

## NEW CONCEPTS OF WELL-POSEDNESS FOR OPTIMIZATION PROBLEMS WITH VARIATIONAL INEQUALITY CONSTRAINT

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## Abstract

In this note we present a new concept of well-posedness for Optimization Problems with constraints described by parametric Variational Inequalities or parametric Minimum Problems. We investigate some classes of operators and functions that ensure this type of well-posedness.

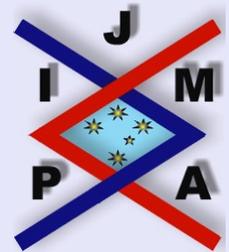
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# 1. Introduction

Let  $E$  be a reflexive Banach space with dual  $E^*$ ,  $A$  be an operator from  $E$  to  $E^*$  and  $K \subseteq E$  be a nonempty, closed, convex set. The Variational Inequality (VI), defined by the pair  $(A, K)$ , consists of finding a point  $u_0$  such that:

$$u_0 \in K \text{ and } \langle Au_0, u_0 - v \rangle \leq 0 \quad \forall v \in K.$$

This problem, introduced by G. Stampacchia in [22], has been recently investigated by many authors including [2], [4], [8], [9] and [15].

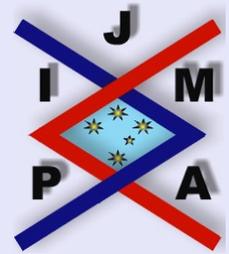
If  $(X, \tau)$  is a topological space, one can consider the parametric Variational Inequality  $(VI)(x)$ , defined by the pair  $(A(x, \cdot), H(x))$ , where, for all  $x \in X$ ,  $A(x, \cdot)$  is an operator from  $E$  to  $E^*$  and  $H$  is a set-valued function from  $X$  to  $E$  with nonempty and convex values.

The interest in this study is twofold: one is to study the behavior of perturbations of  $(VI)$ , another is to consider the parameter  $x$  as a decision variable in a multilevel optimization problem. More precisely, the solution set to  $(VI)(x)$  can be seen as the constraint set  $T(x)$  of the following Optimization Problem with Variational Inequality Constraints:

$$(OPVIC) \quad \inf_{x \in X} \inf_{u \in T(x)} f(x, u),$$

where  $f : X \times E \rightarrow \mathbb{R} \cup \{+\infty\}$ .

The problems **OPVIC** (often termed Mathematical Programming with Equilibrium Constraints **MPEC**) have been investigated by many authors (see for example [13], [14], [17], [19] and [21]) since they describe many economic or engineering problems (see for example [18]) such as:



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- The price setting problem
- Price setting of telecommunication networks
- Yield management in airline industry
- Traffic management through link tolls.

Assuming that  $(VI)(x)$  has a unique solution, a well-posedness concept for *OPVIC*, inspired from numerical methods, has been considered in [13]. However, in many applications, the problems  $(VI)(x)$  do not always have a unique solution.

So, in this paper, motivated from a numerical method for Variational Inequalities (M. Fukushima [7]), we introduce and study, for  $\alpha \geq 0$ , the concepts of  $\alpha$ -well-posedness and  $\alpha$ -well-posedness in the generalized sense for a family of Variational Inequalities  $(VI) = \{(VI)(x), x \in X\}$  and for *OPVIC*. The particular case of variational inequalities arising from minimum problems is also considered.

The paper is organized as follows. In Section 2 we review some basic notions for variational inequalities and present some new results on  $\alpha$ -well-posedness for unparametric variational inequalities. Section 3 is devoted to introducing and investigating the concept of  $\alpha$ -well-posedness for parametric variational inequalities and Section 4 to parametric minimum problems. Finally, some new concepts of well-posedness for *OPVIC* is presented and investigated in Section 5.




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## 2. Definitions and Background

In this section, some notions of *well-posedness* for variational inequalities (VI) introduced in [13] and in [15] and their connections with optimization problems are presented, together with equivalent characterizations.

Let  $E$  be a reflexive Banach space with dual  $E^*$ ,  $\sigma$  be a convergence on  $E$ , and  $K$  be a nonempty, closed and convex subset of  $E$ .

**Definition 2.1.** [5, 23]. Let  $h : K \rightarrow \mathbb{R} \cup \{+\infty\}$ . The minimization problem (2.1):

$$(2.1) \quad \min_{v \in K} h(v)$$

is Tikhonov well-posed (resp. well-posed in the generalized sense) with respect to  $\sigma$  if there exists a unique solution  $u_0$  to (2.1) and every minimizing sequence  $\sigma$ -converges to  $u_0$  (resp. if (2.1) has at least a solution and every minimizing sequence has a subsequence  $\sigma$ -converging to a minimum point).

For an operator  $A$  from  $E$  to  $E^*$ , we consider the following Variational Inequality (VI) defined by the pair  $(A, K)$ :

$$\text{find } u_0 \in K \text{ such that } \langle Au_0, u_0 - v \rangle \leq 0 \quad \forall v \in K.$$

**Definition 2.2.** [13, 15] Let  $\alpha \geq 0$ . A sequence  $(u_n)_n$  is  $\alpha$ -approximating for (VI) if:

i)  $u_n \in K \quad \forall n \in \mathbb{N}$ ;



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ii) there exists a sequence  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , decreasing to 0 such that

$$\langle Au_n, u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n \quad \forall v \in K \quad \forall n \in \mathbb{N}.$$

A variational inequality (VI) is termed  $\alpha$ -well-posed with respect to  $\sigma$ , if it has a unique solution  $u_0$  and every  $\alpha$ -approximating sequence  $(u_n)_n$   $\sigma$ -converges to  $u_0$ . If  $\sigma$  is the strong convergence  $s$  (resp. the weak convergence  $w$ ) on  $E$ , (VI) will be termed strongly  $\alpha$ -well-posed (resp. weakly  $\alpha$ -well-posed).

The above concept originated from the notion of Tikhonov well-posedness for the following minimization problem (2.2):

$$(2.2) \quad \min_{u \in K} g_\alpha(u),$$

where

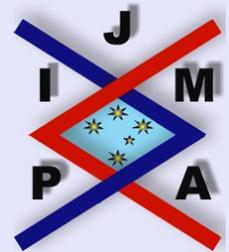
$$g_\alpha(u) = \sup_{v \in K} \left( \langle Au, u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \right).$$

Indeed, the following result holds:

**Proposition 2.1.** *Let  $\alpha \geq 0$ . The variational inequality problem (VI) is  $\alpha$ -well-posed if and only if the minimization problem (2.2) is Tikhonov well-posed.*

*Proof.* If (VI) is  $\alpha$ -well-posed there exists a unique solution  $u_0$  for (VI), that is:

$$u_0 \in K \quad \text{and} \quad g_0(u_0) = \sup_{v \in K} \langle Au_0, u_0 - v \rangle \leq 0$$



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and, consequently,  $g_\alpha(u_0) \leq g_0(u_0) \leq 0$ . Since  $g_\alpha(u) \geq 0$  for every  $u \in K$ ,  $g_\alpha(u_0) = 0$  and  $u_0$  is a minimum point for  $g_\alpha$ . In order to prove that (2.2) has a unique solution, consider  $u' \in K$  such that  $g_\alpha(u') = g_\alpha(u_0) = 0$ . For every  $v \in K$  consider the point  $w = \lambda u' + (1 - \lambda)v$ ,  $\lambda \in [0, 1]$ , which belongs to  $K$ . Since  $g_\alpha(u') = 0$  one has:

$$\langle Au', u' - w \rangle - \frac{\alpha}{2} \|u' - w\|^2 = (1 - \lambda)\langle Au', u' - v \rangle - \frac{\alpha}{2}(1 - \lambda)^2 \|u' - v\|^2 \leq 0$$

which implies:

$$\langle Au', u' - v \rangle - \frac{\alpha}{2}(1 - \lambda) \|u' - v\|^2 \leq 0 \quad \forall \lambda \in [0, 1].$$

So, when  $\lambda$  converges to 1, one gets:

$$\langle Au', u' - v \rangle \leq 0 \quad \forall v \in K.$$

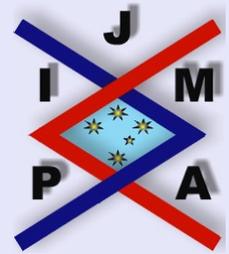
Then also  $u'$  solves (VI) and it must coincide with  $u_0$ .

As the family of minimizing sequences for (2.2) coincides with the family of  $\alpha$ -approximating sequence for (VI), the first part is proved.

Now, assume that (2.2) is well-posed and  $u_\alpha$  is the unique solution for (2.2), that is  $u_\alpha \in K$  and  $g_\alpha(u_\alpha) = 0$ .

With the same arguments used in the first part of this proof it can be proved that  $u_\alpha$  solves also the variational inequality (VI) (this has been already proved in [7] with other arguments). In order to prove that  $u_\alpha$  is the unique solution to (VI), let  $u'$  be another solution to (VI). Since  $g_\alpha(u') \leq g_0(u') = 0$ , the point  $u'$  should be a solution to (2.2), thus it has to coincide with  $u_\alpha$ .

