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# OPERATOR-SPLITTING METHODS FOR GENERAL MIXED VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we use the technique of updating the solution to suggest and analyze a class of new splitting methods for solving general variational inequalities. It is shown that these modified methods converge for pseudomonotone operators, which is a weaker condition than monotonicity. Our method includes the two-step forward-backward splitting and extragradient methods for solving various classes of variational inequalities and complementarity problems as special cases.

*Key words and phrases:* Variational inequalities, Resolvent operators, Iterative methods, Convergence, Fixed points.

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## 1. INTRODUCTION

Variational inequalities theory is a branch of mathematics with a wide range of applications in industrial, physical, regional, social, pure and applied sciences, see [1] – [18]. Variational inequalities have been extended and generalized in many different directions using new and novel techniques. A useful and significant generalization is called the general mixed variational inequality or variational inequality of the second type. In recent years, several numerical methods for solving variational inequalities have been developed. It is a well known fact that the projection method and its variant forms including the Wiener-Hopf equations cannot be extended for mixed variational inequalities involving the nonlinear terms. These facts motivated us to use the technique of the resolvent operators. In this technique, the given operator is decomposed into the sum of two (or more) monotone operators, whose resolvents are easier to evaluate then the resolvent of the original operator. Such type of methods are called the operators splitting methods. This can lead to the development of very efficient methods, since one can treat each part of the original operator independently. In the context of variational inequalities, Noor [9, 10] has used the resolvent operator technique to suggest and analyze some two-step forward-backward splitting methods. A useful feature of the forward-backward splitting methods for solving variational inequalities is that the resolvent step involves the subdifferential of

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the proper, convex and lower semicontinuous part only, the other part facilitate the problem decomposition. If the nonlinear term involving the general mixed variational inequalities is proper, convex and lower-semicontinuous, then it has been shown [8] – [10] that the general mixed variational inequalities are equivalent to the fixed point and resolvent equations. These alternative formulations have been used to develop a number of iterative type methods for solving mixed variational inequalities. Noor [9, 10] used the technique of updating the solution in conjunction with resolvent operator techniques to suggest a number of splitting type algorithms for various classes of variational inequalities. It has been shown [13] that the convergence of such type of splitting and predictor-corrector type algorithms requires the partially relaxed strongly monotonicity condition, which is weaker than cocoercivity. In this paper, we suggest and analyze a class of forward-backward splitting algorithms for a class of general mixed variational inequalities by modifying the associated fixed-point equation. The new splitting methods are self-adaptive type methods involving the line search strategy, where the step size depends upon the resolvent equation and the direction of searching is a combination of the resolvent residue and the modified extraresolvent direction. Our results include the previous results of Noor [9, 10], Wang et al. [18] and Han and Lo [4] for solving different classes of variational inequalities as special cases. Our results can be viewed as novel applications of the technique of updating the solution as well as a refinement and improvement of previously known results.

## 2. PRELIMINARIES

Let $H$ be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let $K$ be a nonempty closed convex set in $H$. Let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a function.

For given nonlinear operators $T, g : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, g(v) - g(u)\rangle + \varphi(g(v)) - \varphi(g(u)) \geq 0, \qquad \text{for all } g(v) \in H. \tag{2.1}$$

The inequality of type (2.1) is called the general mixed variational inequality or the general variational inequality of the second kind. If the function $\varphi(\cdot)$ is a proper, convex and lower semicontinuous function, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in Tu + \partial\varphi(g(u)),$$

which is known as the problem of finding a zero of the sum of two (maximal) monotone operators and has been studied extensively in recent years. We remark that if $g \equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \qquad \text{for all } v \in H, \tag{2.2}$$

which are called the mixed variational inequalities. It has been shown that a wide class of linear and nonlinear problems arising in finance, economics, circuit and network analysis, elasticity, optimization and operations research can be studied via the mixed variational inequalities (2.1) and (2.2). For the applications, numerical methods and formulations of general mixed variational inequalities, see [1] – [3], [7] – [10], [13], [17] and the references therein.

We note that if $\varphi$ is the indicator function of a closed convex set $K$ in $H$, that is,

$$\varphi(u) \equiv I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.1) is equivalent to finding $u \in H,\ g(u) \in K$ such that

$$\langle Tu, g(v) - g(u)\rangle \geq 0, \qquad \text{for all } g(v) \in K. \tag{2.3}$$

The inequality of the type (2.3) is known as the *general variational inequality,* which was introduced and studied by Noor [5] in 1988. It turned out that the odd-order and nonsymmetric free, unilateral, obstacle and equilibrium problems can be studied using the general variational inequality (2.3), see [5, 6, 12, 14, 15].

