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A NOTE ON ACZÉL TYPE INEQUALITIES

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ABSTRACT. The main result here is a simple general-purpose numerical inequality that can be used to produce a variety of Aczél type inequalities with little effort.

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While reading Dragomir and Mond's paper [3], I noticed that a basic and elementary inequality principle could have been the essential tool for deriving not only all the inequalities proved there (I am referring to the case $p = q = 2$ of Theorem 2 below), but even most of the classical ones mentioned in their introduction. Since the same idea can be used to give immediate proofs of a large variety of Aczél type inequalities (including the classical Aczél Inequality — see Corollary 3, case $p = q = 2$), I thought it worthwhile to present it here in more detail, together with some examples that hopefully will show the implicit power of the tool.

Our first lemma is a simple consequence of Hölder's inequality, and so we state it without proof:

Lemma 1. *Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and $r, s, h, k \geq 0$. Then*

$$h^{1/p}k^{1/q} \leq (r+h)^{1/p}(s+k)^{1/q} - r^{1/p}s^{1/q}\,, \tag{1}$$

with equality if and only if $rk = sh$.

A direct corollary of this and our main tool to prove Aczél type inequalities is then

Theorem 2. *Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and $a, b, c, \alpha, \beta, \gamma \geq 0$ satisfy $b \geq a^{1/p}c^{1/q}$ and $\alpha^{1/p}\gamma^{1/q} \geq \beta$. Then, if $a \geq \alpha$ and $c \geq \gamma$ we have*

$$(a-\alpha)^{1/p}(c-\gamma)^{1/q} \leq b - \beta\,, \tag{2}$$

with equality if and only if $c\alpha = a\gamma$, $b = a^{1/p}c^{1/q}$ and $\beta := \alpha^{1/p}\gamma^{1/q}$.

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Proof. To see this, note that by the assumptions we have $b \geq \beta$, and this means that the right hand side in (2) is smallest by choosing $b := a^{1/p}c^{1/q}$ and $\beta := \alpha^{1/p}\gamma^{1/q}$: but then (2) reduces to

$$(a-\alpha)^{1/p}(c-\gamma)^{1/q} \leq a^{1/p}c^{1/q} - \alpha^{1/p}\gamma^{1/q}\,,$$

and this is exactly the inequality (1) in Lemma 1 if we set there $r := \alpha$, $s := \gamma$, $h := a - \alpha$ and $k := c - \gamma$. The identity condition follows immediately from the proof and from Lemma 1. □

Let us now see how to easily derive the classical Aczél Inequality [1] in a generalized form due to Popoviciu [7] (see also the discussion after Theorem 5 below, and [6, p. 118]):

Corollary 3 (Popoviciu [7])**.** *Let $p, q \geq 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, and $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ be two sequences of positive real numbers such that*

$$a_1^p - a_2^p - \ldots - a_n^p > 0 \ \text{ and } \ b_1^q - b_2^q - \ldots - b_n^q > 0\,. \tag{3}$$

Then

$$(a_1^p - a_2^p - \ldots - a_n^p)^{1/p}(b_1^q - b_2^q - \ldots - b_n^q)^{1/q} \leq a_1b_1 - a_2b_2 - \ldots - a_nb_n\,. \tag{4}$$

with equality if and only if the sequences are proportional (the classical Aczél Inequality is the special case $p = q = 2$).

Proof. Let $\vec{a}$ and $\vec{b}$ be as in the statement, and apply Theorem 2 with $a := a_1^p$, $b := a_1b_1$, $c := b_1^q$, $\alpha := a_2^p + \ldots + a_n^p$, $\beta := a_2b_2 + \ldots + a_nb_n$, $\gamma := b_2^q + \ldots + b_n^q$. Then the hypotheses $b \geq a^{1/p}c^{1/q}$ and $\alpha^{1/p}\gamma^{1/q} \geq \beta$ are easily verified (the second one is the classical Hölder inequality), and so we derive that (2) must hold for this choice of parameters, and that takes exactly the form of inequality (4).

As for the equality condition, from Theorem 2 we get the identities

$$\begin{gathered} b_1^p(a_2^p + \ldots + a_n^p) = a_1^p(b_2^p + \ldots + b_n^p) \\ (a_2b_2 + \ldots + a_nb_n)^p = (a_2^p + \ldots + a_n^p)(b_2^p + \ldots + b_n^p)\,. \end{gathered}$$

The second identity says that we have equality in the Hölder inequality for the $(n-1)$-tuples $a_2, \ldots, a_n$ and $b_2, \ldots, b_n$, which means that these must be proportional (see [5, p. 50]). The first identity then tells us that to have equality in (4) the two full sequences $\vec{a}$ and $\vec{b}$ must be proportional. □

This may be a good moment to notice that in a sense Corollary 3 is the "right" generalization of Aczél's inequality, as opposed to the other, more restrictive Popoviciu's generalization (see Corollary 6 below). In fact, the latter is occasionally quoted with a mistake in its statement and is really only valid for a small range of exponents (see Corollary 6 and the remark following it). Mostly for the sake of illustration of the flexibility of our method, here are a variant Lemma and Theorem that should be compared to Lemma 1 and Theorem 2:

Lemma 4. *Let $1 \leq p \leq 2$ and $r, s, h, k \geq 0$. Then*

$$h^{1/p}k^{1/p} \leq (r+h)^{1/p}(s+k)^{1/p} - r^{1/p}s^{1/p}\,, \tag{5}$$

with equality if and only if $rk = sh$.