Then the result follows as in the first part. □



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The gap function  $g_\alpha$ , which provides an optimization problem formulation for (VI), is, for  $\alpha = 0$ , the gap function introduced by Auslender in [1], and, for  $\alpha > 0$ , the merit function introduced by Fukushima in [7] for numerical purposes.

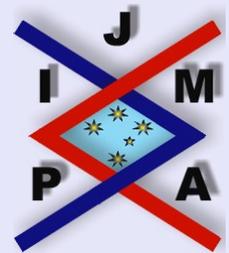
As it is well known, when the set  $K$  is not bounded, the set  $T$  of the solutions to (VI) may be empty, even in finite dimensional spaces. This does not happen when the operator  $A$  satisfies some of the following well known properties.

**Definition 2.3.** *The operator  $A$  is said to be:*

- *monotone on  $K$  if  $\langle Au - Av, u - v \rangle \geq 0$  for every  $u$  and  $v \in K$ ,*
- *pseudomonotone on  $K$  if for every  $u$  and  $v \in K$   $\langle Au, u - v \rangle \leq 0 \Rightarrow \langle Av, u - v \rangle \leq 0$ ;*
- *strongly monotone on  $K$  (with modulus  $\beta$ ) if  $\langle Au - Av, u - v \rangle \geq \beta \|u - v\|^2$  for every  $u$  and  $v \in K$ ;*
- *hemicontinuous on  $K$  if it is continuous from every segment of  $K$  to  $E^*$  endowed with the weak topology.*

It is well known (see for example [2]) that the variational inequality (VI) has a unique solution if the operator  $A$  is strongly monotone and hemicontinuous, while there exists at least a solution for (VI) if the operator  $A$  is pseudomonotone and hemicontinuous and some coerciveness condition is satisfied (see for example [8]).

We recall some continuity properties for set-valued functions that will be used later on:



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**Definition 2.4.** A set-valued function  $F$  from a topological space  $(X, \tau)$  to a convergence space  $(Y, \sigma)$  (see [11]) is:

- sequentially  $\sigma$ -lower semicontinuous at  $x \in X$  if, for every sequence  $(x_n)_n$   $\tau$ -converging to  $x$  and every  $y \in F(x)$ , there exists a sequence  $(y_n)_n$   $\sigma$ -converging to  $y$  such that  $y_n \in F(x_n) \forall n \in \mathbb{N}$ ;
- sequentially  $\sigma$ -subcontinuous at  $x \in X$  if, for every sequence  $(x_n)_n$   $\tau$ -converging to  $x$ , every sequence  $(y_n)_n$ ,  $y_n \in F(x_n) \forall n \in \mathbb{N}$ , has a  $\sigma$ -convergent subsequence;
- sequentially  $\sigma$ -closed at  $x \in X$  if for every sequence  $(x_n)_n$   $\tau$ -converging to  $x$ , for every sequence  $(y_n)_n$   $\sigma$ -converging to  $y$ ,  $y_n \in F(x_n) \forall n \in \mathbb{N}$ , one has  $y \in F(x)$ .

We have chosen to deal with sequential continuity notions for set-valued functions since our well-posedness concepts are defined in a sequential way. However, for brevity, from now on the term *sequentially* will be omitted.

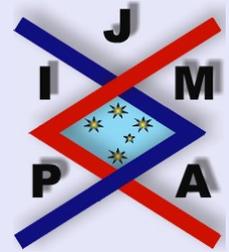
Let  $\varepsilon > 0$ . The following approximate solutions set, introduced in [15],

$$\mathcal{T}_{\alpha, \varepsilon} = \left\{ u \in K : \langle Au, u - v \rangle \leq \varepsilon + \frac{\alpha}{2} \|u - v\|^2 \quad \forall v \in K \right\} \quad \text{for } \varepsilon > 0$$

can be used to provide a characterization of  $\alpha$ -well-posedness in line with [13, Prop. 2.3 bis] and [5].

**Proposition 2.2.** Let  $\alpha \geq 0$  and assume that the operator  $A$  is hemicontinuous and monotone on  $K$  and that (VI) has a unique solution. The variational inequality (VI) is strongly  $\alpha$ -well-posed if and only if

$$\mathcal{T}_{\alpha, \varepsilon} \neq \emptyset \quad \forall \varepsilon > 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \text{diam}(\mathcal{T}_{\alpha, \varepsilon}) = 0.$$



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*Proof.* Assume that  $(VI)$  is strongly  $\alpha$ -well-posed and

$$\lim_{\varepsilon \rightarrow 0} \text{diam} \mathcal{T}_\alpha(\varepsilon) > 0.$$

Then there exists a positive number  $\beta$  such that, for every sequence  $(\varepsilon_n)_n$  decreasing to 0,  $\varepsilon_n > 0$ , there exist two sequences  $(y_n)_n$  and  $(v_n)_n$  in  $K$  such that

$$y_n \in \mathcal{T}_{\alpha, \varepsilon_n}, \quad v_n \in \mathcal{T}_{\alpha, \varepsilon_n} \quad \text{and} \quad \|y_n - v_n\| > \beta \quad \text{for } n \text{ sufficiently large.}$$

Since  $(VI)$  is strongly  $\alpha$ -well-posed, the sequences  $(y_n)_n$  and  $(v_n)_n$  must converge to the unique solution  $u_0$ , so

$$\lim_n \|y_n - v_n\| = 0$$

which gives a contradiction.

Conversely, let  $(y_n)_n$  be an  $\alpha$ -approximating sequence for  $(VI)$ , that is  $y_n \in \mathcal{T}_{\alpha, \varepsilon_n}$  for a sequence  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , decreasing to 0. Being  $\lim_n \text{diam} \mathcal{T}_{\alpha, \varepsilon_n} = 0$ , for every positive number  $\beta$  there exists a positive integer  $m$  such that  $\|y_n - y_p\| < \beta \quad \forall n \geq m \text{ and } p \geq m$ .

Therefore  $(y_n)_n$  is a Cauchy sequence and has to converge to a point  $u_0 \in K$ . Since  $A$  is monotone one has:

$$\begin{aligned} \langle Av, u_0 - v \rangle &= \lim_n \langle Av, y_n - v \rangle \\ &\leq \liminf_n \langle Ay_n, y_n - v \rangle \\ &\leq \lim_n \frac{\alpha}{2} \|y_n - v\|^2 = \frac{\alpha}{2} \|u_0 - v\|^2 \quad \forall v \in K. \end{aligned}$$



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Since  $A$  is monotone and hemicontinuous, the following equivalence holds:

$$\begin{aligned} \langle Av, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 &\leq 0 \quad \forall v \in K \\ \Leftrightarrow \langle Au_0, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 &\leq 0 \quad \forall v \in K. \end{aligned}$$

In fact, assume that

$$\langle Av, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0 \quad \forall v \in K.$$

If  $v$  is a point of  $K$ , for every number  $t \in [0, 1]$  the point  $v_t = tv + (1 - t)u_0$  belongs to  $K$ , so:

$$\langle Av_t, u_0 - v_t \rangle - \frac{\alpha}{2} \|u_0 - v_t\|^2 = t \langle Av_t, u_0 - v \rangle - t^2 \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0 \quad \forall t \in [0, 1].$$

So one has:

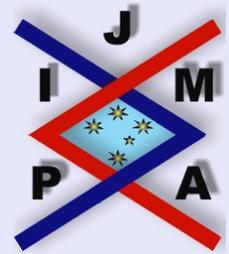
$$\lim_{t \rightarrow 0} \left( \langle Av_t, u_0 - v \rangle - \frac{\alpha}{2} t \|u_0 - v\|^2 \right) \leq 0$$

and, in light of the hemicontinuity of  $A$ :

$$\langle Au_0, u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 \leq \langle Au_0, u_0 - v \rangle \leq 0 \quad \forall v \in K.$$

The converse is an easy consequence of the monotonicity of  $A$ .

So  $g_\alpha(u_0) = 0$  and, arguing as in Proposition 2.1, it can be proved that  $u_0$  coincides with the unique solution to (VI). This completes the proof.  $\square$



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### 3. Parametrically $\alpha$ -Well-Posed Variational Inequalities

In what follows we shall consider a topological space  $(X, \tau)$ , a convergence  $\sigma$  on  $E$  and, for every  $x \in X$ , a parametric variational inequality on  $E$ ,  $(VI)(x)$ , defined by the pair  $(A(x, \cdot), H(x))$ , where  $A$  is an operator from  $X \times E$  to  $E^*$  and  $H$  is a set-valued function from  $X$  to  $E$  which is assumed to be nonempty, convex and closed-valued. In many situations  $H(x)$  is described by a finite number of inequalities:  $H(x) = \{u \in E : g_i(x, u) \leq 0, \forall i = 1, \dots, n\}$ , where  $g_i$  is a real-valued function, for  $i = 1, \dots, n$ , satisfying suitable assumptions.

