Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 4, Issue 1, Article 3, 2003

POWER-MONOTONE SEQUENCES AND INTEGRABILITY OF TRIGONOMETRIC SERIES

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Received 13 June, 2002; accepted 20 September, 2002
Communicated by A. Babenko

ABSTRACT. The theorem proved in this paper is a generalization of some results, concerning integrability of trigonometric series, due to R.P. Boas, L. Leindler, etc. This result can be considered as an example showing the utility of the notion of power-monotone sequences.

Key words and phrases: Trigonometrical series; Integrability; Quasi power-monotone sequence.

2000 *Mathematics Subject Classification.* 42A32.

1. INTRODUCTION

Several authors have studied the integrability of the series

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{1.1}$$

requiring certain conditions on the sequence $\{b_n\}$ (see [1] – [6] and [9] – [14]).

For example R.P. Boas in [2] proved the following result for (1.1):

Theorem 1.1. *If $b_n \downarrow 0$ then for $0 \le \gamma \le 1$, $x^{-\gamma} g(x) \in L[0,\pi]$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1} b_n$ converges.*

This theorem had previously been proved for $\gamma = 0$ by W.H. Young [14] and was later extended by P. Heywood [6] for $1 < \gamma < 2$.

Further generalization was given by Aljančić, R. Bojanić and M. Tomić in [1], by using the so called slowly varying functions.

A positive, continuous function defined on $[0,\infty)$ is called slowly varying if $\frac{L(tx)}{L(x)} \to 1$, if $x \to \infty$ for all $t > 0$.

They proved among others the following results:

ISSN (electronic): 1443-5756

The author was partially supported by the Hungarian National Foundation for Scientific Research under Grant # T 029094.

073-02

Theorem 1.2. *Let $L(x)$ be a convex, non-decreasing showly varying function. If $b_n \downarrow 0$, then $L(1/x)g(x) \in L(0,\pi]$ if and only if $\sum_{n=1}^{\infty} n^{-1}L(n)b_n$ converges.*

Theorem 1.3. *Let $L(x)$ be a slowly varying function. If $b_n \downarrow 0$ and $0 < \gamma < 2$, then $x^{-\gamma}L\left(\frac{1}{x}\right) g(x) \in L[0,\pi]$ if and only if $\sum_{n=1}^{\infty} n^{\gamma-1}L(n)b_n$ converges.*

Later the monotonicity condition on $\{b_n\}$ was changed to more general ones by S.M. Shah [11] and L. Leindler [9]. Before formulating a result of this type we need a definition due to L. Leindler.

A sequence $\mathbf{c} := \{c_n\}$ of positive numbers tending to zero is of rest bounded variation, or briefly R_0^+BVS, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c})c_m \tag{1.2}$$

for all natural number m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

Using this notion L. Leindler ([9]) proved

Theorem 1.4. *Let $\{b_n\} \in R_0^+BVS$. If $0 \le \gamma \le 1$ and*

$$\sum_{n=1}^{\infty} n^{\gamma-1}b_n < \infty, \tag{1.3}$$

then $x^{-\gamma}g(x) \in L(0;\pi)$.

The aim of the present note is to give further generalization of above mentioned theorems by using the concept of the so called quasi β-power-monotone sequence changing the function x^{γ} to more general one. We deal only with the sufficiency of the conditions because only this point of the proofs has interest in showing up the utility of the quasi β-power-monotone sequences, the proof of the necessity, in general, goes on the same way as in the earlier cited papers.

First we need some definitions before formulating our result and the lemmas used in the proof.

Following L. Leindler we shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is quasi β-power-monotone increasing (decreasing) if there exists a natural number $N := N(\beta,\gamma)$ and constant $K := K(\beta,\gamma) > 1$ such that

$$Kn^{\beta}\gamma_n \ge m^{\beta}\gamma_m \quad (n^{\beta}\gamma_n \le Km^{\beta}\gamma_m) \tag{1.4}$$

holds for any $n \ge m \ge N$.

Here and in the sequel, K and K_i denote positive constants that are not necessarily the same of each occurrence.

If (1.4) holds with $\beta = 0$ then we omit the attribute "β-power".

Furthermore, we shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is quasi geometrically increasing (decreasing) if there exist natural numbers $\mu := \mu(\gamma)$, $N := N(\gamma)$ and a constant $K := K(\gamma) \ge 1$ such that

$$\gamma_{n+\mu} \ge 2\gamma_n \quad \text{and} \quad \gamma_n \le K\gamma_{n+1} \quad \left(\gamma_{n+\mu} \le \frac{1}{2}\gamma_n \text{ and } \gamma_{n+1} \le K\gamma_n\right) \tag{1.5}$$

hold for all $n \ge N$.

