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A MONOTONICITY PROPERTY OF RATIOS OF SYMMETRIC HOMOGENEOUS MEANS

PETER A. HÄSTÖ

DEPARTMENT OF MATHEMATICS,
P.O. BOX 4,
00014 UNIVERSITY OF HELSINKI.
peter.hasto@helsinki.fi
URL: http://www.helsinki.fi/~hasto/

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ABSTRACT. We study a certain monotonicity property of ratios of means, which we call a strong inequality. These strong inequalities were recently shown to be related to the so-called relative metric. We also use the strong inequalities to derive new ordinary inequalities. The means studied are the extended mean value of Stolarsky, Gini's mean and Seiffert's mean.

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1. INTRODUCTION AND MAIN RESULTS

In this paper we study a certain monotonicity property of ratios of symmetric homogeneous means of two variables. In this setting the monotonicity property can be interpreted as a strong version of an inequality. The means considered are the extended mean value of Stolarsky [18], Gini's mean [6] and Seiffert's mean [15].

These kind of strong inequalities were shown in [7] to provide sufficient conditions for the so-called relative distance to be a metric. This aspect is described in Section 7, which also contains the new relative metrics found in this paper. A question by H. Alzer on whether the results of [7], specifically Lemma 4.2, could be generalized was the main incentive for the present paper. Another motivation for this work was that monotonicity properties of ratios have been found useful in several studies related to gamma and polygamma functions, see for instance [5], [10], [1] and [2]. Such inequalities have also been used, implicitly, in studying means by M. Vamanamurthy and M. Vuorinen in the paper [20], an aspect further exposed in Section 2.2.

Let us next introduce some terminology in order to state the main results. Denote $\mathbb{R}^> := (0,\infty)$ and let $f, g\colon [1,\infty) \to \mathbb{R}^>$ be arbitrary functions. We say that f is *strongly greater than*

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or equal to g, in symbols $f \succeq g$, if $x \mapsto f(x)/g(x)$ is increasing. By a *symmetric homogeneous increasing mean* (of two variables) we understand a symmetric function $M\colon \mathbb{R}^> \times \mathbb{R}^> \to \mathbb{R}^>$ which satisfies

$$\min\{x, y\} \le M(x, y) \le \max\{x, y\}$$

and $M(sx, sy) = sM(x, y)$ for all $s, x, y \in \mathbb{R}^>$ and for which $t_M(x) := M(x, 1)$ is increasing for $x \in [1, \infty)$. The function t_M is called the *trace* of M and uniquely determines M, since $M(x, y) = yt_M(x/y)$. If M and N are symmetric homogeneous increasing means we say that M is strongly greater than or equal to N, $M \succeq N$, if $t_M \succeq t_N$.

Let us next introduce the means that will be considered in this paper. The extended mean value, $E_{s,t}$, was first considered by Stolarsky in [18] and later by Leach and Scholander, [11], who gave several basic properties of the mean. It is defined for distinct $x, y \in \mathbb{R}^>$ and distinct $s, t \in \mathbb{R} \setminus \{0\}$ by

$$E_{s,t}(x, y) := \left(\frac{t}{s} \frac{x^s - y^s}{x^t - y^t} \right)^{1/(s-t)}$$

and $E_{s,t}(x, x) := x$. The extended mean value is defined for the parameter values $s = 0$ and $s = t$ by continuous extension, see Section 3.2. Let us also define the power means by $A_s := E_{2s,s}$, see also Section 3.1.

In the paper [12] Leach and Scholander provided a complete description of the values $s, t, p, q \in \mathbb{R}$ for which $E_{s,t} \ge E_{p,q}$. The next theorem is the corresponding result for strong inequalities. Notice that this result is a generalization of [7, Lemma 4.2], which in turn is the strong version of Pittenger's inequality, see [14]. We also state a corollary containing the ordinary inequalities implied by the theorem.

Theorem 1.1. *Let* $s, t, p, q \in \mathbb{R}^+ := [0, \infty)$. *Then* $E_{s,t} \succeq E_{p,q}$ *if and only if* $s + t \ge p + q$ *and* $\min\{s, t\} \ge \min\{p, q\}$.

Corollary 1.2. *Let* $s, t, p, q \in \mathbb{R}^>$, $s > t$ *and* $p > q$. *If* $p + q \ge s + t$ *and* $t \ge q$ *then*

$$E_{s,t} \le E_{p,q} \le (q/p)^{1/(p-q)} (s/t)^{1/(s-t)} E_{s,t}.$$

Both inequalities are sharp.

Remark 1.3. Let M and N be symmetric homogeneous increasing means. The inequality $M \le N$ is understood to mean that the real value inequality $M(x, y) \le N(x, y)$ holds for all $x, y \in \mathbb{R}^>$. The inequality $M \le cN$ is said to be *sharp* if the constant cannot be replaced by a smaller one. Notice that this does not necessarily mean that the inequality cannot be improved, for instance the previous one could possibly be replaced by $M \le cN - \log\{1 + N\}$.