Throughout this section we will consider the following family of variational inequalities:

$$(VI) = \{(VI)(x), x \in X\}.$$

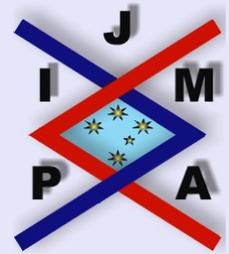
Let  $\alpha \geq 0$  and  $\varepsilon > 0$ . In the sequel, we shall denote by  $T$  (resp.  $T_{\alpha, \varepsilon}$ ) the map which associates to every  $x \in X$  the solution set (resp. the approximate solution set) to  $(VI)(x)$  :

$$T(x) = \{u \in H(x) : \langle A(x, u), u - v \rangle \leq 0 \quad \forall v \in H(x)\}$$

$$(\text{resp. } T_{\alpha, \varepsilon}(x) = \left\{ u \in H(x) : \langle A(x, u), u - v \rangle \leq \varepsilon + \frac{\alpha}{2} \|u - v\|^2 \quad \forall v \in H(x) \right\}).$$

Now, we introduce the notion of parametric  $\alpha$ -well-posedness for the family  $(VI)$ .

**Definition 3.1.** Let  $x \in X$  and  $(x_n)_n$  be a sequence converging to  $x$ . A sequence  $(u_n)_n$  is said to be  $\alpha$ -approximating for  $(VI)(x)$  (with respect to  $(x_n)_n$ ) if:



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i)  $u_n \in H(x_n) \quad \forall n \in \mathbb{N}$ ,

ii) there exists a sequence  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , decreasing to 0 such that

$$\langle A(x_n, u_n), u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n \quad \forall v \in H(x_n) \quad \forall n \in \mathbb{N}.$$

**Definition 3.2.** The family of variational inequalities (VI) is termed parametrically  $\alpha$ -well-posed with respect to  $\sigma$  if:

- for every  $x \in X$ ,  $(VI)(x)$  has a unique solution  $u_x$ ;
- for every sequence  $(x_n)_n$  converging to  $x$ , every  $\alpha$ -approximating sequence  $(u_n)_n$  for  $(VI)(x)$  (with respect to  $(x_n)_n$ )  $\sigma$ -converges to  $u_x$ .

If  $\sigma$  is the strong convergence  $s$  (resp. the weak convergence  $w$ ) on  $E$ , (VI) will be termed parametrically strongly  $\alpha$ -well-posed (resp. parametrically weakly  $\alpha$ -well-posed).

Observe that for  $\alpha = 0$  the above definition amounts to Definition 2.3 in [13].

**Definition 3.3.** The family of variational inequalities (VI) is termed parametrically  $\alpha$ -well-posed in the generalized sense with respect to  $\sigma$  if, for every  $x \in X$ ,  $(VI)(x)$  has at least a solution and for every sequence  $(x_n)_n$  converging to  $x$ , every  $\alpha$ -approximating sequence  $(u_n)_n$  for  $(VI)(x)$  (with respect to  $(x_n)_n$ ) has a subsequence  $\sigma$ -convergent to a solution to  $(VI)(x)$ .



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For a parametric variational inequality it is natural to consider the following parametric gap function  $g_\alpha(x, u)$ :

$$g_\alpha(x, u) = \sup_{v \in H(x)} \left( \langle A(x, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \right)$$

and with the same arguments as in Proposition 2.1 one can prove the following two propositions:

**Proposition 3.1.** *Let  $\alpha \geq 0$  and  $x \in X$ . A point  $u_x$  solves the variational inequality (VI)( $x$ ) if and only if:*

$$u_x \in H(x) \text{ and } g_\alpha(x, u_x) = \inf_{u \in H(x)} g_\alpha(x, u) = 0,$$

that is:

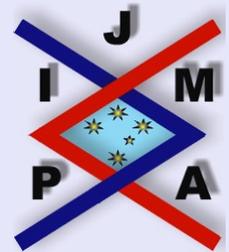
$$\langle A(x, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \leq 0 \quad \forall v \in H(x).$$

**Proposition 3.2.** *The family of variational inequality (VI) is parametrically  $\alpha$ -well-posed (resp. parametrically- $\alpha$ -well-posed in the generalized sense) with respect to  $\sigma$  if and only if, for every  $x \in X$ , the minimization problem*

$$(3.1) \quad \min_{u \in H(x)} g_\alpha(x, u)$$

*is parametrically Tikhonov well-posed (resp. parametrically Tikhonov well-posed in the generalized sense) with respect to  $\sigma$ , that is:  $g_\alpha$  is bounded from below, (3.1) has a unique solution (resp. has at least a solution)  $u_x$  and for every sequence  $(x_n)_n$  converging to  $x$ , every sequence  $(u_n)_n$  such that*

$$\inf_{u \in H(x)} g_\alpha(x, u) \geq \liminf_n g_\alpha(x_n, u_n)$$



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$\sigma$ -converges (resp. has a subsequence  $\sigma$ -convergent) to  $u_x$  (see Definition 2.3 in [13]).

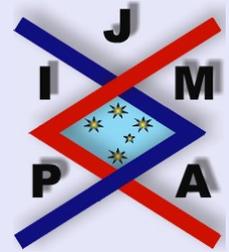
The connection between parametric  $\alpha$ -well-posedness and the convergence to 0 of the diameters of  $T_{\alpha,\varepsilon}(x)$  is given by the following result.

**Proposition 3.3.** *Let  $\alpha \geq 0$ . If the family of variational inequalities (VI) is strongly parametrically  $\alpha$ -well-posed, then, for every  $x \in X$ , every sequence  $(x_n)_n$  converging to  $x$  and every sequence  $(\varepsilon_n)_n$  of positive real numbers decreasing to 0, one has:*

$$T_{\alpha,\varepsilon}(x) \neq \emptyset \quad \forall \varepsilon > 0 \quad \text{and} \quad \lim_n \text{diam}(T_{\alpha,\varepsilon_n}(x_n)) = 0.$$

*Proof.* In light of the assumption, the set  $T_{\alpha,\varepsilon}(x)$  is nonempty since  $\{u_x\} = T(x) \subseteq T_{\alpha,\varepsilon}(x)$ . Assume that  $\lim_n \text{diam}(T_{\alpha,\varepsilon_n}(x_n)) > 0$ . Then there exist  $\eta > 0$  and two sequences  $(u_n)_n$  and  $(y_n)_n$  such that  $u_n \in T_{\alpha,\varepsilon_n}(x_n)$ ,  $y_n \in T_{\alpha,\varepsilon_n}(x_n)$  and  $\|y_n - u_n\| > \eta$ , for  $n$  sufficiently large. But, being  $(u_n)_n$  and  $(y_n)_n$  sequences  $\alpha$ -approximating for (VI)( $x$ ) (with respect to  $(x_n)_n$ ), they must converge to  $u_x$ , and this gives a contradiction.  $\square$

In order to achieve a similar result for generalized  $\alpha$ -well-posedness, one can consider the non compactness measure  $\mu$ , introduced by Kuratowski in [11]: if  $(S, d)$  is a metric space and  $B$  is a bounded subset of  $S$ ,  $\mu(B)$  is defined as the infimum of  $\varepsilon > 0$  such that  $B$  can be covered by a finite number of open sets having diameter less than  $\varepsilon$ . The following proposition, whose proof is in line with previous results concerning generalized well-posedness for minimum problems (see [5]), gives the link between the noncompactness measure



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of  $T_{\alpha, \varepsilon_n}(x)$  and the generalized  $\alpha$ -well-posedness, when the set-valued function  $H$  is constant:

**Proposition 3.4.** *Let  $\alpha \geq 0$ . Assume that for every  $u \in E$  the operator  $A(\cdot, u)$  is continuous from  $X$  to  $(E^*, w)$  and the set-valued function  $H$  is constant, that is  $H(x) = K$ , where  $K$  is a nonempty, closed convex subset of  $E$ . If the family of variational inequalities (VI) is parametrically strongly  $\alpha$ -well-posed in the generalized sense, then, for every  $x \in X$ , every sequence  $(x_n)_n$  converging to  $x$  and every sequence  $(\varepsilon_n)_n$  of positive real numbers decreasing to 0, one has:*

$$T_{\alpha, \varepsilon}(x) \neq \emptyset \quad \forall \varepsilon > 0 \quad \text{and} \quad \lim_n \mu(T_{\alpha, \varepsilon_n}(x_n)) = 0.$$

*Proof.* Let  $(\varepsilon_n)_n$  be a sequence of positive real numbers, let  $x \in X$  and  $(x_n)_n$  be a sequence converging to  $x$ .

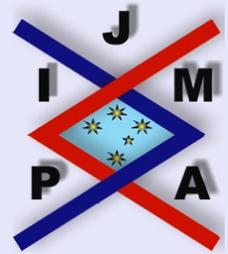
We start by proving that  $\lim_n h(T_{\alpha, \varepsilon_n}(x_n), T(x)) = 0$ , where  $h(T_{\alpha, \varepsilon_n}(x_n), T(x)) = h_n$  is the Hausdorff distance [11] between  $T_{\alpha, \varepsilon_n}(x_n)$  and the set of solutions to  $(VI)(x)$ , that is:

$$h_n = \max \left\{ \sup_{u \in T_{\alpha, \varepsilon_n}(x_n)} d(u, T(x)), \sup_{v \in T(x)} d(T_{\alpha, \varepsilon_n}(x_n), v) \right\}.$$

By the assumptions, every  $u \in T(x)$  belongs to  $T_{\alpha, \varepsilon_n}(x_n)$ , for  $n$  sufficiently large.

