\overline{x} \in C$ is a Nash equilibrium if, and only if, $\overline{x}$ solves ($\mathcal{E}P$).

Finally, the problem of finding the *saddle point* of a convex-concave function, namely, the point $(\bar{x}, \bar{p})$ that satisfies the inequalities

($\mathcal{S}P$) $\qquad L(\bar{x}, p) \le L(\bar{x}, \bar{p}) \le L(x, \bar{p}),$

for all $x \in Q$ and $p \in P$, where P and Q are two closed and convex sets, can also be stated as ($\mathcal{E}P$). Indeed, let us introduce the normalized function $F(w,v) = L(z,p) - L(x,y)$, where $w = (z,y)$ and $v = (x,p)$ and set $C = Q \times P$, it follows that ($\mathcal{S}P$) is equivalent to ($\mathcal{E}P$) and that their sets of solutions coincide.

It is worth mentioning that the property $F(x,x) = 0$ for all $x \in C$ is trivially satisfied for all the above examples. Furthermore, this reflects the name of the class of games of n persons with zero sum.

The following definitions will be needed in the sequel (see for example [5]).

Definition 1.1. Let $F : C \times C \to \mathbb{R}$ be a real valued bifunction.

(i) F is said to be monotone, if

$$F(x, y) + F(y, x) \leq 0, \quad \text{for each} \quad x, y \in C.$$

(ii) F is said to be strictly monotone if

$$F(x, y) + F(y, x) < 0, \quad \text{for each} \quad x, y \in C, \text{ with } x \neq y.$$

(iii) F is upper-hemicontinuous, if for all $x, y, z \in C$

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \leq F(x, y).$$

One approach to solving ($\mathcal{E}P$) is the proximal method (see [4] or [7]), which generates the next iterates x_{k+1} by solving the subproblem

$$F(x_{k+1}, x) + \lambda_k^{-1} \langle x_{k+1} - x_k, x - x_{k+1} \rangle \geq 0 \quad \forall x \in C. \tag{1.1}$$

In the light of Antipin's research, we propose the following iterative method which works as follows. Given $x_{k-1}, x_k \in C$ and two parameters $\alpha_k \in [0, 1[$ and $\lambda_k > 0$, find $x_{k+1} \in C$ such that

$$F(x_{k+1}, x) + {\lambda_k}^{-1} \langle x_{k+1} - x_k - \alpha_k (x_k - x_{k-1}), x - x_{k+1} \rangle \geq 0 \quad \forall x \in C. \tag{1.2}$$

It is well known that the proximal iteration may be interpreted as a first order implicit discretisation of differential inclusion

$$\frac{du}{dt}(t) \in P_{Tx}(-\partial F(u(t), \cdot) u(t), \tag{1.3}$$

where $Tx = c\mathbb{R}(C - x)$ is the tangent cone of C at $x \in C$ and the operator P_K stands for the orthogonal projection onto a closed convex set K. While the inspiration for (1.2) comes from the implicit discretization of the differential system of the second-order in time, namely

$$\frac{d^2u}{dt^2}(t) + \gamma \frac{du}{dt}(t) \in P_{Tx}(-\partial F(u(t), \cdot) u(t), \tag{1.4}$$

where $\gamma > 0$ is a damping or a friction parameter.

Under appropriate conditions on α_k and λ_k we prove that if the solution set S is nonempty, then for every sequence $\{x_k\}$ generated by our algorithm, there exists an $\bar{x} \in S$ such that $\{x_k\}$ converges to $\bar{x}$ weakly in $\mathcal{H}$ as $k \to \infty$.

Now, for developing implementable computational techniques, it is of particular importance to treat the case when (1.2) is solved approximately. To this end, we propose an approximate method based on a notion which is inspired by the approximate subdifferential and more generally by the ε-enlargement of a monotone operator (see for example [10]). This allows an extra degree of freedom, which is very useful in various applications. On the other hand, by setting $\varepsilon_k = 0$, the exact method can also be treated. More precisely, we consider the following scheme: find $x_{k+1} \in C$ such that

$$F(x_{k+1}, x) + {\lambda_k}^{-1} \langle x_{k+1} - y_k, x - x_{k+1} \rangle \geq -\varepsilon_k \quad \forall x \in C, \tag{1.5}$$

where $y_k := x_k + \alpha_k(x_k - x_{k-1}), \lambda_k, \alpha_k, \varepsilon_k$ are nonnegative real numbers.

