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ON MULTIDIMENSIONAL OSTROWSKI AND GRÜSS TYPE FINITE DIFFERENCE INEQUALITIES

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ABSTRACT. The aim of this paper is to establish some new multidimensional finite difference inequalities of the Ostrowski and Grüss type using a fairly elementary analysis.

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1. INTRODUCTION

The most celebrated Ostrowski inequality can be stated as follows (see [5, p. 469]).

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$, then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty , \tag{1.1}$$

for all $x \in [a,b]$.

Another remarkable inequality established by Grüss (see [4, p. 296]) in 1935 states that

$$\left| \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \right| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that f and g are two integrable functions on $[a,b]$ and satisfy the conditions $m \le f(x) \le M$, $n \le g(x) \le N$ for all $x \in [a,b]$, where m, M, n, N are constants.

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Many papers have been written dealing with generalisations, extensions and variants of the inequalities (1.1) and (1.2), see [1] – [10] and the references cited therein. It appears that, the finite difference inequalities of the Ostrowski and Grüss type are more difficult to establish and require more effort. The main purpose of the present paper is to establish the Ostrowski and Grüss type finite difference inequalities involving functions of many independent variables and their first order forward differences. An interesting feature of the inequalities established here is that the analysis used in their proofs is quite elementary and provides new estimates on these types of inequalities.

2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ and $\mathbb{N}$ denote the sets of real and natural numbers respectively. Let $N_i[0,a_i] = \{0,1,2,\dots,a_i\}$, $a_i \in \mathbb{N}$. $i = 1,2,\dots,n$ and $B = \prod_{i=1}^{n} N_i[0,a_i]$. For a function $z(x) : B \to \mathbb{R}$ we define the first order forward difference operators as

$$\Delta_1 z(x) = z(x_1+1, x_2, \dots, x_n) - z(x), \dots, \Delta_n z(x) = z(x_1, \dots, x_{n-1}, x_n + 1) - z(x)$$

and denote the $n-$fold sum over B with respect to the variable $y = (y_1, \dots, y_n) \in B$ by

$$\sum_y z(y) = \sum_{y_1=0}^{a_1-1} \cdots \sum_{y_n=0}^{a_n-1} z(y_1, \dots, y_n).$$

Clearly $\sum_y z(y) = \sum_x z(x)$ for $x, y \in B$. The notation

$$\sum_{t_i=y_i}^{x_i-1} \Delta_i z(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n), \quad x_i, y_i \in N_i[0,a_i]$$

for $i = 1,2,\dots,n$ we mean for $i = 1$ it is $\sum_{t_1=y_1}^{x_1-1} \Delta_1 z(t_1, x_2, \dots, x_n)$ and so on and for $i = 1$ it is $\sum_{t_n=y_n}^{x_n-1} \Delta_n z(y_1, \dots, y_{n-1}, t_n)$. We use the usual convention that the empty sum is taken to be zero.

Our main results are given in the following theorems.

Theorem 2.1. *Let f, g be real-valued functions defined on B and $\Delta_i f$, $\Delta_i g$ are bounded, i.e.,*

$$\|\Delta_i f\|_\infty = \sup_{x\in B} |\Delta_i f(x)| < \infty,$$
$$\|\Delta_i g\|_\infty = \sup_{x\in B} |\Delta_i g(x)| < \infty.$$

Let w be a real-valued nonnegative function defined on B and $\sum_y w(y) > 0$. Then for $x, y \in B$,

$$\left| f(x) g(x) - \frac{1}{2M} g(x) \sum_y f(y) - \frac{1}{2M} f(x) \sum_y g(y) \right| \leq \frac{1}{2M} \sum_{i=1}^{n} \left[|g(x)| \|\Delta_i f\|_\infty + |f(x)| \|\Delta_i g\|_\infty \right] H_i(x), \tag{2.1}$$

$$\left|f(x)g(x)-\frac{g(x)\sum_y w(y)f(y)+f(x)\sum_y w(y)g(y)}{2\sum_y w(y)}\right| \leq \frac{\sum_y w(y)\sum_{i=1}^{n}\left[|g(x)|\,\|\Delta_i f\|_\infty+|f(x)|\,\|\Delta_i g\|_\infty\right]|x_i-y_i|}{2\sum_y w(y)}, \tag{2.2}$$

where $M=\prod_{i=1}^{n}a_i$ *and* $H_i(x)=\sum_y |x_i-y_i|$.