Indeed  $u \in T(x)$  if and only if  $\langle A(x, u), u - v \rangle \leq 0 \quad \forall v \in K$  and, consequently:

$$\langle A(x, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 \leq 0 \quad \forall v \in K.$$



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If

$$v \neq u, \langle A(x, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 < 0 = \lim_n \varepsilon_n$$

and in light of continuity of  $A(\cdot, u)$  one gets

$$\langle A(x_n, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 < \varepsilon_n$$

for  $n$  sufficiently large.

If  $v = u$ , the result is obvious since

$$\langle A(x_n, u), u - v \rangle - \frac{\alpha}{2} \|u - v\|^2 = 0 < \varepsilon_n \text{ for every } n \in \mathbb{N}.$$

So, if  $\limsup_n h(T_{\alpha, \varepsilon_n}(x_n), T(x)) > c > 0$ , there exists a sequence  $(u_n)_n$  :

$$u_n \in T_{\alpha, \varepsilon_n}(x_n) \text{ and } d(u_n, T(x)) > c \text{ for } n \text{ sufficiently large.}$$

Since  $(u_n)_n$  is  $\alpha$ -approximating, there is a subsequence  $(u_{n_k})_k$  converging to  $u_x \in T(x)$  and one gets:

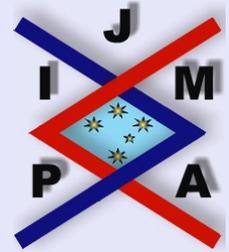
$$0 = d(u_x, T(x)) \geq \limsup_k d(u_{n_k}, T(x)) > c,$$

which gives a contradiction.

In order to complete the proof, it takes only to observe that  $T_{\alpha, \varepsilon_n}(x_n) \subseteq B(T(x), h_n)$  (the ball of radius  $h_n$  around  $T(x)$ ) and  $\mu(T(x)) = 0$ , so the following inequality holds (see, for example [5]):

$$\mu(T_{\alpha, \varepsilon_n}(x_n)) \leq 2h_n + \mu(T(x)) = 2h_n.$$

□



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The next lemma is in the spirit of the Minty's Lemma and will be used to characterize  $\alpha$ -well-posedness for parametric variational inequalities. The proof is omitted since it is similar to the proof given in Proposition 2.2 for unparametric variational inequalities.

**Lemma 3.5.** *Let  $\alpha \geq 0$ . If, for every  $x \in X$ , the operator  $A(x, \cdot)$  is hemicontinuous and monotone on  $H(x)$ , then the following conditions are equivalent:*

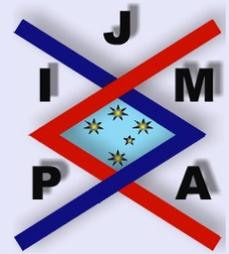
- i)  $u_0 \in H(x)$  and  $\langle A(x, u_0), u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0$  for every  $v \in H(x)$ ,
- ii)  $u_0 \in H(x)$  and  $\langle A(x, v), u_0 - v \rangle - \frac{\alpha}{2} \|u_0 - v\|^2 \leq 0$  for every  $v \in H(x)$ .

The next proposition proves that in finite dimensional spaces the parametric  $\alpha$ -well-posedness is equivalent to the uniqueness of solutions to  $(VI)(x)$ , for every  $\alpha \geq 0$ .

**Proposition 3.6.** *Let  $\alpha \geq 0$  and  $E = R^k$ . If the following conditions hold:*

- i) *the set-valued function  $H$  is lower semicontinuous, closed and subcontinuous;*
- ii) *for every  $x \in X$ ,  $A(x, \cdot)$  is monotone and hemicontinuous;*
- iii) *for every  $u \in R^k$ ,  $A(\cdot, u)$  is continuous on  $X$ ;*
- iv)  *$A$  is uniformly bounded on  $X \times R^k$ , that is there exists  $k > 0$  such that for every converging sequence  $(x_n, u_n)_n$  one has  $\|A(x_n, u_n)\| \leq k$  for every  $n \in \mathbb{N}$ ;*

*then (VI) is parametrically  $\alpha$ -well-posed if and only if, for every  $x \in X$ ,  $(VI)(x)$  has a unique solution  $u_x$ .*




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*Proof.* For  $x \in X$ , let  $(x_n)_n$  be a sequence converging to  $x$  and  $(u_n)_n$  be an  $\alpha$ -approximating sequence (with respect to  $(x_n)_n$ ), that is:

$$u_n \in H(x_n) \text{ and } \langle A(x_n, u_n), u_n - v \rangle \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - v\|^2 \quad \forall v \in H(x_n),$$

where  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , is a sequence decreasing to 0.

Since  $H$  is closed and subcontinuous there exists a subsequence  $(u_{n_k})_k$  of  $(u_n)_n$  converging to a point  $\tilde{u}_x \in H(x)$ . Moreover, in light of the lower semi-continuity of  $H$ , for every  $v \in H(x)$  there exists a sequence  $(v_n)_n$  converging to  $v$  such that  $v_n \in H(x_n)$  for every  $n \in \mathbb{N}$ .

The monotonicity of  $A(x_{n_k}, \cdot)$  implies:

$$\begin{aligned} \langle A(x_{n_k}, v), u_{n_k} - v \rangle &\leq \langle A(x_{n_k}, u_{n_k}), u_{n_k} - v_{n_k} \rangle + \langle A(x_{n_k}, u_{n_k}), v_{n_k} - v \rangle \\ &\leq \varepsilon_{n_k} + \frac{\alpha}{2} \|u_{n_k} - v_{n_k}\|^2 + \|A(x_{n_k}, u_{n_k})\| \|v_{n_k} - v\| \end{aligned}$$

for every  $k \in \mathbb{N}$ .

Since  $A(\cdot, v)$  is continuous at  $x$  and  $A$  is uniformly bounded one has:

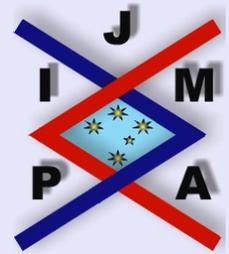
$$\langle A(x, v), \tilde{u}_x - v \rangle \leq \frac{\alpha}{2} \|\tilde{u}_x - v\|^2$$

and applying the previous lemma:

$$\langle A(x, \tilde{u}_x), \tilde{u}_x - v \rangle \leq \frac{\alpha}{2} \|\tilde{u}_x - v\|^2.$$

But, from Proposition 3.1, this inequality is equivalent to:

$$\langle A(x, \tilde{u}_x), \tilde{u}_x - v \rangle \leq 0 \quad \forall v \in H(x)$$



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that is  $\tilde{u}_x$  solves  $(VI)(x)$ .

Since  $(VI)(x)$  has a unique solution, the point  $\tilde{u}_x$  must coincide with  $u_x$  and the whole sequence  $(u_n)_n$  has to converge to  $u_x$ .  $\square$

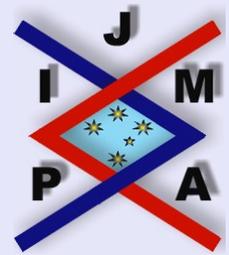
A similar result could be obtained in infinite dimensional spaces if one modifies the assumptions: in iii)  $A(\cdot, u)$  should be continuous from  $X$  to  $(E^*, s)$ , but in i)  $H$  should be assumed to be  $s$ -lower semicontinuous,  $w$ -closed and  $s$ -subcontinuous, which unfortunately lead to the strong compactness of  $H(x)$  for every  $x \in X$ .

**Remark 3.1.** *If the set-valued function  $H$  is constant, that is  $H(x) = K \ \forall x \in X$ , the same result holds assuming that the set  $K$  is compact and convex,  $A(x, \cdot)$  is monotone and hemicontinuous on  $K$  for every  $x \in X$ , and  $A(\cdot, u)$  is continuous on  $X$  for every  $u \in K$ . Indeed, arguing as in Proposition 3.6, for every  $v \in K$  one has:*

$$\begin{aligned} \langle A(x_{n_k}, v), \tilde{u} - v \rangle &= \langle A(x_{n_k}, v), \tilde{u} - u_{n_k} \rangle + \langle A(x_{n_k}, v), u_{n_k} - v \rangle \\ &\leq \langle A(x_{n_k}, v), \tilde{u} - u_{n_k} \rangle + \langle A(x_{n_k}, u_{n_k}), u_{n_k} - v \rangle \\ &\leq \langle A(x_{n_k}, v), \tilde{u} - u_{n_k} \rangle + \varepsilon_{n_k} + \frac{\alpha}{2} \|u_{n_k} - v\|^2, \end{aligned}$$

and for  $k$  converging to  $+\infty$  the result follows.

**Example 3.1.** *If  $E$  is an infinite dimensional space, the previous result may fail to be true when  $K$  is only weakly compact, that is: there are variational inequalities with a unique solution which are not  $\alpha$ -well-posed. Indeed, the following example (already considered in [5]) holds: let  $E$  be a separable Hilbert space with an ortonormal basis  $(e_n)_n$ ,  $B$  be the unitary closed ball of  $E$ . Consider the*



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operator  $\nabla h(u)$ , where  $h(u) = \sum_n \frac{\langle u, e_n \rangle}{n^2}$  and the variational inequality (VI) defined by:  $v \in B$  and  $\langle \nabla h(u), u - v \rangle \leq 0 \quad \forall v \in B$ .