Proof. To see this, let $2 \leq q < \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and proceed just as in Lemma 1 using Hölder's Inequality to get

$$h^{1/p}k^{1/p} + r^{1/p}s^{1/p} \leq (r+h)^{1/p}(s^{q/p} + k^{q/p})^{1/q}\,.$$

Now, by an inequality due to Jensen [2, p.18] and since $q \geq p$, we have

$$(s^{q/p} + k^{q/p})^{1/q} \leq (s^{p/p} + k^{p/p})^{1/p} = (s+k)^{1/p}$$

which readily gives the claim. We leave verification of the equality condition to the reader. □

The following Theorem is now derived from Lemma 4 exactly in the way as Theorem 2 was derived from Lemma 1 (we omit the proof):

Theorem 5. *Let* $1 \le p \le 2$ *and* $a, b, c, \alpha, \beta, \gamma \ge 0$ *satisfy* $b \ge a^{1/p} c^{1/p}$ *and* $\alpha^{1/p}\gamma^{1/p} \ge \beta$. *Then, if* $a \ge \alpha$ *and* $c \ge \gamma$ *we have*

$$(a-\alpha)^{1/p}(c-\gamma)^{1/p} \le b - \beta\,, \tag{6}$$

with equality if and only if $c\alpha = a\gamma$, $b = a^{1/p}c^{1/p}$ *and* $\beta := \alpha^{1/p}\gamma^{1/p}$.

We can now state and easily prove another generalization of Aczél's Inequality that is also attributed to Popoviciu in [5]. Note that the quotation of this result in [5] mistakenly states that the following is true for all $p \ge 1$:

Corollary 6 (Popoviciu [7])**.** *Let* $1 \le p \le 2$ *and* $\vec{a} = (a_1, \ldots, a_n)$, $\vec{b} = (b_1, \ldots, b_n)$ *be two sequences of positive real numbers such that*

$$a_1^p - a_2^p - \ldots - a_n^p > 0 \ \text{ and } \ b_1^p - b_2^p - \ldots - b_n^p > 0\,. \tag{7}$$

Then

$$(a_1^p - a_2^p - \ldots - a_n^p)^{1/p}(b_1^p - b_2^p - \ldots - b_n^p)^{1/p} \le a_1b_1 - a_2b_2 - \ldots - a_nb_n\,. \tag{8}$$

with equality if and only if the sequences are proportional (again, the classical Aczél Inequality is the special case $p = 2$*, and here the* a_j *and* b_j *don't need to be positive).*

To quickly see why inequality (8) cannot be always true for $p > 2$, just consider the following special case where we assume $a_1 = b_1 = 1$ and $a_2 = b_2 = a \in (0, 1)$:

$$(1-a^p)^{2/p} = (1-a^p)^{1/p}(1-a^p)^{1/p} \le 1 - a^2$$

which is always false if $p > 2$.

Next, as an example of the applications of Theorem 2 to vector spaces, we have the following trivial consequence:

Corollary 7. *Let* u, v, w, z *be vectors in a complex Hilbert space such that* $\|u\| \ge \|w\|$ *and* $\|v\| \ge \|z\|$. *Then we have*

$$(\|u\|^2 - \|w\|^2)(\|v\|^2 - \|z\|^2) \le (\|u\|\|v\| - |\langle w, z\rangle|)^2 \tag{9}$$

Proof. Just use Theorem 2 with $p = q = 2$ and define $a := \|u\|^2$, $b := \|u\|\|v\|$, $c := \|v\|^2$, $\alpha := \|w\|^2$, $\beta := |\langle w, z\rangle|$, $\gamma := \|z\|^2$, and notice that the hypothesis $\alpha^{1/2}\gamma^{1/2} \ge \beta$ is just the familiar Cauchy-Schwarz inequality. □

As harmless as the previous result may seem, note how it can give a trivial proof to inequalities that sometimes are presented in a way that makes them look much more difficult than they are. As an example we prove the complex version of an old inequality stated in the real case by Kurepa [4] (also quoted in [6, p. 602] and [3]):