The following result is a consequence of Theorem 2.1.

Corollary 2.2. *Let* $g(x)=1$ *in Theorem 2.1 and hence* $\Delta_i g(x)=0$, *then for* $x,y\in B$,

$$\left|f(x)-\frac{1}{M}\sum_y f(y)\right|\leq\frac{1}{M}\sum_{i=1}^{n}\|\Delta_i f\|_\infty H_i(x), \tag{2.3}$$

$$\left|f(x)-\frac{\sum_y w(y)f(y)}{\sum_y w(y)}\right|\leq\frac{\sum_y w(y)\sum_{i=1}^{n}\|\Delta_i f\|_\infty|x_i-y_i|}{\sum_y w(y)}, \tag{2.4}$$

where M,w *and* $H_i(x)$ *are as in Theorem 2.1.*

Remark 2.3. It is interesting to note that the inequalities (2.3) and (2.4) can be considered as the finite difference versions of the inequalities established by Milovanović [3, Theorems 2 and 3]. The one independent variable version of the inequality given in (2.3) is established by the present author in [10].

Theorem 2.4. *Let* $f,g,\Delta_i f,\Delta_i g$ *be as in Theorem 2.1. Then for every* $x,y\in B$,

$$\left|f(x)g(x)-\frac{1}{M}g(x)\sum_y f(y)-\frac{1}{M}f(x)\sum_y g(y)+\frac{1}{M}\sum_y f(y)g(y)\right| \leq\frac{1}{M}\sum_y\left[\sum_{i=1}^{n}\|\Delta_i f\|_\infty|x_i-y_i|\right]\left[\sum_{i=1}^{n}\|\Delta_i g\|_\infty|x_i-y_i|\right], \tag{2.5}$$

$$\left|f(x)g(x)-\frac{1}{M}g(x)\sum_y f(y)-\frac{1}{M}f(x)\sum_y g(y)+\frac{1}{M^2}\left(\sum_y f(y)\right)\left(\sum_y g(y)\right)\right| \leq\frac{1}{M^2}\left(\sum_{i=1}^{n}\|\Delta_i f\|_\infty H_i(x)\right)\left(\sum_{i=1}^{n}\|\Delta_i g\|_\infty H_i(x)\right), \tag{2.6}$$

where M *and* $H_i(x)$ *are as defined in Theorem 2.1.*

Remark 2.5. In [8, 9] the discrete versions of Ostrowski type integral inequalities established therein are given. Here we note that the inequalities in Theorem 2.4 are different and the analysis used in the proof is quite elementary.

Theorem 2.6. *Let* $f,g,\Delta_i f,\Delta_i g$ *be as in Theorem 2.1. Then*

$$\left|\frac{1}{M}\sum_x f(x)g(x)-\left(\frac{1}{M}\sum_x f(x)\right)\left(\frac{1}{M}\sum_x g(x)\right)\right| \tag{2.7}$$

$$\le \frac{1}{2M^2}\sum_x\left(\sum_y\left[\sum_{i=1}^n \|\Delta_i f\|_\infty |x_i-y_i|\right]\left[\sum_{i=1}^n \|\Delta_i g\|_\infty |x_i-y_i|\right]\right),$$

$$\left|\frac{1}{M}\sum_x f(x)g(x)-\left(\frac{1}{M}\sum_x f(x)\right)\left(\frac{1}{M}\sum_x g(x)\right)\right| \le \frac{1}{2M^2}\sum_x\left(\sum_{i=1}^n\left[|g(x)|\,\|\Delta_i f\|_\infty+|f(x)|\,\|\Delta_i g\|_\infty\right]H_i(x)\right), \tag{2.8}$$

where M and $H_i(x)$ are as defined in Theorem 2.1.

Remark 2.7. In [4] and the references cited therein, many generalisations of Grüss inequality (1.2) are given. Multidimensional integral inequalities of the Grüss type were recently established in [6, 7]. We note that the inequality (2.8) can be considered as the finite difference analogue of the inequality recently established in [7, Theorem 2.3].