We will impose the following tolerance criteria on the term ε_k which is standard in the literature:

$$\sum_{k=1}^{+\infty} \lambda_k \varepsilon_k < +\infty, \tag{1.6}$$

and which is typically needed to establish global convergence.

The remainder of the paper is organized as follows: In Section 2, we present a weak convergence result for the sequence generated by (1.5) under criterion (1.6). In Section 3, we present an application to convex minimization and monotone inclusion cases.

2. THE MAIN RESULT

Theorem 2.1. *Let $\{x_k\} \subset C$ be a sequence generated by (1.5) under criterion (1.6), where F is monotone, upper hemicontinuous such that $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$. Assume that the solution set of ($\mathcal{E}P$) is nonempty and the parameters α_k, λ_k and ε_k satisfy:*

(1) $\exists \lambda > 0$ *such that* $\forall k \in \mathbb{N}^*, \lambda_k \geq \lambda$.
(2) $\exists \alpha \in [0, 1[$ *such that* $\forall k \in \mathbb{N}^*, 0 \leq \alpha_k \leq \alpha$.
(3) $\sum_{k=1}^{+\infty} \lambda_k \varepsilon_k < +\infty$.

If the following condition holds

$$\sum_{k=1}^{+\infty} \alpha_k |x_k - x_{k-1}|^2 < +\infty, \tag{2.1}$$

then, there exists $\tilde{x}$ which solves ($\mathcal{E}P$) and such that $\{x_k\}$ weakly converges to $\tilde{x}$ as $k \to +\infty$.

Proof. Let $\bar{x}$ be a solution of ($\mathcal{E}P$). By setting $x = x_{k+1}$ in ($\mathcal{E}P$) and taking into account the monotonicity of F, we get $-F(x_{k+1}, \bar{x}) \geq 0$. This combined with (1.5) gives

$$\langle x_{k+1} - x_k - \alpha_k(x_k - x_{k-1}), x_{k+1} - \bar{x} \rangle \leq \lambda_k \varepsilon_k.$$

Define the auxiliary real sequence $\varphi_k := \frac{1}{2}|x_k - \bar{x}|^2$. It is direct to check that

$$\langle x_{k+1} - x_k - \alpha_k(x_k - x_{k-1}), x_{k+1} - \bar{x} \rangle = \varphi_{k+1} - \varphi_k + \frac{1}{2}|x_{k+1} - x_k|^2 \\ - \alpha_k \langle x_k - x_{k-1}, x_{k+1} - \bar{x} \rangle,$$

and since

$$\langle x_k - x_{k-1}, x_{k+1} - \bar{x} \rangle = \langle x_k - x_{k-1}, x_k - \bar{x} \rangle + \langle x_k - x_{k-1}, x_{k+1} - x_k \rangle \\ = \varphi_k - \varphi_{k-1} + \frac{1}{2}|x_k - x_{k-1}|^2 + \langle x_k - x_{k-1}, x_{k+1} - x_k \rangle,$$

it follows that

$$\varphi_{k+1} - \varphi_k - \alpha_k(\varphi_k - \varphi_{k-1}) \leq -\frac{1}{2}|x_{k+1} - x_k|^2 + \alpha_k \langle x_k - x_{k-1}, x_{k+1} - x_k \rangle \\ + \frac{\alpha_k}{2}|x_k - x_{k-1}|^2 + \lambda_k \varepsilon_k \\ = -\frac{1}{2}|x_{k+1} - y_k|^2 + \frac{\alpha_k + \alpha_k^2}{2}|x_k - x_{k-1}|^2 + \lambda_k \varepsilon_k.$$