Remark 1.4. The first inequality in the previous corollary follows directly from the result of Leach and Scholander, and is not as good (in terms of the assumptions on p, q, s and t). The upper bound does not follow from their result, however.

The Gini mean was introduced in [6] as a generalization of the power means. It is defined by

$$G_{s,t}(x, y) := \left(\frac{x^s + y^s}{x^t + y^t} \right)^{1/(s-t)}$$

for $x, y \in \mathbb{R}^>$ and distinct $s, t \in \mathbb{R}$. Like the extended mean value, the Gini mean is continuously extended to $s = t$, see Section 3.3.

The Gini means turn out to be less well behaved than the other means that we consider in terms of strong inequalities. We give here two main results on inequalities of Gini means, however, the reader may also want to view the summary of results presented in Section 5.3. The following theorem gives a sufficient condition for the Gini means to be strongly greater than or equal to an extended mean value and is also a generalization of [7, Lemma 4.2].

Theorem 1.5. *Let $a, p, q \in \mathbb{R}^+$. Then $G_{s,t} \succeq E_{p,q}$ for all $s, t \geq 0$ with $s + t = a$ if and only if $p + q \leq 3a$ and $\min\{p, q\} \leq a$.*

If the parameters of the Gini mean are of similar magnitude then we are able to give a characterization of the extended mean values that are strongly less than the Gini mean:

Theorem 1.6. *Let $s, t \in \mathbb{R}^>$ with $1/3 \leq s/t \leq 3$ and $p, q \in \mathbb{R}^+$. Then $G_{s,t} \succeq E_{p,q}$ if and only if $p + q \leq 3(s + t)$*

Again we have a corollary of ordinary inequalities:

Corollary 1.7. *Let $s, t, p, q \in \mathbb{R}^>$, $p > q$ and $p+q \leq 3(s+t)$. Assume also that $1/3 \leq s/t \leq 3$ or $q \leq s + t$. Then*

$$E_{p,q} \leq G_{s,t} \leq (p/q)^{1/(p-q)} E_{p,q}.$$

Both inequalities are sharp.

Remark 1.8. Contrary to the corollaries of the other theorems, this one provides, to the best knowledge of the author, new inequalities.

The Seiffert mean was introduced in [15] and is defined by

$$P(x, y) := \frac{x - y}{4 \arctan(\sqrt{x/y}) - \pi}$$

for distinct $x, y \in \mathbb{R}^>$ and $P(x, x) := x$. The next theorem provides a characterization of those Stolarsky means which are strongly less than the Seiffert mean. Notice that the Stolarsky mean is of particular interest to us, since it has been implicated in finding relative metrics, as is described in Section 7.

Theorem 1.9. *Denote $S_\alpha := E_{1,1-\alpha}$ for $0 < \alpha \leq 1$. Then $P \succeq S_\alpha$ if and only if $\alpha \geq 1/2$.*

Remark 1.10. We will call $S_\alpha = E_{1,1-\alpha}$ *Stolarsky means* following [20] and [7], since this particular form of the extended mean value was studied in depth by Stolarsky in [19] and call the family $E_{s,t}$ extended mean values, even though they too originated from [18] by Stolarsky.

The previous theorem has the following corollary containing the corresponding ordinary inequalities.

Corollary 1.11. *If $1/2 \leq \alpha \leq 1$ then*

$$S_\alpha \leq P \leq \frac{1}{\pi}(1 - \alpha)^{-1/\alpha} S_\alpha.$$

Both inequalities are sharp.

Remark 1.12. In the previous corollary the lower bound is decreasing and the upper bound is increasing in α (for any fixed x). Hence the best estimate for P given by the previous corollary is

$$\frac{(\sqrt{x} + \sqrt{y})^2}{4} \leq P(x, y) \leq \frac{(\sqrt{x} + \sqrt{y})^2}{\pi},$$

since $S_{1/2} = A_{1/2}$. Notice also that the first of these inequalities was given by A. A. Jager in [15] in order to solve H.-J. Seiffert's problem $E_{0,1} \leq P \leq E_{1,1}$. Once again however, the upper bound is new. For another inequality of P, see Corollary 6.4.

The structure of the rest of this paper is as follows: in the next section we state some basic properties of strong inequalities and show how the corollaries in this section follow from their respective theorems. In Section 3 we present the complete definition of the means studied as well as some simple results on their derivatives. Section 4 contains the complete characterization of strong inequalities between extended mean values, that is the proof of Theorem 1.1. In Section 5 we present the proofs of Theorems 1.5 and 1.6, relating extended mean values and Gini means as well as some additional results summarized in Section 5.3. Section 6 contains the characterizations of strong inequalities between Seiffert's mean and the Stolarsky means. In Section 7 we present a brief summary of the result regarding relative metrics from [7] and show how the theorems of this paper yield new families of metrics.