It has as unique solution  $u_0 = 0$ , but  $(e_n)_n$  is an approximating (and consequently  $\alpha$ -approximating for every  $\alpha > 0$ ) sequence that does not strongly converge to 0.

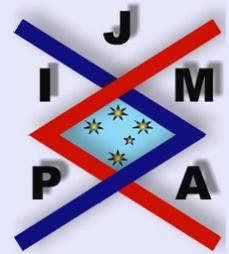
The next result and the following remark, concerning  $\alpha$ -well-posedness in the generalized sense, can be easily proved with the same arguments as in Proposition 3.6 and Remark 3.1.

**Proposition 3.7.** *Let  $E = R^k$  and  $\alpha \geq 0$ . If the assumptions of Proposition 3.6 hold, then the family (VI) is parametrically  $\alpha$ -well-posed in the generalized sense.*

*Proof.* Since under assumption i) the set  $H(x)$  is compact,  $(VI)(x)$  has at least a solution for every  $x \in X$  (see for example [10] or [2]), so the result can be easily proved as in Proposition 3.6.  $\square$

The previous proposition says nothing else that, under conditions i) to iv), in finite dimensional spaces, the parametric  $\alpha$ -well-posedness in the generalized sense is equivalent to the existence of solutions.

**Remark 3.2.** *If the set-valued function  $K$  is constant, that is  $H(x) = K \quad \forall x \in X$ , the same result holds assuming that the set  $K$  is compact and convex, for every  $x \in X$   $A(x, \cdot)$  is monotone and hemicontinuous on  $H$ , and, for every  $u \in K$   $A(\cdot, u)$  is continuous on  $X$ .*



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The following propositions furnish classes of operators for which the corresponding variational inequalities are parametrically  $\alpha$ -well-posed or parametrically  $\alpha$ -well-posed in the generalized sense.

**Proposition 3.8.** *Assume that the following conditions are satisfied:*

i) *the operator  $A$  is strongly monotone on  $E$  in the variable  $u$ , uniformly with respect to  $x$ , that is:*

$$\begin{aligned} \exists \beta > 0 \text{ such that } \langle A(x, u) - A(x, v), u - v \rangle \\ \geq \beta \|u - v\|^2 \quad \forall x \in X, \forall u \in E, \forall v \in E; \end{aligned}$$

ii) *for every  $u \in E$ ,  $A(\cdot, u)$  is continuous from  $(X, \tau)$  to  $(E^*, s)$ ;*

iii) *for every  $x \in X$ ,  $A(x, \cdot)$  is hemicontinuous on  $H(x)$ ;*

iv)  *$A$  is uniformly bounded on  $X \times E$ ;*

v) *the set-valued function  $H$  is  $w$ -closed,  $w$ -subcontinuous and  $s$ -lower semicontinuous.*

Then (VI) is parametrically strongly  $\alpha$ -well-posed for every  $\alpha$  such that  $0 \leq \alpha \leq 2\beta$ .

*Proof.* First of all, for every  $x \in X$ , the variational inequality (VI)( $x$ ) has a unique solution  $u_x$  (see, for example, [10] or [2]).

To prove that, for  $0 \leq \alpha \leq 2\beta$ , every  $\alpha$ -approximating sequence is strongly convergent, let  $x \in X$ ,  $(x_n)_n$  be a sequence converging to  $x$  and  $(u_n)_n$  be an  $\alpha$ -approximating sequence for (VI) with respect to  $(x_n)_n$ .



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Since  $H$  is  $w$ -closed and  $w$ -subcontinuous, the sequence  $(u_n)_n$  has a subsequence, still denoted by  $(u_n)_n$ , which weakly converges to  $\tilde{u}_x \in H(x)$ . To prove that  $\tilde{u}_x = u_x$ , consider a point  $v \in H(x)$  and a sequence  $(v_n)_n$  strongly converging to  $v$  such that  $v_n \in H(x_n)$  for every  $n \in \mathbb{N}$  (such sequence exists in virtue of the lower semicontinuity of  $H$ ). One has, for every  $n \in \mathbb{N}$ :

$$\begin{aligned} & \langle A(x_n, v), u_n - v \rangle \\ & \leq \langle A(x_n, u_n), u_n - v \rangle - \beta \|u_n - v\|^2 \\ & = \langle A(x_n, u_n), u_n - v_n \rangle + \langle A(x_n, u_n), v_n - v \rangle - \beta \|u_n - v\|^2 \\ & \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - v_n\|^2 - \beta \|u_n - v\|^2 + \|A(x_n, u_n)\| \|v_n - v\|. \end{aligned}$$

Since  $\frac{\alpha}{2} \leq \beta$ , one gets:

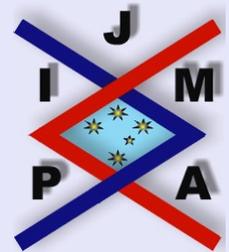
$$\begin{aligned} & \langle A(x_n, v), u_n - v \rangle \\ & \leq \varepsilon_n + \beta (\|v_n - v\|^2 + 2 \|u_n - v\| \|v_n - v\|) + \|A(x_n, u_n)\| \|v_n - v\| \end{aligned}$$

and in light of assumptions ii) and iv):

$$\langle A(x, v), \tilde{u}_x - v \rangle \leq 0.$$

The last inequality, for the arbitrariness of  $v$ , implies, by Minty's Lemma (see, for example, [2]), that  $\tilde{u}_x$  solves  $(VI)(x)$ , so  $\tilde{u}_x = u_x$ .

To prove that the sequence  $(u_n)_n$  strongly converges to  $u_x$ , let  $(w_n)_n$  be a sequence strongly converging to  $u_x$ ,  $w_n \in H(x_n) \forall n \in \mathbb{N}$  (such a sequence



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exists since  $H$  is  $s$ -lower semicontinuous). Observe that:

$$\begin{aligned} & \beta \|u_n - u_x\|^2 \\ & \leq \langle A(x_n, u_n) - A(x_n, u_x), u_n - u_x \rangle \\ & = \langle A(x_n, u_n), u_n - w_n \rangle + \langle A(x_n, u_n), w_n - u_x \rangle - \langle A(x_n, u_x), u_n - u_x \rangle \\ & \leq \varepsilon_n + \frac{\alpha}{2} \|u_n - w_n\|^2 + \|A(x_n, u_n)\| \|w_n - u_x\| \\ & \quad - \langle A(x_n, u_x), u_n - u_x \rangle \quad \forall n \in \mathbb{N}. \end{aligned}$$

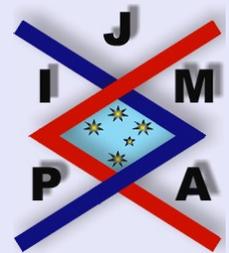
Since  $\|w_n - u_n\|^2 \leq (\|w_n - u_x\| + \|u_n - u_x\|)^2$ , one gets, for every  $n \in \mathbb{N}$ :

$$\begin{aligned} 0 & \leq \left( \beta - \frac{\alpha}{2} \right) \|u_n - u_x\|^2 \\ & \leq \varepsilon_n + \frac{\alpha}{2} \|u_x - w_n\|^2 + \alpha \|u_n - u_x\| \|u_x - w_n\| \\ & \quad + \|A(x_n, u_n)\| \|w_n - u_x\| - \langle A(x_n, u_x), u_n - u_x \rangle \end{aligned}$$

and this implies that  $\lim_n \|u_n - u_x\| = 0$ . So, we have proved that every weakly converging subsequence of  $(u_n)_n$  is also strongly converging to the unique solution for  $(VI)(x)$ . Then the whole sequence  $(u_n)_n$  strongly converges to  $u_x$ .  $\square$

**Remark 3.3.** *If the set-valued function  $H$  is constant, that is  $H(x) = K \forall x \in X$ , the same result can be established assuming that:*

- i) the operator  $A$  is strongly monotone in the variable  $u$  on  $E$  (with modulus  $\beta$ ), uniformly with respect to  $x$ ;*
- ii) for every  $u \in K$ ,  $A(\cdot, u)$  is continuous from  $(X, \tau)$  to  $(E^*, s)$ ;*



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iii) for every  $x \in X$ ,  $A(x, \cdot)$  is hemicontinuous on  $H(x)$ ;

iv) the set  $K$  is convex, closed and bounded.

For what concerning parametric  $\alpha$ -well-posedness in the generalized sense, we have the following result for  $\alpha = 0$  :

**Proposition 3.9.** Assume that the following conditions are satisfied:

i) for every  $x \in X$ ,  $A(x, \cdot)$  is monotone on  $H(x)$ ;

ii) for every  $u \in H$ ,  $A(\cdot, u)$  is continuous from  $(X, \tau)$  to  $(E^*, s)$ ;

iii) for every  $x \in X$ ,  $A(x, \cdot)$  is hemicontinuous on  $H(x)$ ;

iv)  $A$  is uniformly bounded on  $X \times E$ ;

v) the set-valued function  $H$  is  $w$ -closed,  $w$ -subcontinuous and  $s$ -lower semicontinuous.