Corollary 8 (Kurepa [4])**.** *Let* u, v, w *be vectors in a complex Hilbert space. Define* $u_0 := u - \langle u, w\rangle w$ *and* $v_0 := v - \langle v, w\rangle w$. *If we have* $\|w\| = 1$,

$$|\langle u, w\rangle| \ge \|u_0\|$$

and

$$|\langle v, w\rangle| \ge \|v_0\|\ ,$$

then we also have

$$\left(|\langle u, w\rangle|^2 - \|u_0\|^2\right)\left(|\langle v, w\rangle|^2 - \|v_0\|^2\right) \le \left(|\langle u, w\rangle|\,|\langle v, w\rangle| - |\langle u_0, v_0\rangle|\right)^2\ .$$

Proof. To prove this using Corollary 7 define $u_1 := \langle u, w\rangle w$ and $v_1 := \langle v, w\rangle w$, to note that $u = u_0 + u_1$ and $v = v_0 + v_1$ are just the orthogonal decompositions with respect to the direction of w (since $\|w\| = 1$). Given this, just observe that $\|u_1\| = |\langle u, w\rangle|$, $\|v_1\| = |\langle v, w\rangle|$, and so u_1, u_0, v_1, v_0 satisfy the hypotheses in Corollary 7 of (respectively) u, w, v, z. □

To any reader who grasped how immediate Lemma 1 and Theorem 2 are, it is probably not going to be a surprise that a similarly easy proof (which we omit) can be given to the following generalization of Theorem 2:

Theorem 9. *Let $p_1, \ldots, p_n \geq 1$ be such that $\sum_j \frac{1}{p_j} = 1$, and $a_1, \ldots, a_n$, $\alpha_1, \ldots, \alpha_n$, b and β be all non-negative numbers that satisfy $b \geq \prod_j a_j^{1/p_j}$ and $\prod_j \alpha_j^{1/p_j} \geq \beta$. Then, if $a_j \geq \alpha_j$ for all j, we have*

$$\prod_j (a_j - \alpha_j)^{1/p_j} \leq b - \beta\,, \tag{10}$$

with equality if and only if $a_i\alpha_j = a_j\alpha_i$ for all $i, j \in \{1, 2, \ldots, n\}$, and $b = \prod_j a_j^{1/p_j}$ and $\beta := \prod_j \alpha_j^{1/p_j}$.

As an example for the many possible applications (we omit the obvious extension of Corollary 3 to more than two sequences), let me give a companion to Dragomir and Mond's Theorem 4 in [3]. Let H be a complex Hilbert space, and $x_1, \ldots, x_n$ be arbitrary vectors in H. Define the Gramian of these vectors as $\Gamma(x) := \det[\langle x_i, x_j\rangle]_{ij}$ (recall that $\Gamma(x)$ is always non-negative). We have the following:

Corollary 10. *Let $a_1, \ldots, a_n$ be non-negative numbers. Then, for every n-tuple $x_1, \ldots, x_n$ of vectors in a complex Hilbert space such that*

$$\|x_j\| \leq a_j \qquad \forall\, j = 1, \ldots, n$$

we have the inequality

$$\prod_j\, (a_j - \|x_j\|) \leq \left[\left(\prod_j a_j \right)^{1/n} - \Gamma(x)^{1/(2n)} \right]^n . \tag{11}$$

Proof. All we need to see is that we let $p_j = n$ in Theorem 9, and set $b := (\prod_j a_j)^{1/n}$, $\alpha_j := \|x_j\|$ and $\beta := \Gamma(x)^{1/(2n)}$ there. Then, the hypothesis $\prod_j \alpha_j^{1/p_j} \geq \beta$ translates into

$$\prod_j \|x_j\| \geq \Gamma(x)^{1/2}$$

which is the well-known Hadamard Inequality for Gramians [6, p. 597]. □

REFERENCES

[1] J. ACZÉL, Some general methods in the theory of functional equations in one variable, New applications of functional equations (Russian), *Uspehi Mat. Nauk (N.S.)* 11, **69**(3) (1956), 3–68.

[2] E.E. BECKENBACH AND R. BELLMAN, *Inequalities*, 4^{th} printing, Springer-Verlag, 1983.

[3] S.S. DRAGOMIR AND B. MOND, Some inequalities of Aczél type for Gramians in inner product spaces, *Nonlinear Funct. Anal. & Appl.*, **6** (2001), 411–424.

[4] S. KUREPA, On the Buniakowski-Cauchy-Schwarz inequality, *Glasnik Mat.* Ser. III, **21**(1) (1966) 147–158.

[5] D.S. MITRINOVIĆ, *Analytic Inequalities*, Springer-Verlag, 1970.

[6] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer, 1993.

[7] T. POPOVICIU, Sur quelques inégalités, *Gaz. Mat. Fiz.* Ser. A, **11** (64) (1959) 451–461.