2. STRONG INEQUALITIES

In this section we will consider some basic properties of strong inequalities and show how the corollaries stated in the introduction are derived from their respective theorems.

2.1. Basic Properties of Strong Inequalities.

Recall from the introduction that we say that f is strongly greater than or equal to g, $f \succeq g$, if $x \mapsto f(x)/g(x)$ is increasing, where $f, g\colon [1,\infty) \to \mathbb{R}^>$ are arbitrary functions. The relation $f \preceq g$ is defined to hold if and only if $g \succeq f$. The following lemma follows immediately from the definition since x^s is increasing if and only if x is increasing, for $s > 0$.

Lemma 2.1. *Let $f, g\colon [1,\infty) \to \mathbb{R}^>$ be arbitrary functions and $s > 0$. Define $f_s(x) := f(x^s)$ and $g_s(x) := g(x^s)$. Then following conditions are equivalent:*

(1) $f \succeq g$,
(2) $f_s \succeq g_s$ *and*
(3) $f^s \succeq g^s$.

Suppose next that $f, g\colon [1,\infty) \to \mathbb{R}^>$ are differentiable functions. Then $f \succeq g$ if and only if $d(f/g)/dx \geq 0$ if and only if

$$0 \leq \frac{d\log\{f/g\}}{dx} = \frac{d\log f}{dx} - \frac{d\log g}{dx}.$$

We see that in this situation the strong inequality is equivalent to an ordinary inequality between the logarithmic derivatives.

We end this subsection by showing that $\succeq$ is a partial order, as is suggested by its symbol. A binary relation $\trianglelefteq \subset X \times X$ is called a *partial order* in the set X if

(1) $x \trianglelefteq x$ for all $x \in X$ (reflexivity),
(2) if $x \trianglelefteq y$ and $y \trianglelefteq x$ then $x = y$ (antisymmetry) and
(3) if $x \trianglelefteq y$ and $y \trianglelefteq z$ then $x \trianglelefteq z$ (transitivity). [17, Section 3.1].

Let $f, g, h\colon [1,\infty) \to \mathbb{R}^>$ be arbitrary functions. Then $f \succeq f$, since $f/f = 1$ is increasing, hence the property of reflexivity is satisfied. If $f \succeq g$ and $g \succeq h$ then f/g and g/h are increasing, hence so is their product, f/h, which means that $f \succeq h$, hence $\succeq$ is transitive. The antisymmetry condition is not quite satisfied, though – if $f = cg$ with $c > 1$ then $f \succeq g$ and $g \succeq f$ but $f \neq g$. One easily sees that the antisymmetry condition holds in the set of symmetric homogeneous means, hence $\succeq$ is a partial order in this set, which is the one that will concern us in what follows.

2.2. Ordinary Inequalities from Strong Inequalities.

In this section we will see how strong inequalities imply ordinary inequalities. The method to be presented has been used in the context of gamma and polygamma functions by several investigators, as noted in the introduction and by M. Vamanamurthy and M. Vuorinen ([20]) in the context of means.

If M and N are symmetric homogeneous means then $t_M(1) = t_N(1) = 1$. Hence, if $M \succeq N$ then

$$t_M(x)/t_N(x) \geq t_M(1)/t_N(1) = 1$$

for $x \geq 1$. To get an upper bound we observe that if $t_M(x)/t_N(x)$ is increasing on $[1,\infty)$ then

$$\frac{t_M(x)}{t_N(x)} \leq \lim_{x\to\infty} \frac{t_M(x)}{t_N(x)} =: c,$$

and so $t_M(x) \le ct_N(x)$. Notice also that the constant in neither of the two inequalities can be improved. Since both M and N were assumed to be homogeneous, the previous inequalities imply that

$$N(x,y) = yt_N(x/y) \le yt_M(x/y) = M(x,y) \le cyt_N(x/y) = cN(x,y),$$

where $x, y \in \mathbb{R}^>$. Notice in particular that the relation $\succeq$ implies the relation $\ge$, which is the reason for the terminology "strong inequality".

Applying this reasoning to the Theorems 1.1, 1.9 and 1.5 and 1.6 gives the Corollaries 1.2, 1.11 and 1.7, respectively, since

$$E_{s,t}(x,1) \sim (s/t)^{1/(s-t)}x,\ G_{s,t}(x,1) \sim x \text{ and } P(x,1) \sim 2x/\pi$$

as $x \to \infty$ for distinct $s, t \in \mathbb{R}^>$.

3. THE MEANS

In this section we give the precise definitions of the means that are studied. We will also define and calculate a certain variety of their derivatives.

3.1. Classical Means.

In this subsection we define some classical means and prove an inequality between them that is needed in Section 4.