A sequence $\{\gamma_n\}$ is said to be bounded by blocks if the inequalities

$$\alpha_1\Gamma_m^{(k)} \le \gamma_n \le \alpha_2\Gamma_M^{(k)}, \quad 0 < \alpha_1 \le \alpha_2 < \infty \tag{1.6}$$

hold for any $2^k \le n \le 2^{k+1}$, $k = 1,2,\ldots$, where

$$\Gamma_m^{(k)} := \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \quad \text{and} \quad \Gamma_M^{(k)} := \max(\gamma_{2^k}, \gamma_{2^{k+1}}).$$

Finally, for a given sequence $\{\gamma_n\}$, $\gamma(x)$ will denote the following function:

$$\gamma(x) = \gamma_n \quad \text{if} \quad x = \frac{1}{n}, n \geq 1; \ \text{ linear on the interval } \ \left[\frac{1}{n+1}, \frac{1}{n}\right].$$

2. RESULT

Theorem 2.1. *Let $\{b_n\} \in R_0^+ BVS$ and $\{\gamma_n\}$ such that $\gamma_n n^{-2+\varepsilon}$ is quasi-monotone decreasing for some $\varepsilon > 0$. If*

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} b_n < \infty \tag{2.1}$$

then $\gamma(x)g(x) \in L(0;\pi]$.

Remark 2.2. This result is a generalization of Theorem 1.4 since in (1.3) n^γ is replaced by γ_n and the case $0 \leq \gamma \leq 1$ is extending to $0 \leq \gamma < 2$. Furthermore the sufficiency parts of Theorem 1.2 and 1.3 are also special cases of our Theorem in a few respects: namely the monotonicity of $\{b_n\}$ is changed to the property of $R_0^+ BVS$ and as we will prove later for any slowly varying function $L(x)$ and for $0 \leq \gamma < 2$ the sequence $\{n^\gamma L(n) n^{-2+\varepsilon}\}$ is quasi-monotone decreasing for some $\varepsilon(> 0)$ therefore $n^\gamma L(n)$ can be replaced by $\{\gamma_n\}$ $(0 \leq \gamma < 2)$. Moreover using our result it turns out that the convexity and monotonicity conditions on $L(x)$ can be dropped in the case of Theorem 1.2. For example our result contains statement of the type $\sum_{n=2}^{\infty} \frac{1}{n \log n} b_n < \infty \Rightarrow \frac{1}{\log \frac{1}{x}} g(x) \in L[0;\pi]$, too.

3. LEMMAS

We need the following lemmas.

Lemma 3.1. *([8]) A positive sequence $\{\gamma_n\}$ bounded by blocks is quasi ε-power-monotone decreasing with a certain positive exponent ε if and only if the sequence $\{\gamma_{2^n}\}$ is quasi geometrically decreasing.*

Lemma 3.2. *([7]) For any positive sequence $\gamma := \{\gamma_n\}$ the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \leq K\gamma_m \quad (m = 1, 2, \ldots; K \geq 1), \tag{3.1}$$

or

$$\sum_{n=1}^{m} \gamma_n \leq K\gamma_m \quad (m = 1, 2, \ldots; K \geq 1), \tag{3.2}$$

hold if and only if γ is quasi geometrically decreasing or increasing, respectively.

Lemma 3.3. *If $\{\gamma_n\}$ has the same property as in Theorem 2.1, then*

$$\gamma_n \leq K \cdot \sum_{k=1}^{n} \frac{\gamma_k}{k} \quad \textit{for all} \quad n, \tag{3.3}$$

and

$$\gamma_n \leq K \cdot n^2 \quad \textit{for all} \quad n. \tag{3.4}$$

These statements immediately follow from the definition of $\{\gamma_n\}$.

Lemma 3.4. *If $\{b_n\} \in R_0^+ BVS$ and (2.1) is satisfied then*

$$\sum_{n=1}^{\infty} |b_n - b_{n+1}| \sum_{k=1}^{n} \frac{\gamma_k}{k} < \infty. \tag{3.5}$$

Proof. Using the definition of R_0^+BVS (see (1.2)) we have from (2.1) that

$$\sum_{n=1}^{\infty}\frac{\gamma_n}{n}\sum_{k=n}^{\infty}|b_k-b_{k+1}|<\infty.$$

Now changing the order of summation we get (3.5). □

4. PROOFS

Proof of Theorem 2.1. Since $\{b_n\}$ is of bounded variation, the function $g(x)$ is continuous except perhaps at 0 ([15, p. 4]), so we are concerned only with a neighbourhood of 0.

We shall write $c(x) := 1-\cos x$. Then by Abel transformation ([3, p. 5]) we have

$$\begin{aligned}\frac{1}{2}g(x) &= \frac{1}{\sin x}\sum_{n=0}^{\infty}b_{n+1}[c\{(n+2)x\}-c(nx)]\\ &= \frac{1}{\sin x}\left[-b_2c(x)+b_3c(2x)+\sum_{n=3}^{\infty}(b_{n-1}-b_{n+1})c(nx)\right].\end{aligned}$$

Since $\sin x\sim x$ as $x\to 0$, so it is enough to prove the existence of

$$\int_0^1\gamma(x)\cdot\frac{1}{x}\sum_{n=3}^{\infty}|b_{n-1}-b_{n+1}|c(nx)dx. \tag{4.1}$$

and the integrability of $\gamma(x)c(x)/x$ and $\gamma(x)c(2x)/x$.