3. Proof of Theorem 2.1

For $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ in B, it is easy to observe that the following identities hold:

$$f(x)-f(y)=\sum_{i=1}^n\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}, \tag{3.1}$$

$$g(x)-g(y)=\sum_{i=1}^n\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}. \tag{3.2}$$

Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding we get

$$2f(x)g(x)-g(x)f(y)-f(x)g(y) = g(x)\sum_{i=1}^n\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\} + f(x)\sum_{i=1}^n\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}. \tag{3.3}$$

Summing both sides of (3.3) with respect to y over B, using the fact that $M>0$ and rewriting we have

$$f(x)g(x)-\frac{1}{2M}g(x)\sum_y f(y)-\frac{1}{2M}f(x)\sum_y g(y) = \frac{1}{2M}\left[g(x)\sum_y\left[\sum_{i=1}^n\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}\right] + f(x)\sum_y\left[\sum_{i=1}^n\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}\right]\right]. \tag{3.4}$$

From (3.4) and using the properties of modulus we have

$$\left|f(x)\,g(x)-\frac{1}{2M}g(x)\sum_{y}f(y)-\frac{1}{2M}f(x)\sum_{y}g(y)\right|$$
$$\leq \frac{1}{2M}\left[|g(x)|\sum_{y}\left[\sum_{i=1}^{n}\left\{\left|\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right|\right\}\right]\right.$$
$$\left.+|f(x)|\sum_{y}\left[\sum_{i=1}^{n}\left\{\left|\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right|\right\}\right]\right]$$
$$\leq \frac{1}{2M}\left[|g(x)|\sum_{y}\left[\sum_{i=1}^{n}\left\{\|\Delta_i f\|_\infty\left|\sum_{t_i=y_i}^{x_i-1}1\right|\right\}\right]\right.$$
$$\left.+|f(x)|\sum_{y}\left[\sum_{i=1}^{n}\left\{\|\Delta_i g\|_\infty\left|\sum_{t_i=y_i}^{x_i-1}1\right|\right\}\right]\right]$$
$$=\frac{1}{2M}\sum_{i=1}^{n}\left[|g(x)|\,\|\Delta_i f\|_\infty+|f(x)|\,\|\Delta_i g\|_\infty\right]\left(\sum_{y}|x_i-y_i|\right)$$
$$=\frac{1}{2M}\sum_{i=1}^{n}\left[|g(x)|\,\|\Delta_i f\|_\infty+|f(x)|\,\|\Delta_i g\|_\infty\right]H_i(x)\,.$$

The proof of the inequality (2.1) is complete.

Multiplying both sides of (3.4) by $w(y)$, $y\in B$ and summing the resulting identity with respect to y on B and following the proof of inequality (2.1), we get the desired inequality in (2.2).

4. PROOF OF THEOREM 2.4

From the hypotheses, as in the proof of Theorem 2.1, the identities (3.1) and (3.2) hold. Multiplying the left sides and right sides of (3.1) and (3.2) we get

$$f(x)\,g(x)-g(x)\,f(y)-f(x)\,g(y)+f(y)\,g(y)$$
$$=\left[\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}\right]$$
$$\times\left[\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}\right]. \tag{4.1}$$

Summing both sides of (4.1) with respect to y on B and rewriting we have

$$f(x)\,g(x)-\frac{1}{M}g(x)\sum_{y}f(y)-\frac{1}{M}f(x)\sum_{y}g(y)+\frac{1}{M}\sum_{y}f(y)\,g(y)$$
$$=\frac{1}{M}\sum_{y}\left[\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}\right]$$
$$\times\left[\sum_{i=1}^{n}\left\{\sum_{t_i=y_i}^{x_i-1}\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)\right\}\right]. \tag{4.2}$$

From (4.2) and using the properties of modulus we have

$$\left| f(x)g(x) - \frac{1}{M}g(x)\sum_y f(y) - \frac{1}{M}f(x)\sum_y g(y) + \frac{1}{M}\sum_y f(y)g(y) \right|$$
$$\leq \frac{1}{M}\sum_y \left[\sum_{i=1}^{n} \left| \left\{ \sum_{t_i=y_i}^{x_i-1} |\Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)| \right\} \right| \right]$$
$$\times \left[\sum_{i=1}^{n} \left| \left\{ \sum_{t_i=y_i}^{x_i-1} |\Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n)| \right\} \right| \right]$$
$$\leq \frac{1}{M}\sum_y \left[\sum_{i=1}^{n} \|\Delta_i f\|_\infty |x_i - y_i| \right] \left[\sum_{i=1}^{n} \|\Delta_i g\|_\infty |x_i - y_i| \right],$$

which is the required inequality in (2.5).