The *Arithmetic*, *Geometric*, *Harmonic* and *Logarithmic means* are defined for $x, y \in \mathbb{R}^>$ by

$$A(x,y) := \frac{x+y}{2},\ G(x,y) := \sqrt{xy},\ H(x,y) := \frac{2xy}{x+y}$$

and

$$L(x,y) := \frac{x-y}{\log\{x/y\}},\ x \ne y,\ L(x,x) := x,$$

respectively. Moreover, we denote by A_s the power mean of order s: $A_s(x,y) = [A(x^s,y^s)]^{1/s}$ for $s \in \mathbb{R} \setminus \{0\}$ and $A_0 = G$. The next lemma is an improvement over the well known relation $L \ge G$, since $A \ge G$.

Lemma 3.1. *We have* $L \succeq A^{1/3}G^{2/3}$.

Proof. We need to prove that

$$f(x) := \frac{L^3(x,1)}{A(x,1)G^2(x,1)} = \frac{(x-1)^3}{(x+1)x\log^3 x}$$

is increasing in x for $x \ge 1$ (we used Lemma 2.1(3) with $s = 3$). A calculation gives

$$f'(x) = \frac{(x^2+4x+1)\log\{x\} - 3(x^2-1)}{(x+1)^2x^2\log^4\{x\}}(x-1)^2.$$

Hence $f'(x) \ge 0$ if and only if

$$g(x) := \log x - 3\frac{x^2-1}{x^2+4x+1} \ge 0.$$

Since clearly $g(1) = 0$, it suffices to show that g is increasing, which follows from

$$\begin{aligned}(x^2+4x+1)^2xg'(x) &= (x^2+4x+1)^2 - 3x(2x(x^2+4x+1) - (x^2-1)(2x+4))\\ &= (x-1)^4 \ge 0.\end{aligned}$$

□

3.2. **The Extended Mean Value.** Let $x, y \in \mathbb{R}^{>}$ be distinct and $s, t \in \mathbb{R} \setminus \{0\}$, $s \neq t$. We define the *extended mean value* with parameters s and t by

$$E_{s,t}(x,y) := \left(\frac{t}{s} \frac{x^s - y^s}{x^t - y^t} \right)^{1/(s-t)},$$

and also

$$E_{s,s}(x,y) := \exp\left(\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s - y^s} \right),$$

$$E_{s,0}(x,y) := \left(\frac{x^s - y^s}{s \log\{x/y\}} \right)^{1/s} \text{ and } E_{0,0}(x,y) := \sqrt{xy}.$$

Regardless of whether s and t are distinct we also define $E_{s,t}(x,x) := x$. Notice that all the cases are continuous continuations of the first general expression for $E_{s,t}(x,y)$ (this was proved to be possible in [18]).

It should also be noted that $E_{2,1} = A$, $E_{0,0} = G$, $E_{-1,-2} = H$ and $E_{1,0} = L$, and more generally, $A_s = E_{2s,s}$ for $s \in \mathbb{R}$. Hence we see that all these classical means belong to the family of extended mean values.

Let us next calculate the following variety of the logarithmic derivative:

$$e_{s,t}(x) := x \frac{\partial \log E_{s,t}(x,1)}{\partial x} - 1.$$

The reason for choosing this form has to do with the strong inequality (the logarithm, as was seen in Section 2.1) and simplicity of form (multiplying by x and subtracting 1). Assume that $x > 1$ and also $s, t \in \mathbb{R} \setminus \{0\}$, $s \neq t$. Then

$$e_{s,t}(x) = \frac{1}{s-t} \left(\frac{s}{x^s - 1} - \frac{t}{x^s - 1} \right),$$

$$e_{s,s}(x) = \frac{1}{x^s - 1} - \frac{s x^s \log x}{(x^s - 1)^2},$$

$$e_{s,0}(x) = \frac{1}{x^s - 1} - \frac{1}{s \log x}$$

and

$$e_{0,0}(x) = -1/2.$$

Note that for all $s, t \in \mathbb{R}$ we have $e_{s,t}(1+) := \lim_{x \to 1} e_{s,t}(x) = -1/2$. It will be of much use to us that

$$e_{s,s}(x) = \lim_{t \to s} e_{s,t}(x),\ e_{s,0}(x) = \lim_{t \to 0} e_{s,t}(x) \text{ and } e_{0,0}(x) = \lim_{t,s \to 0} e_{s,t}(x),$$

since this will allow us to consider only the general formula (with distinct $s, t \in \mathbb{R} \setminus \{0\}$) and have the remaining cases follow by continuity. Let us record the following simple result which will be needed further on.