From now onward, we assume that the operator $g$ is onto $K$ and $g^{-1}$ exists unless otherwise specified.

If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \text{for all } v \in K\}$ is a polar cone of a convex cone $K$ in $H,$ then problem (2.3) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^*, \text{and} \quad \langle Tu, g(u)\rangle = 0, \tag{2.4}$$

which is known as the *general complementarity problem,* which was introduced and studied by Noor [5] in 1988. We note that if $g(u) = u - m(u),$ where $m$ is a point -to-point mapping, then problem (2.4) is called the quasi(implicit) complementarity problem, see the references for the formulation and numerical methods.

For $g \equiv I,$ the identity operator, problem (2.3) collapses to: find $u \in K$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \text{for all } v \in K, \tag{2.5}$$

which is called the standard variational inequality, introduced and studied by Stampacchia [16] in 1964. For recent results, see [1] – [18].

It is clear that problems (2.2) – (2.5) are special cases of the general mixed variational inequality (2.1). In brief, for a suitable and appropriate choice of the operators $T,$ $g,$ $\varphi$ and the space $H,$ one can obtain a wide class of variational inequalities and complementarity problems. This clearly shows that problem (2.1) is quite general and unifying. Furthermore, problem (2.1) has important applications in various branches of pure and applied sciences, see [1] – [18].

We now recall some well known concepts and results.

**Definition 2.1.** For all $u, v, z \in H$, an operator $T : H \to H$ is said to be:

**(i).** $g$*-monotone,* if

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq 0$$

**(ii).** $g$*-pseudomonotone,*

$$\langle Tu, g(v) - g(u)\rangle \geq 0 \quad \text{implies} \quad \langle Tv, g(v) - g(u)\rangle \geq 0.$$

For $g \equiv I,$ where $I$ is the identity operator, Definition 2.1 reduces to the classical definition of monotonicity and pseudomonotonicity. It is known that monotonicity implies pseudomonotonicity but the converse is not true, see [2]. Thus we conclude that the concept of pseudomonotonicity is weaker than monotonicity.

**Definition 2.2.** If $A$ is a maximal monotone operator on $H,$ then for a constant $\rho > 0,$ the resolvent operator associated with $A$ is defined as

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \text{for all } v \in H,$$

where $I$ is the identity operator. It is well known that the operator $A$ is maximal monotone if and only if the resolvent operator $J_A$ is defined everywhere on the space. The operator $J_A$ is single-valued and nonexpansive.

**Remark 2.1.** It is well known that the subdifferential $\partial\varphi$ of a proper, convex and lower semicontinuous function $\varphi : H \longrightarrow \mathbb{R} \cup \{\infty\}$ is a maximal monotone operator, so

$$J_\varphi(u) = (I + \partial\varphi)^{-1}(u), \quad \text{for all } u \in H,$$

is the resolvent operator associated with $\partial\varphi$ and is defined everywhere.

**Lemma 2.2.** *For a given $z \in H$, $u \in H$ satisfies*

$$\langle u - z, v - u \rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \textit{for all } v \in H \tag{2.6}$$

*if and only if*

$$u = J_\varphi z,$$

*where $J_\varphi$ is the resolvent operator.*

We remark that if the proper, convex and lower semicontinuous function $\varphi$ is an indicator function of a closed convex set $K$ in $H$, then $J_\varphi \equiv P_K$, the projection of $H$ onto $K$. In this case Lemma 2.2 is equivalent to the projection lemma, see [8].

Related to the general mixed variational inequality (2.1), we now consider the resolvent equations. Let $R_\varphi = I - J_\varphi$, where $J_\varphi$ is the resolvent operator. For given nonlinear operators $T, g : H \longrightarrow H$, we consider the problem of finding $z \in H$ such that

$$Tg^{-1}J_\varphi z + \rho^{-1}R_A z = 0. \tag{2.7}$$

Equation of the type (2.7) is called the resolvent equation, which was introduced and studied by Noor [8]. Note that if $\varphi(\cdot)$ is an indicator function of a closed convex set $K$ in $H,$ then the resolvent equations are equivalent to the Wiener-Hopf equation. For applications and numerical methods of the resolvent equations, see [8] – [10] and the references contained therein.

## 3. MAIN RESULTS

In this section, we use the resolvent equations technique to suggest a modified resolvent method for solving general mixed variational inequalities (2.1). For this purpose, we need the following result, which can be proved by using Lemma 2.2.