Then (VI) is parametrically weakly well-posed in the generalized sense.

*Proof.* First of all, for every  $x \in X$ , the variational inequality (VI)( $x$ ) has at least a solution (see, for example, [10] or [2]), since under our assumptions the set  $H(x)$  is compact with respect to the weak convergence.

Let  $x \in X$ ,  $(x_n)_n$  be a sequence converging to  $x$ , and  $(u_n)_n$  be an approximating sequence for (VI) with respect to  $(x_n)_n$ .

Since  $H$  is  $w$ -closed and  $w$ -subcontinuous, the sequence  $(u_n)_n$  has a subsequence, still denoted by  $(u_n)_n$ , which weakly converges to  $u_x \in H(x)$ . To prove that  $u_x \in T(x)$ , consider a point  $v \in H(x)$ , a sequence  $(v_n)_n$  strongly



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converging to  $v$  such that  $v_n \in H(x_n)$  for every  $n \in \mathbb{N}$  (such sequence exists in virtue of the lower semicontinuity of  $H$ ). Since:

$$\begin{aligned} \langle A(x_n, v), u_n - v \rangle &\leq \langle A(x_n, u_n), u_n - v \rangle \\ &= \langle A(x_n, u_n), u_n - v_n \rangle + \langle A(x_n, u_n), v_n - v \rangle \\ &\leq \varepsilon_n + \langle A(x_n, u_n), v_n - v \rangle \\ &\leq \varepsilon_n + \|A(x_n, u_n)\| \|v_n - v\| \quad \forall n \in \mathbb{N} \end{aligned}$$

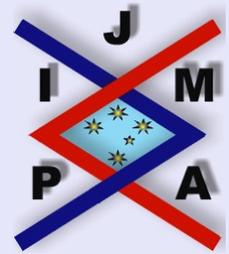
and assumptions ii) and iv) hold, one gets:

$$\langle A(x, v), u_x - v \rangle \leq 0 \quad \forall v \in H(x),$$

that, for the Minty's Lemma, is equivalent to say that  $u_x$  solves  $(VI)(x)$ .  $\square$

**Remark 3.4.** *If the set-valued function  $H$  is constant, that is  $H(x) = K, \forall x \in X$ , the same result can be established assuming that:*

- i) the operator  $A(x, \cdot)$  is hemicontinuous on  $H$ ;*
- ii) the operator  $A(x, \cdot)$  is monotone;*
- iii) for every  $u \in K$ ,  $A(\cdot, u)$  is continuous on  $X$ ;*
- iv) the set  $K$  is convex, closed and bounded.*



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## 4. Parametrically $\alpha$ -Well-Posed Minimum Problems

In this section we consider variational inequalities arising from parametric minimum problems and we investigate, for  $\alpha > 0$ , the links between parametric  $\alpha$ -well-posedness of such problems and parametric  $\alpha$ -well-posedness of the corresponding variational inequalities. The case  $\alpha = 0$  can be found in [13].

Let  $h$  be a function from  $X \times E$  to  $\mathbb{R} \cup \{+\infty\}$  and  $H$  be a set-valued function from  $X$  to  $E$ , which is assumed to be nonempty, convex and closed-valued. If, for every  $x \in X$ , the function  $h(x, \cdot)$  is Gâteaux differentiable, bounded from below and convex on  $H(x)$ , the minimum problem:

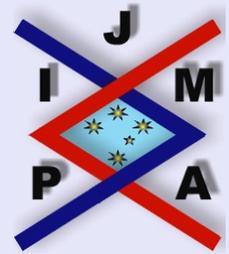
$$((P)(x)) \quad \inf_{u \in H(x)} h(x, u)$$

is equivalent to the following variational inequality problem:

$$((VI)(x)) \quad \text{find } u \in H(x) \text{ such that } \left\langle \frac{\partial h}{\partial u}(x, u), u - v \right\rangle \leq 0 \quad \forall v \in H(x),$$

where  $\frac{\partial h}{\partial u}$  is the derivative of the function  $h$  with respect to the variable  $u$  (see [2]). Then, it is natural to introduce the notion of parametric  $\alpha$ -well-posedness for a family of minimization problems  $\mathbf{P} = \{ (P)(x), x \in X \}$  and compare it with the parametric  $\alpha$ -well-posedness for the family  $\mathbf{VI} = \{ (VI)(x), x \in X \}$ .

**Definition 4.1.** Let  $x \in X$ ,  $(x_n)_n$  be a sequence converging to  $x$ ; the sequence  $(u_n)_n$  is termed  $\alpha$ -minimizing for  $(P)(x)$  (with respect to  $(x_n)_n$ ) if:



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i)  $u_n \in H(x_n) \forall n \in \mathbb{N}$ ,

ii) there exists a sequence  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , decreasing to 0 such that:

$$h(x_n, u_n) \leq h(x_n, v) + \frac{\alpha}{2} \|u_n - v\|^2 + \varepsilon_n \quad \forall v \in H(x_n) \text{ and } \forall n \in \mathbb{N}.$$

**Definition 4.2.** The family of minimum problems  $\mathbf{P}$  is called parametrically  $\alpha$ -well-posed, with respect to  $\sigma$ , if:

i) for every  $x \in X$ ,  $h(x, \cdot)$  is bounded from below,

ii) for every  $x \in X$ ,  $(P)(x)$  has a unique solution  $u_x$ ,

iii) for every sequence  $(x_n)_n$  converging to a point  $x$ , every  $\alpha$ -minimizing sequence  $(u_n)_n$  for  $(P)(x)$  (with respect to  $(x_n)_n$ )  $\sigma$ -converges to  $u_x$ .

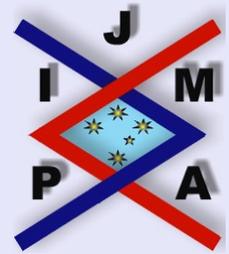
**Definition 4.3.** The family of minimum problems  $\mathbf{P}$  is called parametrically  $\alpha$ -well-posed in the generalized sense, with respect to  $\sigma$ , if:

i) for every  $x \in X$ ,  $h(x, \cdot)$  is bounded from below,

ii) for every  $x \in X$ ,  $(P)(x)$  has at least a solution  $u_x$ ,

iii) for every sequence  $(x_n)_n$  converging to a point  $x$ , every  $\alpha$ -minimizing sequence  $(u_n)_n$  for  $(P)(x)$  (with respect to  $(x_n)_n$ ) has a subsequence  $\sigma$ -convergent to a solution for  $(P)(x)$ .

The following two propositions give, under suitable assumptions, the equivalence between parametric  $\alpha$ -well-posedness for a minimization problem and the corresponding variational inequality.



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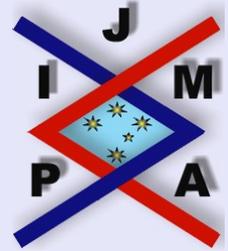


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**Proposition 4.1.** Assume that, for all  $x \in X$ , the function  $h(x, \cdot)$  is bounded from below, convex and Gâteaux differentiable on  $H(x)$  and the family of problems  $\mathbf{P}$  is parametrically  $\alpha$ -well-posed (resp. in the generalized sense) with respect to  $\sigma$ . Then the family of variational inequalities defined by

$$((VI)(x)) \quad \text{find } u \in H(x) \text{ such that } \left\langle \frac{\partial h}{\partial u}(x, u), u - v \right\rangle \leq 0 \quad \forall v \in H(x),$$

is parametrically  $\alpha$ -well-posed (resp. in the generalized sense) with respect to  $\sigma$ .

*Proof.* Under the above assumptions, for all  $x \in X$ , the problems  $(VI)(x)$  and  $(P)(x)$  have the same solutions. Consider a point  $x \in X$ , a sequence  $(x_n)_n$  converging to  $x$  and an  $\alpha$ -approximating sequence  $(u_n)_n$  for  $(VI)(x)$ , with respect to  $(x_n)_n$ , that is:

$$u_n \in H(x_n) \quad \text{and} \quad \left\langle \frac{\partial h}{\partial u}(x_n, u_n), u_n - v \right\rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n$$

$$\forall v \in H(x_n) \quad \forall n \in \mathbb{N},$$

where  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , decreases to 0. Since  $h(x_n, \cdot)$  is convex one has:

$$h(x_n, u_n) - h(x_n, v) \leq \left\langle \frac{\partial h}{\partial u}(x_n, u_n), u_n - v \right\rangle$$

$$\leq \frac{\alpha}{2} \|u_n - v\|^2 + \varepsilon_n \quad \forall v \in H(x_n) \quad \forall n \in \mathbb{N},$$

that is  $(u_n)_n$  is  $\alpha$ -minimizing for  $(P)(x)$  (with respect to  $(x_n)_n$ ) and the result then follows.  $\square$

**Proposition 4.2.** *Let  $E$  be a real Hilbert space. Assume that, for all  $x \in X$ , the function  $h(x, \cdot)$  is lower semicontinuous, bounded from below and Gâteaux differentiable on  $H(x)$  and the family of variational inequalities (VI) is parametrically strongly 0–well-posed. If the range  $H(X)$  is a bounded subset of  $E$ , then the family of minimum problems  $\mathbf{P}$  is strongly parametrically  $\alpha$ –well-posed for every  $\alpha > 0$ .*

*Proof.* Under the assumptions above, every solution to  $(P)(x)$  has to coincide with the unique solution to  $(VI)(x)$ ,  $\forall x \in X$ .