Lemma 3.2. *For every pair $s, t \in \mathbb{R}$ we have $e_{s,t}(x) \leq 0$ for all $x \in (1, \infty)$.*

Proof. It suffices to show this for distinct $s, t \in \mathbb{R} \setminus \{0\}$. Assume further that $s > t$. We have to show that

$$\frac{s}{x^s - 1} \leq \frac{t}{x^t - 1}.$$

If $t > 0$ we just multiply by $(x^t - 1)(x^s - 1)$, whereupon the claim is clear, since $sx^t - tx^s$ is decreasing in x and hence less than or equal to $s - t$. Next if $s > 0 > t$ we have to prove that

$$\frac{s}{x^s - 1} \leq \frac{-tx^{-t}}{x^{-t} - 1}.$$

or, equivalently, $s - t \le (-t)x^s + sx^t$. Since the right hand side is increasing in x this is clear. The case $0 > s > t$ follows like the case $t > 0$, since $(x^t - 1)(x^s - 1)$ is again positive. □

We conclude this subsection by stating that for all $s, t \in \mathbb{R}$ we have

$$\lim_{x\to 1+} \frac{\partial e_{s,t}(x)}{\partial x} = \frac{s+t}{12},$$

a fact which is easy, though tedious, to check (differentiate and use l'Hospital's rule four times; the proof is quite similar to that of Lemma 3.3).

3.3. **The Gini Mean.** The Gini mean was introduced in [6] and is a generalization of the power means. It is defined by

$$G_{s,t}(x, y) := \left(\frac{x^s + y^s}{x^t + y^t}\right)^{1/(s-t)},$$

where $x, y \in (0, \infty)$ and $s, t \in \mathbb{R}$ are distinct. We also define

$$G_{s,s}(x, y) := \exp\left(\frac{x^s \log x + y^s \log y}{x^s + y^s}\right).$$

Notice that the power means are the elements $G_{s,0} = A_s$ in this family of means. The logarithmic mean is not part of the Gini mean family, in fact, Alzer and Ruscheweyh have recently shown that the only means common to the extended mean value and the Gini mean familes are the power means, [3].

We easily find that

$$g_{s,t}(x) := x\frac{\partial \log G_{s,t}(x, 1)}{\partial x} - 1 = \frac{1}{s-t}\left(\frac{t}{x^t + 1} - \frac{s}{x^s + 1}\right),$$

for $s \neq t$ and $x > 1$ and

$$g_{s,s}(x) = \frac{sx^s \log x}{(x^s + 1)^2} - \frac{1}{x^s + 1}.$$

As with the extended mean value we find that $g_{s,s} = \lim_{t\to s} g_{s,t}$. We again have $g_{s,t}(1) = -1/2$ and it is easily derived that $g'_{s,t}(1) = (s + t)/4$.

3.4. **The Seiffert Mean.** The Seiffert mean was introduced in [15] and is defined by

$$P(x, y) := \frac{x - y}{4\arctan(\sqrt{x/y}) - \pi} = \frac{x - y}{2\arcsin((x - y)/(x + y))}$$

for distinct $x, y \in \mathbb{R}^>$ and $P(x, x) := x$. For this mean we have

$$p(x) := x\frac{\partial \log P(x, 1)}{\partial x} - 1 = \frac{1}{x-1} - \frac{2\sqrt{x}}{x+1}\frac{1}{4\arctan(\sqrt{x/y}) - \pi},$$

for $x > 1$. Also, it can be calculated that $p(1+) = -1/2$. Let us for once explicitly calculate the limiting value of the derivative at 1:

Lemma 3.3. *We have*

$$\lim_{x\to 1+} \frac{dp(x)}{dx} = \frac{1}{6}.$$

Proof. A direct calculation gives

$$\begin{aligned} p'(x) &= -\frac{1}{(x-1)^2} + \frac{x-1}{\sqrt{x}(x+1)^2}\frac{1}{4\arctan(\sqrt{x}) - \pi} + \frac{4}{(x+1)^2}\frac{1}{(4\arctan(\sqrt{x}) - \pi)^2} \\ &= \left(\frac{2}{x+1}\frac{1}{4a-\pi} - \frac{1}{x-1}\right)\left(\frac{2}{x+1}\frac{1}{4a-\pi} + \frac{1}{x-1}\right) + \frac{x-1}{\sqrt{x}(x+1)^2}\frac{1}{4a-\pi}, \end{aligned}$$

where we have denoted $a := \arctan(\sqrt{x})$. Hence, when we write $4\arctan(\sqrt{x}) - \pi = c(x-1)$, we have

$$p'(1+) = \lim_{x\to 1} \left(\frac{2}{x+1} \frac{1}{4\arctan(\sqrt{x}) - \pi} - \frac{1}{x-1} \right) \frac{1}{x-1} \left(\frac{2}{c(x+1)} + 1 \right) + \frac{1}{c\sqrt{x}(x+1)^2}$$
$$= \lim_{x\to 1} 2 \left(\frac{2}{x+1} \frac{1}{4\arctan(\sqrt{x}) - \pi} - \frac{1}{x-1} \right) \frac{1}{x-1} + \frac{1}{4},$$