**Lemma 3.1.** *The general mixed variational inequality (2.1) has a solution $u \in H$ if and only if $u \in H$ satisfies*

$$g(u) = J_\varphi[g(u) - \rho Tu], \tag{3.1}$$

*where $J_\varphi(u) = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.*

Lemma 3.1 implies that problems (2.1) and (3.1) are equivalent. This alternative equivalent formulation has played an important part in suggesting several iterative methods for solving general mixed variational inequalities and related problems, see [8, 9].

We define the resolvent residue vector by

$$R(u) = g(u) - J_\varphi[g(u) - \rho Tu]. \tag{3.2}$$

Invoking Lemma 3.1, one can easily show that $u \in H$ is a solution of (2.1) if and only if $u \in H$ is a zero of the equation

$$R(u) = 0. \tag{3.3}$$

We also need the following known result, which can be proved by using Lemma 3.1, which shows that the general mixed variational inequality (2.1) is equivalent to the resolvent equation (2.7).

**Lemma 3.2.** [6] *The general mixed variational inequality (2.1) has a unique solution $u \in H$ if and only if $z \in H$ is a unique solution of the resolvent equation (2.7), where*

$$g(u) = J_\varphi z, \quad \textit{and} \quad z = g(u) - \rho Tu. \tag{3.4}$$

Using (3.2) and (3.4), the resolvent equation (2.7) can be written in the form:

$$R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(u) - \rho Tu] = 0. \tag{3.5}$$

Invoking Lemma 3.1, one can show that $u \in H$ is a solution of (2.1) if and only if $u \in H$ is a zero of the equation (3.5).

Using the technique of updating the solution, one can rewrite the equation (3.1) in the form;

$$g(u) = J_{\varphi}[J_{\varphi}[g(u) - \rho Tu] - \rho Tg^{-1}J_{\varphi}[g(u) - \rho Tu]], \tag{3.6}$$

or equivalently,

$$g(u) = J_{\varphi}[g(w) - \rho Tw], \tag{3.7}$$

$$g(w) = J_{\varphi}[g(u) - \rho Tu]. \tag{3.8}$$

Invoking Lemma 3.1, one can easily show that $u \in H$ is solution of equation (2.1) if and only if $u \in H$ is a zero of the equation

$$g(u) - J_{\varphi}[J_{\varphi}[g(u) - \rho Tu] - \rho Tg^{-1}J_{\varphi}[g(u) - \rho Tu]] = g(u) - J_{\varphi}[g(w) - \rho Tw] = 0. \tag{3.9}$$

The fixed-point formulation (3.6) – (3.8) can be used to suggest and analyze the following iterative methods for solving general variational inequalities (2.1).

**Algorithm 3.1.** For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative schemes

$$g(w_n) = J_{\varphi}[g(u_n) - \rho Tu_n],$$
$$g(u_{n+1}) = J_{\varphi}[g(w_n) - \rho Tw_n], \quad n = 0, 1, 2\ldots,$$

which is known as the predictor-corrector method, see Noor [9, 13].

**Algorithm 3.2.** For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative schemes

$$\begin{aligned} g(u_{n+1}) &= J_{\varphi}[J_{\varphi}[g(u_n) - \rho Tu_n] - \rho Tg^{-1}J_{\varphi}[g(u_n) - \rho Tu_n]] \\ &= J_{\varphi}[I - \rho Tg^{-1}]J_{\varphi}[I - \rho Tg^{-1}]g(u_n), \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is known as the two-step forward-backward splitting algorithm. Note that the order of $T$ and $J_{\varphi}$ has not been changed. For the convergence analysis of Algorithm 3.2, see Noor [9, 13].

By rearranging the terms, one can use the fixed-point formulation (3.6) to suggest and analyze the following method for solving the general mixed variational inequalities (2.1).

**Algorithm 3.3.** For a given $u_0 \in H$, compute $u_{n+1}$ by the iterative scheme

$$g(u_{n+1}) = (I + \rho Tg^{-1})^{-1}\{J_{\varphi}[I - \rho Tg^{-1}]J_{\varphi}[I - \rho Tg^{-1}] + \rho Tg^{-1}\}g(u_n), \quad n = 0, 1, 2\ldots,$$

which is again a two-step forward-backward splitting type method and is similar to that of Tseng [17]. Noor [13] has studied the convergence analysis of Algorithms 3.1 – 3.3 for the partially relaxed strongly monotone operator, which is a weaker condition than cocoercivity.