Hence

$$\varphi_{k+1} - \varphi_k - \alpha_k(\varphi_k - \varphi_{k-1}) \leq -\frac{1}{2}|x_{k+1} - y_k|^2 + \alpha_k |x_k - x_{k-1}|^2 + \lambda_k \varepsilon_k. \tag{2.2}$$

Setting $\theta_k := \varphi_k - \varphi_{k-1}$ and $\delta_k := \alpha_k |x_k - x_{k-1}|^2 + \lambda_k \varepsilon_k$, we obtain

$$\theta_{k+1} \leq \alpha_k \theta_k + \delta_k \leq \alpha_k [\theta_k]_+ + \delta_k,$$

where $[t]_+ := max(t, 0)$, and consequently

$$[\theta_{k+1}]_+ \leq \alpha [\theta_k]_+ + \delta_k,$$

with $\alpha \in [0, 1[$ given by (2).

The latter inequality yields

$$[\theta_{k+1}]_+ \leq \alpha^k[\theta_1]_+ + \sum_{i=0}^{k-1} \alpha^i \delta_{k-i},$$

and therefore

$$\sum_{k=1}^{\infty}[\theta_{k+1}]_{\leq} \frac{1}{1-\alpha}([\theta_1]_+ + \sum_{k=1}^{\infty} \delta_k),$$

which is finite thanks to (3) and (2.1). Consider the sequence defined by $t_k := \varphi_k - \sum_{i=1}^{k}[\theta_i]_+$. Since $\varphi_k \geq 0$ and $\sum_{i=1}^{k}[\theta_i]_+ < +\infty$, it follows that t_k is bounded from below. But

$$t_{k+1} = \varphi_{k+1} - [\theta_{k+1}]_+ - \sum_{i=1}^{k}[\theta_i]_+ \leq \varphi_{k+1} - \varphi_{k+1} + \varphi_k - \sum_{i=1}^{k}[\theta_i]_+ = t_k,$$

so that $\{t_k\}$ is nonincreasing. We thus deduce that $\{t_k\}$ is convergent and so is $\{\varphi_k\}$. On the other hand, from (2.2) we obtain the estimate

$$\frac{1}{2}|x_{k+1} - y_k|^2 \leq \varphi_k - \varphi_{k+1} + \alpha[\theta_k]_+ + \delta_k.$$

Passing to the limit in the latter inequality and taking into account that $\{\varphi_k\}$ converges, $[\theta_k]_+$ and δ_k go to zero as k tends to $+\infty$, we obtain

$$\lim_{k\to+\infty}(x_{k+1} - y_k) = 0.$$

On the other hand, from (1.5) and monotonicity of F we derive

$$\langle x_{k+1} - y_k, x - x_{k+1}\rangle + \lambda_k\varepsilon_k \geq F(x, x_{k+1}) \quad \forall x \in C.$$

Now let $\tilde{x}$ be a weak cluster point of $\{x_k\}$. There exists a subsequence $\{x_\nu\}$ which converges weakly to $\tilde{x}$ and satisfies

$$\langle x_{\nu+1} - y_\nu, x - x_{\nu+1}\rangle + \lambda_\nu\varepsilon_\nu \geq F(x, x_{\nu+1}) \quad \forall x \in C.$$

Passing to the limit, as $\nu \to +\infty$, taking into account the lower semicontinuity of F, we obtain $0 \geq F(x, \tilde{x}) \quad \forall x \in C$. Now, let $x_t = tx + (1-t)\tilde{x},\ 0 < t \leq 1$, from the properties of F follows then for all t

$$\begin{aligned} 0 &= F(x_t, x_t) \\ &\leq tF(x_t, x) + (1-t)F(x_t, \tilde{x}) \\ &\leq tF(x_t, x). \end{aligned}$$

Dividing by t and letting $t \downarrow 0$, we get $x_t \to \tilde{x}$ which together with the upper hemicontinuity of F yields

$$F(\tilde{x}, x) \geq 0 \quad \forall x \in C,$$

that is, any weak limit point $\tilde{x}$ is solution to the problem $(\mathcal{E}P)$. The uniqueness of such a limit point is standard (see for example [10, Theorem 1]). □

Remark 2.2. Under assumptions of Theorem 2.1 and in view of its proof, it is clear that $\{x_k\}$ is **bounded** if, and only if, there exists at least one solution to $(\mathcal{E}P)$.