since $c \to 1$ as $x \to 1+$ and all the factors are continuous. It remains to evaluate

$$\lim_{x\to 1+} \frac{2\frac{x-1}{x+1} - 4\arctan(\sqrt{x}) + \pi}{(x-1)^2(4\arctan(\sqrt{x}) - \pi)} = \lim_{y\to \pi/4+} \frac{\pi - 4y - 2\cos(2y)}{4\cos^2(2y)(4y-\pi)} \cos^4 y,$$

where we used the substitution $y = \arctan(\sqrt{x})$. We have, using l'Hospital's rule and the substitution $z := 2y$

$$\lim_{z\to\pi/2} \frac{\pi - 2z - 2\cos z}{(2z-\pi)(1+\cos(2z))} = \lim_{z\to\pi/2} \frac{-2 + 2\sin z}{2(1+\cos(2z)) - 2(2z-\pi)\sin(2z)}$$
$$= \lim_{z\to\pi/2} \frac{\cos z}{-4\sin(2z) - 2(2z-\pi)\cos(2z)}$$
$$= \lim_{z\to\pi/2} \frac{-\sin z}{-12\cos(2z) + 4(2z-\pi)\sin(2z)} = -\frac{1}{12}.$$

Since $\lim_{y\to\pi/4} \cos^4(y) = 1/2$ we find that $p'(1+) = 2(-1/12)(1/2) + 1/4 = 1/6$, as claimed. □

Let us also introduce another mean of Seiffert's, from [16], for which we will prove just one inequality. Define

$$T(x,y) := \frac{x-y}{2\arctan\frac{x-y}{x+y}}$$

for distinct $x, y \in \mathbb{R}$ and $T(x,x) = x$. This mean satisfies $A \le T \le A_2$, see [16]. We have

$$t(x) := x\frac{\partial \log T(x,1)}{\partial x} - 1 = \frac{1}{x-1} - \frac{x}{x^2+1} \left(\arctan\frac{x-1}{x+1} \right)^{-1}.$$

4. THE EXTENDED MEAN VALUE

In this section we will prove Theorem 1.1, which is the used in the proof of the other theorems. The proof consists essentially of two lemmas which show that the extended mean value behaves nicely with respect to the strong inequality as we move in the parameter plane. We start with the horizontal direction and then go for the diagonal.

Lemma 4.1. *Let $r, t \in \mathbb{R}$. Then $E_{t,s} \succeq E_{r,s}$ if and only if $t \ge r$.*

Proof. It suffices to show that $e_{r,s}$ is increasing in r. We differentiate with respect to r and find that $e_{r,s}$ is increasing when

$$0 \le (r-s)^2 \frac{\partial e_{r,s}}{\partial r} = (r-s)\frac{x^r - 1 - x^r\log x^r}{(x^r-1)^2} + \frac{s}{x^s-1} - \frac{r}{x^r-1} =: f(s).$$

We have $f(r) = 0$, hence it suffices to show that $f'(s) \le 0$ if and only if $s \le r$. Differentiating with respect to s gives

$$f'(s) = \frac{x^r\log x^r - x^r + 1}{(x^r-1)^2} - \frac{x^s\log x^s - x^s + 1}{(x^s-1)^2}.$$

Since $x^s \le x^r$ if and only if $s \le r$ it suffices to show that $g(y) = (y \log y - y + 1)(y-1)^{-2}$ is decreasing. We calculate

$$g'(y) = \frac{(y-1)\log y - 2(y\log y - y + 1)}{(y-1)^3} = \frac{2(y-1) - (y+1)\log y}{(y-1)^3}.$$

Hence $g'(y) \le 0$ if and only if $2(y-1) \le (y+1)\log y$ exactly when $y > 1$. Since

$$\log y - 2\frac{y-1}{y+1}$$

is increasing in y and equals 0 for $y = 1$, this is seen to be so. □

Lemma 4.2. *Let $a \ge 2s \ge 2q \ge 0$. Then*

$$E_{a-s,s} \succeq E_{a-q,q}.$$

Proof. We show that $e_{a-s,s}$ is increasing in $s < a/2$, which is clearly equivalent to the claim. Now

$$\frac{\partial e_{a-s,s}(x)}{\partial s} = \frac{2}{(a-2s)^2}\left(\frac{a-s}{x^{a-s}-1} - \frac{s}{x^s-1}\right) + \frac{1}{a-2s}\left(\frac{1-x^{a-s}+(a-s)x^{a-s}\log x}{(x^{a-s}-1)^2} - \frac{x^s - 1 - sx^s\log x}{(x^s-1)^2}\right).$$