Applying Levi's theorem, the existence of (4.1) will follow from

$$\sum_{n=3}^{\infty}|b_{n-1}-b_{n+1}|\int_0^1\gamma(x)\frac{1}{x}c(nx)dx<\infty. \tag{4.2}$$

Divide the integral $\int_0^1\gamma(x)\frac{1}{x}c(nx)dx$ into two parts for a fixed n:

$$\int_0^1\gamma(x)\frac{1}{x}c(nx)dx=\int_0^{1/n}\gamma(x)\frac{1}{x}c(nx)dx+\int_{1/n}^1\gamma(x)\frac{1}{x}c(nx)dx=I_1+I_2. \tag{4.3}$$

In the estimate of I_1 we use that from the property of $\{\gamma_n\}$ assumed in Theorem 2.1 it follows that $\left\{\frac{\gamma_{2n}}{4^n}\right\}$ is geometrically decreasing and so by Lemma 3.1 (3.1) can be applied for this sequence. So, in the last step using (3.3) also, we get that

$$\begin{aligned}I_1 &= \int_0^{1/n}\gamma(x)\frac{1}{x}c(nx)dx\\ &= \int_0^{1/n}\gamma(x)\frac{1}{x}(1-\cos nx)dx\\ &= \int_0^{1/n}\gamma(x)xn^2\frac{1-\cos nx}{n^2x^2}dx\\ &\le K_1n^2\int_0^{1/n}\gamma(x)xdx\end{aligned} \tag{4.4}$$

$$\leq K_2 n^2 \sum_{k=n}^{\infty} \gamma\left(\frac{1}{k}\right)\frac{1}{k^3}$$
$$\leq K_3 n^2 \sum_{l=[\log n]}^{\infty} \gamma\left(\frac{1}{2^\ell}\right)\frac{1}{4^\ell}$$
$$\leq K_4 \gamma\left(\frac{1}{n}\right) = K_4\gamma_n \leq K_5 \sum_{k=1}^{n} \frac{\gamma_k}{k}.$$

Now we estimate I_2:

$$I_2 = \int_{1/n}^{1} \gamma(x)\frac{1}{x}c(nx)dx \tag{4.5}$$
$$\leq 2\int_{1/n}^{1} \gamma(x)\frac{1}{x}dx$$
$$\leq K \sum_{k=1}^{n} \gamma\left(\frac{1}{k}\right)\frac{1}{k}$$
$$= K \sum_{k=1}^{n} \frac{\gamma_k}{k}.$$

Since (4.3), (4.4), (4.5) with (3.5) give (4.2), so the integral (4.1) exists. Finally the integrability of $\gamma(x)c(x)/x$ and $\gamma(x)c(2x)/x$ can be proved by the same way that was used in the estimate of I_1 in (4.4), applying still (3.4).

Thus the proof of Theorem 2.1 is complete. □

Proof of Remark 2.2. The only fact we need to show that the sequence $\{n^\gamma L(n)n^{-2+\varepsilon}\}$ is quasi monotone decreasing for some $\varepsilon(>0)$, where $0 \leq \gamma < 2$ and $L(x)$ is an arbitrary slowly varying function. According to Lemma 3.1 it is enough to prove that the sequence $\{n^{\gamma-2}L(n)\}$ is bounded by blocks and that $\{2^{n(\gamma-2)}L(2^n)\}$ is a quasi geometrically decreasing sequence. It is obvious that for the sequence $\{n^{\gamma-2}L(n)\}$ (1.6) is equivalent to the existence of positive constants K_1, K_2 such that

$$K_1 \leq \frac{L(2^k+\ell)}{L(2^k)} \leq K_2 \tag{4.6}$$

hold for arbitrary k and $1 \leq \ell \leq 2^k$. But since from the definition of $L(x)$

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1 \tag{4.7}$$

is uniformly satisfied in the ratio t on the interval $[1,2]$ (see [1, p. 69]) therefore (4.6) holds.

In order to prove that for the sequence $2^{n(\gamma-2)}L(2^n)$ the properties (1.5) hold it is enough to show that there exist natural numbers μ and N and a constant $K \geq 1$ such that

$$(2^{\gamma-2})^{n+\mu}L(2^{n+\mu}) \leq \frac{1}{2}(2^{\gamma-2})^n L(2^n) \tag{4.8}$$

and

$$(2^{\gamma-2})^{n+1}L(2^{n+1}) \leq K(2^{\gamma-2})^n L(2^n) \tag{4.9}$$

hold if $n > N$.
However, (4.8) is equivalent to

$$\frac{L(2^{n+\mu})}{L(2^n)} \leq \frac{1}{2}(2^{2-\gamma})^{\mu} \tag{4.10}$$

and if $(2^{2-\gamma})^{\mu} > 2$ then by using (4.7), (4.10) holds if n is large enough, which gives (4.8). Finally since (4.9) can be obtained by using a similar argument as before, Remark 2.2 is proved. □

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