Using Lemma 3.2, we can rewrite the resolvent equation (2.7) in the following useful form:

$$D(u) = 0, \tag{3.10}$$

where

$$\begin{aligned} D(u) &= R(u) - \rho Tu + \rho Tg^{-1}J_{\varphi}[g(u) - \rho Tu] \\ &= R(u) - \rho Tu + \rho Tw. \end{aligned} \tag{3.11}$$

Invoking Lemma 3.1, one can show that $u \in H$ is a solution of (2.1) if and only if $u \in H$ is a zero of the equation (3.10).

In this paper , we suggest another method involving the line search strategy, which includes these splitting type methods as special cases. For a given positive constant $\alpha$, we rewrite the equation (3.6), using (3.2), in the following form.

$$\begin{aligned} g(u) &= J_\varphi[g(u) - \alpha\{g(u) - g(w) + \rho Tw\}] \\ &= J_\varphi[g(u) - \alpha\{R(u) + \rho Tw\}] \\ &= J_\varphi[g(u) - \alpha d(u)], \end{aligned} \tag{3.12}$$

where

$$d(u) = R(u) + \rho Tw \equiv R(u) + \rho Tg^{-1}[g(u) - \rho Tu]. \tag{3.13}$$

This fixed-point formulation enables us to suggest the following iterative method for general mixed variational inequalities (2.1).

**Algorithm 3.4.** For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative schemes.
**Predictor step.**

$$g(w_n) = J_\varphi[g(u_n) - \rho_n Tu_n], \tag{3.14}$$

where $\rho_n$ satisfies

$$\rho_n\langle Tu_n - Tw_n, R(u_n)\rangle \leq \sigma\|R(u_n)\|^2, \quad \sigma \in (0,1). \tag{3.15}$$

**Corrector step.**

$$g(u_{n+1}) = J_\varphi[g(u_n) - \alpha_n d(u_n)], \quad n = 0, 1, 2\ldots \tag{3.16}$$

where

$$d(u_n) = R(u_n) + \rho_n Tw_n, \tag{3.17}$$

$$\alpha_n = \frac{\langle R(u_n), D(u_n)\rangle}{\|d(u_n)\|^2}, \tag{3.18}$$

$$D(u_n) = R(u_n) - \rho Tu_n + \rho Tw_n, \tag{3.19}$$

where $\alpha_n$ is the corrector step size. Note that the corrector step size $\alpha_n$, in (3.18) depend upon the resolvent equation (3.10).

If the proper, convex and lower-semicontinuous function $\varphi$ is an indicator function of a closed convex set $K$ in $H$,, then $J_\varphi \equiv P_K$, the projection of $H$ onto $K$ and consequently Algorithm 3.4 collapses to:

**Algorithm 3.5.** For a given $u_0 \in H$, compute the approximate solution $u_{n+1}$ by the iterative schemes
**Predictor step.**

$$g(w_n) = P_K[g(u_n) - \rho_n Tu_n],$$

where $\rho_n$ satisfies

$$\rho_n\langle Tu_n - Tw_n, R(u_n)\rangle \leq \sigma\|R(u_n)\|^2, \quad \sigma \in (0,1).$$

**Corrector step.**

$$g(u_{n+1}) = P_K[g(u_n) - \alpha_n d_1(u_n)], \quad n = 0, 1, 2\ldots$$

where

$$d_1(u_n) = R(u_n) + \rho_n Tw_n$$
$$\alpha_n = \frac{\langle R(u_n), D_1(u_n)\rangle}{\|d_1(u_n)\|^2}$$
$$D_1(u_n) = R(u_n) - \rho Tu_n + \rho Tw_n.$$

Algorithm 3.4 appears to be a new one even for general variational inequalities (2.3). Note that for $\alpha_n = 1,$ Algorithm 3.4 is exactly Algorithm 3.1, which is mainly due to Noor [9, 10]. For $g \equiv I$, the identity operator, we obtain new improved versions of previously known algorithms of Wang et al. [18] and Han and Lo [4] for variational inequalities and related optimization problems. This clearly shows that Algorithm 3.4 is a unifying one and includes several known and new algorithms as special cases.

For the convergence analysis of Algorithm 3.4, we need the following results.

**Lemma 3.3.** *If $\bar{u} \in H$ is a solution of (2.1) and $T$ is $g$-pseudomonotone, then*

(3.20) $$\langle g(u) - g(\bar{u}), d(u)\rangle \geq (1-\sigma)\|R(u)\|^2, \quad \textit{for all } u \in H.$$

*Proof.* Let $\bar{u} \in H$ be a solution of (2.1). Then

$$\langle T\bar{u}, g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi g(\bar{u}) \geq 0, \quad \text{for all } v \in H,$$

which implies

(3.21) $$\langle Tv, g(v) - g(\bar{u})\rangle + \varphi(g(v)) - \varphi g(\bar{u}) \geq 0,$$

since $T$ is $g$-pseudomonotone.