Consider  $x \in X$ , a sequence  $(x_n)_n$  converging to  $x$  and an  $\alpha$ -minimizing sequence  $(u_n)_n$  for  $(P)(x)$ , with respect to  $(x_n)_n$ , that is:

$$u_n \in H(x_n) \text{ and } h(x_n, u_n) \leq h(x_n, v) + \frac{\alpha}{2} \|u_n - v\|^2 + \varepsilon_n \quad \forall v \in H(x_n) \quad \forall n \in \mathbb{N},$$

where  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , is a sequence decreasing to 0.

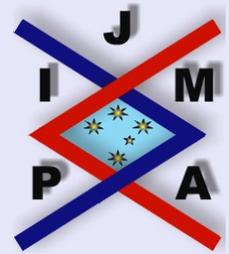
For every  $n \in \mathbb{N}$  define a new function  $f_n$  on  $E$  by:

$$f_n(v) = h(x_n, v) + \frac{\alpha}{2} \|u_n - v\|^2$$

and observe that  $f_n$  is lower semicontinuous, bounded from below, Gâteaux differentiable on  $H(x_n)$  and  $f_n(u_n) = h(x_n, u_n)$ .

Since  $f_n(u_n) \leq f_n(v) + \varepsilon_n \quad \forall v \in H(x_n)$ , from Ekeland Theorem (see [6]), for every  $n \in \mathbb{N}$  there exists  $u'_n \in H(x_n)$  such that:

$$\|u'_n - u_n\| < \sqrt{\varepsilon_n} \quad \text{and} \\ \left\langle \frac{\partial f_n}{\partial u}(u'_n), u'_n - v \right\rangle \leq \sqrt{\varepsilon_n} \|u'_n - v\| \quad \forall v \in H(x_n) \quad \forall n \in \mathbb{N}.$$



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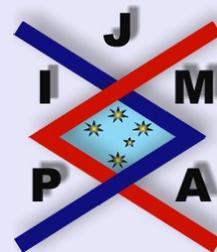
Therefore:

$$\begin{aligned} \left\langle \frac{\partial h}{\partial u}(x_n, u'_n), u'_n - v \right\rangle &= \left\langle \frac{\partial f_n}{\partial u}(u'_n), u'_n - v \right\rangle - \alpha \langle u_n - u'_n, u'_n - v \rangle \\ &\leq \sqrt{\varepsilon_n} \|u'_n - v\| (1 + \alpha) \quad \forall v \in H(x_n). \end{aligned}$$

Since the set-valued function  $H$  has a bounded range, the sequence  $(u'_n)_n$  is 0-approximating for  $(VI)(x)$  and the result follows.  $\square$

**Corollary 4.3.** *Let  $E$  be a real Hilbert space. Assume that, for all  $x \in X$ , the function  $h(x, \cdot)$  is lower semicontinuous, convex, bounded from below and Gâteaux differentiable on  $H(x)$  and the range  $H(X)$  is a bounded subset of  $E$ . Then the family of variational inequalities  $(VI)$  is parametrically strongly  $\alpha$ -well-posed (resp. in the generalized sense) with respect to  $\sigma$ , if and only if the minimum problem  $\mathbf{P}$  is parametrically strongly  $\alpha$ -well-posed (resp. in the generalized sense) with respect to  $\sigma$ .*

**Corollary 4.4.** *Let  $E$  be a real Hilbert space. Assume that, for all  $x \in X$ , the function  $h(x, \cdot)$  is lower semicontinuous, convex, bounded from below and Gâteaux differentiable on  $H(x)$  and the range  $H(X)$  is a bounded subset of  $E$ . Then the family of variational inequalities  $(VI)$  is parametrically strongly 0-well-posed (resp. in the generalized sense) if and only if it is parametrically strongly  $\alpha$ -well-posed (resp. in the generalized sense) for (every)  $\alpha > 0$ .*



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## 5. $\alpha$ -Well-Posedness for *OPVIC*

In this section we consider a convergence  $\sigma$  on  $E$  and the problem introduced in Section 1:

$$(OPVIC) \quad \inf_{x \in X} \inf_{u \in T(x)} f(x, u),$$

where  $f : X \times E \rightarrow \mathbb{R} \cup \{+\infty\}$  is bounded from below,  $H$  is a set-valued function from  $X$  to  $E$ , and, for every  $x \in X$ ,  $A(x, \cdot)$  is an operator from  $E$  to  $E^*$ , while  $T(x)$  is the set of solutions to the parametric variational inequality  $(VI)(x)$  defined by the pair  $(A(x, \cdot), H(x))$ .

In order to obtain sufficient conditions for  $\alpha$ -well-posedness of *OPVIC* we shall assume also that the function  $f$  satisfies a coercivity condition: namely, we say that  $f$  is *equicoercive* on  $(X \times E, (\tau \times \sigma))$  if every sequence  $(x_n, u_n)_n$ , such that  $f(x_n, u_n) \leq k \forall n \in \mathbb{N}$ , has a  $(\tau \times \sigma)$ -convergent subsequence.

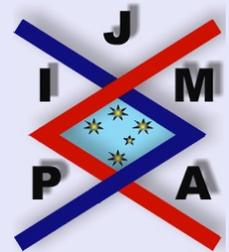
**Definition 5.1.** Let  $\alpha \geq 0$ . A sequence  $(x_n, u_n)_n$  is said to be  $\alpha$ -approximating for *OPVIC* if:

$$i) \liminf_n f(x_n, u_n) \leq \inf_{(x,u) \in X \times E, u \in T(x)} f(x, u);$$

ii) there exists a sequence  $(\varepsilon_n)_n$ ,  $\varepsilon_n > 0$ , decreasing to 0, such that  $u_n \in T_{\alpha, \varepsilon_n}(x_n) \forall n \in \mathbb{N}$ , that is:

$$u_n \in H(x_n) \text{ and } \langle A(x_n, u_n), u_n - v \rangle - \frac{\alpha}{2} \|u_n - v\|^2 \leq \varepsilon_n \quad \forall v \in H(x_n).$$

Observe that for  $\alpha = 0$  the above definition amounts to Definition 3.1 in [13] for *OPVIC* with variational inequalities having a unique solution.



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**Definition 5.2.** An optimization problem with variational inequality constraints *OPVIC* is termed  $\alpha$ –well-posed with respect to  $(\tau \times \sigma)$ , if it has a unique solution  $(x_0, u_0)$  towards which every  $\alpha$ –approximating sequence  $(x_n, u_n)_n$   $(\tau \times \sigma)$ –converges.

**Definition 5.3.** An optimization problem with variational inequality constraints *OPVIC* is termed  $\alpha$ –well-posed in the generalized sense with respect to  $(\tau \times \sigma)$ , if *OPVIC* has at least a solution and every  $\alpha$ –approximating sequence  $(x_n, u_n)_n$  has a subsequence  $\tau \times \sigma$ –convergent to a solution for *OPVIC*.

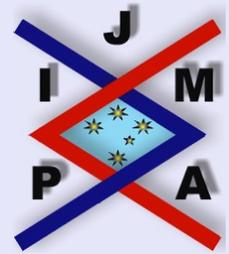
**Remark 5.1.** We point out that the set  $T(x)$  of solutions to  $(VI)(x)$  is not assumed to be always a singleton. In this situation many different types of “approximating” sequences could be considered instead of the ones considered in Definition 5.1 (see [20], where the well-posedness of MinSup problems is investigated).

In order to give sufficient conditions for the  $\alpha$ –well-posedness or  $\alpha$ –well-posedness in the generalized sense of *OPVIC*, we will distinguish the following situations:

- for every  $x \in X$   $(VI)(x)$  has a unique solution;
- there exists  $x \in X$  such that  $(VI)(x)$  has not a unique solution.

**First Case:** for every  $x \in X$   $(VI)(x)$  has a unique solution

Since this case for  $\alpha = 0$  has been already investigated in [13], assume that  $\alpha > 0$ .



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**Theorem 5.1.** *If (VI) is parametrically  $\alpha$ -well-posed with respect to  $\sigma$ ,  $f$  is sequentially lower semicontinuous and equicoercive on  $(X \times E, (\tau \times \sigma))$  and **OPVIC** has a unique solution, then **OPVIC** is  $\alpha$ -well-posed with respect to  $(\tau \times \sigma)$ .*

*Proof.* Let  $(x_n, u_n)_n$  be a sequence  $\alpha$ -approximating for **OPVIC**. Being  $f$  equicoercive, there exists a subsequence of  $(x_n, u_n)_n$ , still denoted by  $(x_n, u_n)_n$ , which  $(\tau \times \sigma)$ -converges to a point  $(x_0, u_0)$ .