3. APPLICATIONS

To begin with, let us recall the following concept (see for example [10]): The ε-enlargement of a monotone operator T, say $T^\varepsilon(x)$, is defined as

$$T^\varepsilon(x) := \{v \in \mathcal{H}; \langle u - v, y - x\rangle \geq -\varepsilon \quad \forall y, u \in T(y)\}, \tag{3.1}$$

where $\varepsilon \geq 0$. Since T is assumed to be maximal monotone, $T^0(x) = T(x)$, for any x. Furthermore, directly from the definition it follows that

$$0 \leq \varepsilon_1 \leq \varepsilon_2 \Rightarrow T^{\varepsilon_1}(x) \subset T^{\varepsilon_2}(x).$$

Thus T^ε is an enlargement of T. The use of elements in T^ε instead of T allows an extra degree of freedom, which is very useful in various applications.

3.1. Convex Optimization.

An interesting case is obtained by taking $F(x, y) = \varphi(y) - \varphi(x)$, φ a proper convex lower-semicontinuous function $f : X \to \mathbb{R}$. In this case $(\mathcal{E}P)$ reduces to the one of finding a minimizer of the function $f := \varphi + i_C$, i_C denoting the indicator function of C and (1.5) takes the following form

$$\lambda_k(\partial f)^{\varepsilon_k}(x_{k+1}) + x_{k+1} - x_k - \alpha_k(x_k - x_{k-1}) \ni 0. \tag{3.2}$$

Since the enlargement of the subdifferential is larger than the approximate subdifferential, i.e. $\partial_\varepsilon f \subset (\partial f)^\varepsilon$, we can write $\partial_{\varepsilon_k} f(x_{k+1}) \subset (\partial f)^{\varepsilon_k}(x_{k+1})$, which leads to the fact that the approximate method

$$\lambda_k \partial_{\varepsilon_k} f(x_{k+1}) + x_{k+1} - x_k - \alpha_k(x_k - x_{k-1}) \ni 0, \tag{3.3}$$

where $\partial_{\varepsilon_k} f$ is the approximate subdifferential of f, is a special case of our algorithm. In the further case where $\alpha_k = 0$ for all $k \in \mathbb{N}$, our method reduces to the proximal method by Martinet and we recover the corresponding convergence result (see [6]).

3.2. Monotone Inclusion.

First, let us recall that by taking $F(x, y) = \sup_{\xi \in Bx} \langle \xi, y - x\rangle \ \forall y, x \in C$, where $B : C \rightrightarrows \mathcal{H}$ is a maximal monotone operator, $(\mathcal{E}P)$ is nothing but the problem of finding a zero of the operator B. On the other hand F is maximal monotone according to Blum's-Oetlli definition, namely, for every $(\zeta, x) \in \mathcal{H} \times C$

$$F(y, x) \leq \langle -\zeta, y - x\rangle \quad \forall y \in C \quad \Rightarrow \quad 0 \leq F(x, y) + \langle -\zeta, y - x\rangle \quad \forall y \in C.$$

It should be noticed that a monotone function which is convex in the second argument and upper hemi-continuous in the first one is maximal monotone.

Moreover, taking $C = \mathcal{H}$, $F(x, y) = \sup_{\xi \in Bx} \langle \xi, y - x\rangle$, leads to

$$x_{k+1} \in (I + \lambda_k B^{\varepsilon_k})^{-1}(x_k - \alpha_k(x_k - x_{k-1})),$$

which reduces in turn, when $\varepsilon_k = 0$ and $\alpha_k = 0$ for all $k \in \mathbb{N}$, to the well-known Rockafellar's proximal point algorithm and we recover its convergence result ([9, Theorem 1]).

It is worth mentioning that the proposed algorithm leads to new methods for finding fixed-points, Nash-equilibria as well as solving variational inequalities.

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