Summing both sides of (3.1) and (3.2) with respect to y and rewriting we get

$$f(x) - \frac{1}{M}\sum_y f(y) = \frac{1}{M}\sum_y \left[\sum_{i=1}^{n} \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n) \right\} \right] \tag{4.3}$$

and

$$g(x) - \frac{1}{M}\sum_y g(y) = \frac{1}{M}\sum_y \left[\sum_{i=1}^{n} \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n) \right\} \right], \tag{4.4}$$

respectively. Multiplying the left sides and right sides of (4.3) and (4.4) we get

$$\begin{aligned} f(x)g(x) &- \frac{1}{M}g(x)\sum_y f(y) - \frac{1}{M}f(x)\sum_y g(y) \\ &+ \frac{1}{M^2}\left(\sum_y f(y)\right)\left(\sum_y g(y)\right) \\ = \frac{1}{M^2} &\left(\sum_y \left[\sum_{i=1}^{n} \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n) \right\} \right] \right) \\ &\times \left(\sum_y \left[\sum_{i=1}^{n} \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n) \right\} \right] \right). \end{aligned} \tag{4.5}$$

From (4.5) and using the properties of modulus we have

$$\left| f(x)g(x) - \frac{1}{M}g(x)\sum_y f(y) - \frac{1}{M}f(x)\sum_y g(y) \right.$$
$$\left. + \frac{1}{M^2}\left(\sum_y f(y)\right)\left(\sum_y g(y)\right) \right|$$

$$\leq \frac{1}{M^2}\left(\sum_y\left[\sum_{i=1}^n\left|\left\{\sum_{t_i=y_i}^{x_i-1}\left|\Delta_i f\left(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n\right)\right|\right\}\right|\right]\right)$$
$$\times\left(\sum_y\left[\sum_{i=1}^n\left|\left\{\sum_{t_i=y_i}^{x_i-1}\left|\Delta_i g\left(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n\right)\right|\right\}\right|\right]\right)$$
$$\leq \frac{1}{M^2}\left(\sum_{i=1}^n\left\|\Delta_i f\right\|_\infty H_i\left(x\right)\right)\left(\sum_{i=1}^n\left\|\Delta_i g\right\|_\infty H_i\left(x\right)\right).$$

This is the desired inequality in (2.6) and the proof is complete.

5. PROOF OF THEOREM 2.6

From the hypotheses, the identities (4.2) and (3.4) hold. Summing both sides of (4.2) with respect to x on $B,$ rewriting and using the properties of modulus we have

$$\left|\frac{1}{M}\sum_x f\left(x\right)g\left(x\right)-\left(\frac{1}{M}\sum_x f\left(x\right)\right)\left(\frac{1}{M}\sum_x g\left(x\right)\right)\right|$$
$$\leq \frac{1}{2M^2}\sum_x\left(\sum_y\left[\sum_{i=1}^n\left|\left\{\sum_{t_i=y_i}^{x_i-1}\left|\Delta_i f\left(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n\right)\right|\right\}\right|\right]\right.$$
$$\left.\times\left[\sum_{i=1}^n\left|\left\{\sum_{t_i=y_i}^{x_i-1}\left|\Delta_i g\left(y_1,\dots,y_{i-1},t_i,x_{i+1},\dots,x_n\right)\right|\right\}\right|\right]\right)$$
$$\leq \frac{1}{2M^2}\sum_x\left(\sum_y\left[\sum_{i=1}^n\left\|\Delta_i f\right\|_\infty\left|x_i-y_i\right|\right]\left[\sum_{i=1}^n\left\|\Delta_i g\right\|_\infty\left|x_i-y_i\right|\right]\right),$$

which proves the inequality (2.7).

Summing both sides of (3.4) with respect to x on B and following the proof of inequality (2.7) with suitable changes we get the required inequality in (2.8). The proof is complete.

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