Let us denote $a - s =: r$. The inequality $\partial e_{a-s,s}/\partial s \ge 0$ becomes

$$\frac{x^r\log x^r}{(x^r-1)^2} + \frac{x^s\log x^s}{(x^s-1)^2} \ge \frac{1}{r-s}\left(2\frac{s}{x^s-1} - 2\frac{r}{x^r-1} + \frac{r-s}{x^s-1} + \frac{r-s}{x^r-1}\right) = \frac{r+s}{r-s}\left(\frac{1}{x^s-1} - \frac{1}{x^r-1}\right).$$

Let us multiply both sides by $(x^s-1)(x^r-1)$. The inequality becomes

$$\frac{x^s-1}{x^r-1}x^r\log x^r + \frac{x^r-1}{x^s-1}x^s\log x^s \ge \frac{r+s}{r-s}(x^r-x^s).$$

Let us next use the equalities $(x^s-1)/(x^r-1) = 1-(x^r-x^s)/(x^r-1)$ and $(x^r-1)/(x^s-1) = 1+(x^r-x^s)/(x^s-1)$ and divide by $x^r - x^s$:

$$f_{r,s}(x) := \left(\frac{sx^s}{x^s-1} - \frac{rx^r}{x^r-1} + \frac{rx^r+sx^s}{x^r-x^s}\right)\log x - \frac{r+s}{r-s} = \left(\frac{s}{x^s-1} - \frac{r}{x^r-1} + \frac{sx^r+rx^s}{x^r-x^s}\right)\log x - \frac{r+s}{r-s} \ge 0.$$

We will demonstrate that this is so by showing that $f_{r,r}(x) = 0$, that

$$\lim_{s\to 0}\frac{\partial f_{r,s}}{\partial r} = 0, \text{ and that } \frac{\partial^2 f_{r,s}}{\partial r\partial s} \ge 0.$$

The last two conditions imply that $\partial f_{r,s}/\partial r \ge 0$. This, together with the first condition implies that $f_{r,s} \ge 0$ if $s \ge 0$, which completes the proof.

We first show that $f_{r,r}(x) = 0$:

$$\lim_{s\to r} f_{r,s}(x) = \lim_{s\to r}\frac{(sx^r+rx^s)(r-s)\log x - (r+s)(x^r-x^s)}{(x^r-x^s)(r-s)} = \lim_{s\to r}\frac{-2(x^r+rx^s\log x)\log x + 2x^s\log x + (r+s)x^s\log^2 x}{2x^s\log x} = 0.$$

Upon calculating $\partial f_{r,s}/\partial r$,

$$\frac{\partial f_{r,s}}{\partial r} = \left(\frac{x^r \log x^r}{(x^r-1)^2} - \frac{1}{x^r-1} - \frac{x^{r+s}\log x^{r+s}}{(x^r-x^s)^2} + \frac{x^s}{x^r-x^s} \right) \log x + \frac{2s}{(r-s)^2},$$

we immediately find that $\partial f_{r,s}/\partial r|_{s=0} = 0$. Next we calculate

$$\begin{aligned} \frac{\partial^2 f_{r,s}}{\partial r \partial s} &= \frac{x^{r+s}\log^2 x}{(x^r-x^s)^2} - \frac{x^{r+s}(x^r-x^s)\log^2 x + (r+s)(x^r+x^s)x^{r+s}\log^3 x}{(x^r-x^s)^3} + 2\frac{r-s+2s}{(r-s)^3} \\ &= -\frac{(r+s)(x^r+x^s)x^{r+s}\log^3 x}{(x^r-x^s)^3} + 2\frac{r+s}{(r-s)^3}. \end{aligned}$$

Therefore $\partial^2 f_{r,s}/\partial r \partial s$ is positive when

$$\frac{2}{(r-s)^3} \ge \frac{(x^r+x^s)x^{r+s}\log^3 x}{(x^r-x^s)^3},$$

where we used that $r+s=a>0$.

Since $x^r \ge x^s$ this last inequality is equivalent with

$$L(x^r,x^s)^3 \ge A(x^r,x^s)G(x^r,x^s)^2,$$

which follows from Lemma 3.1, and so we are done. □

Proof of Theorem 1.1. Let us assume without loss of generality that $s \ge t$ and $p \ge q$.

Suppose first that $E_{s,t} \succeq E_{p,q}$ holds. This is equivalent with the condition $e_{s,t}(x) \ge e_{p,q}(x)$. As $x \to 1+$ there is equality in the inequality. Hence $e'_{s,t}(1+) \ge e'_{p,q}(1+)$, for otherwise $e_{s,t}(x) < e_{p,q}(x)$ in some neighborhood (with respect to $[1,\infty)$) of $x=1$. It follows that $(s+t)/12 \ge (p+q)/12$, or, equivalently, $s+t \ge p+q$. As $x \to \infty$ we have

$$e_{s,t} \sim -\frac{t}{s-t}x^{-t}$$

if $0<t<s$, $e_{t,t} \sim -tx^{-t}\log x$ and $e_{s,0} \sim -1/\log\{x^s\}$. Hence we see that the condition $e_{s,t}(x) \ge e_{p,q}(x)$ implies that $t \ge q$.