Taking $g(v) = J_\varphi[g(u) - \rho Tu] = g(w),$ (where $g(w)$ is defined by (3.8)) in (3.21), we have

$$\langle Tw, g(w) - g(\bar{u})\rangle + \varphi(g(w)) - \varphi(g(\bar{u})) \geq 0,$$

from which, we have

(3.22) $$\langle g(u) - g(\bar{u}), \rho Tw\rangle \geq \rho\langle R(u), Tw\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi(g(w)).$$

Setting $u = g(w), z = g(u) - \rho Tu$ and $v = g(\bar{u})$ in (2.6), we have

$$\langle g(w) - g(u) + \rho Tu, g(\bar{u}) - g(w)\rangle + \rho\varphi(g(\bar{u})) - \rho\varphi(g(w)) \geq 0,$$

from which, we obtain

$$\begin{aligned}\langle g(u) - g(\bar{u}), R(u)\rangle &\geq \langle R(u), R(u) - \rho Tu\rangle - \rho\varphi(g(\bar{u})) + \rho\varphi(g(w)) \\ &\quad + \rho\langle Tu, g(u) - g(\bar{u})\rangle. \\ &\geq \langle R(u), R(u) - \rho Tu\rangle - \rho\varphi(g(\bar{u})) + \rho\varphi(g(w)),\end{aligned} \tag{3.23}$$

where we have used the fact that the operator $T$ is pseudomonotone.

Adding (3.22) and (3.23), we have

$$\begin{aligned}\langle g(u) - g(\bar{u}), R(u) + \rho Tw\rangle &= \langle g(u) - g(\bar{u}), d(u)\rangle \\ &\geq \langle R(u), D(u)\rangle \qquad (3.24) \\ &= \langle R(u), R(u) - \rho Tu + \rho Tw\rangle \\ &\geq \|R(u)\|^2 - \rho\langle R(u), Tu - Tw\rangle \\ &\geq (1-\sigma)\|R(u)\|^2, \quad \text{using (3.15)}, \qquad (3.25)\end{aligned}$$

the required result. □

**Lemma 3.4.** *Let $\bar{u} \in H$ be a solution of (2.1) and let $u_{n+1}$ be the approximate solution obtained from Algorithm 3.4. Then*

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2 - \frac{(1-\sigma)^2\|R(u)\|^4}{\|d(u_n)||^2}. \tag{3.26}$$

*Proof.* From (3.16), (3.20) and (3.24), we have

$$\begin{aligned}
\|g(u_{n+1}) - g(\bar{u})\|^2 &\leq \|g(u_n) - g(\bar{u}) - \alpha_n d(u_n)\|^2 \\
&\leq \|g(u_n) - g(\bar{u})\|^2 - 2\alpha_n \langle g(u) - g(\bar{u}), d(u_n)\rangle \\
&\qquad + \alpha_n^2 \|d(u_n)\|^2 \\
&\leq \|g(u_n) - g(\bar{u})\|^2 - \alpha_n \langle R(u_n), D(u_n)\rangle \\
&\leq \|g(u_n) - g(\bar{u})\|^2 - \alpha_n (1-\sigma)\|R(u_n)\|^2 \\
&\leq \|g(u_n) - g(\bar{u})\|^2 - \frac{(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2},
\end{aligned}$$

the required result. □

**Theorem 3.5.** *Let $g : H \longrightarrow H$ be invertible and let $H$ be a finite dimensional space. If $u_{n+1}$ is the approximate solution obtained from Algorithm 3.4 and $\bar{u} \in H$ is a solution of (2.1), then* $\lim_{n\to\infty} u_n = \bar{u}$.

*Proof.* Let $\bar{u} \in H$ be a solution of (2.1). From (3.25), it follows that the sequence $\{||g(\bar{u}) - g(u_n)||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty} \frac{(1-\sigma)^2\|R(u_n)\|^4}{\|d(u_n)\|^2} \leq ||g(u_0) - g(\bar{u})||^2,$$

which implies that

$$\lim_{n\to\infty} R(u_n) = 0. \tag{3.27}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Since $R(u)$ is continuous, so

$$R(\hat{u}) = \lim_{j\to\infty} R(u_{n_j}) = 0,$$

which implies that $\hat{u}$ solves the general mixed variational inequality (2.1) by invoking Lemma 3.1. From (3.26), it follows that

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since $g$ is invertible, then

$$\lim_{n\to\infty} (u_n) = \hat{u},$$

the required result. □

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