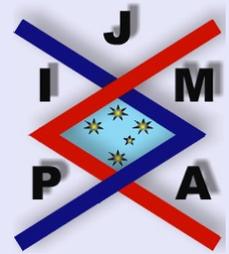
Since the sequence  $(u_n)_n$  is  $\alpha$ -approximating for  $(VI)(x_0)$  with respect to  $(x_n)_n$  and (VI) is parametrically  $\alpha$ -well-posed with respect to  $\sigma$ , the point  $u_0$  must belong to  $T(x_0)$ . Therefore, in light of condition i) in Definition 5.1 and lower semicontinuity of  $f$ , one has:

$$f(x_0, u_0) \leq \inf_{(x,u) \in X \times E, u \in T(x)} f(x, u),$$

that is  $(x_0, u_0)$  is the unique solution to **OPVIC**. Since every  $(\tau \times \sigma)$ -convergent subsequence of  $(x_n, u_n)_n$  converges to the unique solution for **OPVIC**, the whole sequence  $(x_n, u_n)_n$   $(\tau \times \sigma)$ -converges to it.  $\square$

Bearing in mind the proof of Proposition 3.8, a sufficient condition for the strongly  $\alpha$ -well-posedness of **OPVIC** with explicit assumptions on the data can be established.

**Theorem 5.2.** *Assume that  $f$  is sequentially lower semicontinuous and equicoercive on  $(X \times E, (\tau \times w))$ , and **OPVIC** has a unique solution. If the following assumptions are satisfied:*



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i) the operator  $A$  is strongly monotone on  $E$  in the variable  $u$ , uniformly with respect to  $x$ , that is:

$$\exists \beta > 0 \text{ such that } \langle A(x, u) - A(x, v), u - v \rangle \geq \beta \|u - v\|^2 \\ \forall x \in X, \forall u \in E, \forall v \in E;$$

ii) for every  $u \in E$ ,  $A(\cdot, u)$  is continuous from  $(X, \tau)$  to  $(E^*, s)$ ;

iii) for every  $x \in X$ ,  $A(x, \cdot)$  is hemicontinuous on  $H(x)$ ;

iv)  $A$  is uniformly bounded on  $X \times E$ ;

v) the set-valued function  $H$  is  $w$ -closed,  $w$ -subcontinuous,  $s$ -lower semicontinuous and convex-valued.

Then **OPVIC** is  $\alpha$ -well-posed with respect to  $(\tau \times s)$ , for every  $\alpha \leq 2\beta$ .

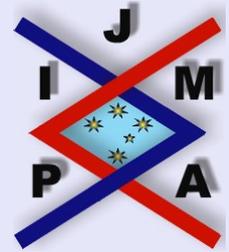
Now we do not assume that **OPVIC** has a unique solution. With the same arguments as in Theorem 5.1 one can prove:

**Theorem 5.3.** If (VI) is parametrically  $\alpha$ -well-posed with respect to  $\sigma$ ,  $f$  is sequentially lower semicontinuous and equicoercive on  $(X \times E, (\tau \times \sigma))$  and **OPVIC** has at least a solution, then **OPVIC** is  $\alpha$ -well-posed in the generalized sense with respect to  $(\tau \times \sigma)$ .

In finite dimensional spaces one obtains:

**Corollary 5.4.** Assume that  $f$  is sequentially lower semicontinuous and equicoercive on  $X \times \mathbb{R}^k$ , **OPVIC** has at least a solution and, for every  $x \in X$ ,  $(VI)(x)$  has a unique solution.

If the following assumptions are satisfied:



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- i) the set-valued function  $H$  is closed, lower semicontinuous, subcontinuous and convex-valued;
- ii) for every  $x \in X$ ,  $A(x, \cdot)$  is monotone and hemicontinuous on  $H(x)$ ;
- iii) for every  $u \in R^k$ ,  $A(\cdot, u)$  is continuous on  $X$ ;
- iv)  $A$  is uniformly bounded on  $X \times R^k$ ;

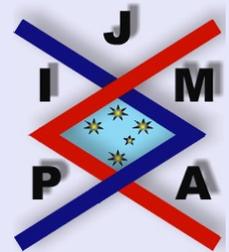
then **OPVIC** is  $\alpha$ -well-posed in the generalized sense. If the set-valued function  $H$  is constant, that is  $H(x) = K \forall x \in X$ , the same result holds assuming ii), iii) and the set  $K$  compact and convex.

**Second Case:** there exists  $x \in X$  such that  $(VI)(x)$  does not have a unique solution.

**Theorem 5.5.** Let  $\alpha \geq 0$ . If  $(VI)$  is parametrically  $\alpha$ -well-posed in the generalized sense with respect to  $\sigma$ ,  $f$  is sequentially lower semicontinuous and equicoercive on  $(X \times E, (\tau \times \sigma))$  and **OPVIC** has at least a solution, then **OPVIC** is  $\alpha$ -well-posed in the generalized sense with respect to  $(\tau \times \sigma)$ .

*Proof.* Let  $(x_n, u_n)_n$  be a sequence  $\alpha$ -approximating for **OPVIC**. From the equicoercivity of  $f$ , there exists a subsequence of  $(x_n, u_n)_n$ , still denoted by  $(x_n, u_n)_n$ , which  $(\tau \times \sigma)$ -converges to a point  $(x_0, u_0)$ .

Since the sequence  $(u_n)_n$  is  $\alpha$ -approximating for  $(VI)$  with respect to  $(x_n)_n$  and  $(VI)$  is parametrically  $\alpha$ -well-posed in the generalized sense with respect to  $\sigma$ ,  $(u_n)_n$  has a subsequence  $(u_{n_k})_{n_k}$   $\sigma$ -converging to a solution  $u_0$  to  $(VI)(x_0)$ . Therefore, from condition i) in Definition 5.1 and in light of the



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lower semicontinuity of  $f$ , one has:

$$f(x_0, u_0) \leq \inf_{(x,u) \in X \times E, u \in T(x)} f(x, u),$$

that is  $(x_0, u_0)$  is a solution to **OPVIC**. □

**Theorem 5.6.** *Under the same assumptions of Theorem 5.5, if, moreover, **OPVIC** has a unique solution, then **OPVIC** is  $\alpha$ -well-posed with respect to  $(\tau \times \sigma)$ .*

*Proof.* Following the proof of the previous theorem, every  $\alpha$ -approximating sequence  $(x_n, u_n)_n$  for **OPVIC** has a subsequence which  $(\tau \times \sigma)$ -converges to the unique solution  $(x_0, u_0)$ . This is sufficient to conclude that the whole sequence  $(x_n, u_n)_n$   $(\tau \times \sigma)$ -converges to  $(x_0, u_0)$ . □

When the variational inequality arises from a minimization problem, **OPVIC** is nothing else than a bilevel optimization problem, also called strong Stackelberg problem (see [16]):

$$\inf_{x \in X} \inf_{u \in M(x)} f(x, u),$$

where

$$M(x) = \text{Argmin } h(x, \cdot) = \left\{ u \in H(x) : h(x, u) \leq \inf_{u' \in H(x)} h(x, u') \right\}.$$

**Theorem 5.7.** *Assume that  $f$  is sequentially lower semicontinuous, equicoercive on  $(X \times E, (\tau \times w))$  and **OPVIC** has a unique solution. If the following assumptions are satisfied:*



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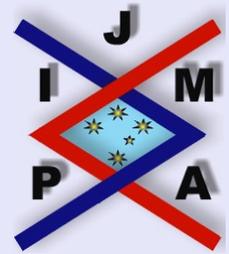
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- i) for every  $x \in X$ , the function  $h(x, \cdot)$  is lower semicontinuous, bounded from below, convex and Gâteaux differentiable on  $H(x)$ ;
- ii) the set-valued function  $H$  is  $w$ -closed,  $w$ -subcontinuous,  $s$ -lower semicontinuous, convex-valued and the range  $H(X)$  is a bounded subset of  $E$ ;
- iii) for every  $u \in E$ ,  $\frac{\partial h}{\partial u}(\cdot, u)$  is continuous on  $X$ ;
- iv) for every  $x \in X$ ,  $\frac{\partial h}{\partial u}(x, \cdot)$  is hemicontinuous on  $H(x)$ ;
- v)  $\frac{\partial h}{\partial u}$  is uniformly bounded on  $X \times E$ ;
- then **OPVIC** is  $\alpha$ -well-posed with respect to  $(\tau \times s)$ .




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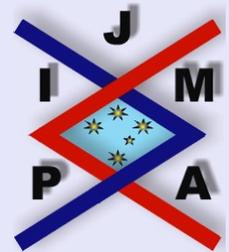
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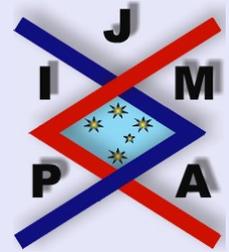
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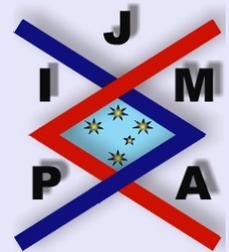
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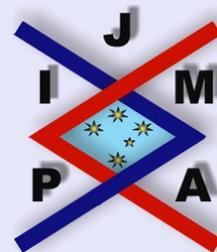
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