Assume conversely that $s+t \ge p+q$ and $t \ge q$. Then we have

$$E_{s,t} \succeq E_{s+t-q,q} \succeq E_{p+q-q,q} = E_{p,q},$$

where the first inequality follows from Lemma 4.2 since $t \ge q$ and the second inequality follows from Lemma 4.1, since $s+t \ge p+q$. □

5. THE GINI MEAN

The Gini mean was defined in Section 3.3. In this section we will derive partial results on when a Gini mean is strongly greater than or equal to an extended mean value. We will see that although the Gini mean was easier to define (required less cases) than the extended mean value, it is a lot more difficult to handle, since it does not satisfy the kinds of lemmas that were proved for the extended mean value in Section 4.

It is well known that $G_{s,q} \ge G_{t,q}$ if and only if $s \ge t$ (proved for instance in [13, Theorem 1.1 (h)]). The next example shows that this inequality does not generalize to a strong inequality.

Example 5.1. Let $s>t>q>0$. Then $G_{s,q}$ and $G_{t,q}$ are not comparable in the partial order $\succeq$. Indeed, $g_{s,q}(x) > g_{t,q}(x)$ holds for small $x>1$, since both have the same limit (viz. $-1/2$) as $x \to 1+$ and $g_{s,q}$ has a greater derivative at $x=1+$, as was shown in Section 3.3. On the other hand $g_{s,q}(x) < g_{t,q}(x)$ for x large enough, since

$$g_{s,q} \sim qx^{-q}/(s-q) < qx^{-q}/(t-q) \sim g_{t,q}$$

as $x \to \infty$.

5.1. **The Easy Case – when there are strong inequalities between Gini means.** Despite the previous example we can derive some strong inequalities between Gini means, which is what we will do next. Note the $G_{s+t,0}$ is the power mean A_{s+t}.

Lemma 5.1. *If $s, t \geq 0$ then $G_{s,t} \succeq G_{s+t,0}$.*

Proof. Assume without loss of generality that $s+t > 0$. Using the transformation $x \mapsto x^{2/(s+t)}$ we may assume that $s+t=2$ (here we use Lemma 2.1(2)). Assume further that $s = 1+d$ and $t = 1-d$ where $d \geq 0$ and for the time being suppose further that $d > 0$. The claim of the lemma is that

$$g_{1+d,1-d}(x) = \frac{1}{2d}\left(\frac{1-d}{x^{1-d}+1} - \frac{1+d}{x^{1+d}+1}\right) \geq -\frac{1}{x^2+1} = g_{2,0}(x).$$

Let us multiply this inequality by $2d(x^{1-d}+1)(x^{1+d}+1)$ (which is obviously positive) to get the equivalent inequality

$$(1-d)(x^{1+d}+1) - (1+d)(x^{1-d}+1) \geq -2d\frac{x^2+x^{1+d}+x^{1-d}+1}{x^2+1}.$$

Collect the terms multiplied by d:

$$\begin{aligned} x^{1+d} - x^{1-d} &= (x^{1+d}+1) - (x^{1-d}+1) \\ &\geq (x^{1+d}+x^{1-d}+2)d - 2d\frac{x^2+x^{1+d}+x^{1-d}+1}{x^2+1} \\ &= (x^{1+d}+x^{1-d})(1-2/(x^2+1))d \\ &= (x^{1+d}+x^{1-d})(x^2-1)d/(x^2+1). \end{aligned}$$

Multiplying the first and the last expression by x^{d-1} gives the inequality

$$x^{2d} - 1 \geq (x^{2d}+1)(x^2-1)d/(x^2+1).$$

Let us set $x^d =: z$ or, equivalently, $d = \log\{z\}/\log x$. Then we get the equivalent inequality

$$\frac{z^2+1}{z^2-1}\log z \leq \frac{x^2+1}{x^2-1}\log x,$$

which is further equivalent with the function $f(y) := (y+1)\log\{y\}/(y-1)$ being increasing, since $x \geq z$. Now

$$f'(y) = \frac{y^2-1-2y\log y}{y(y-1)^2} \geq 0$$

if and only if $y^2 - 1 - 2y\log y \geq 0$, which follows, since $y - y^{-1} - 2\log y$ is increasing in y for $y \geq 1$. This ends the proof for the case $d > 0$. The case $d = 0$ follows, since $g_{1+d,1-d}$ is continuous in d. □

Proof of Theorem 1.5. If $s, t \geq 0$ and $a = s+t$ then

$$G_{s,t} \succeq G_{a,0} = A_a = E_{2a,a},$$

where the strong inequality follows from Lemma 5.1. It then follows from Theorem 1.1 that

$$G_{s,t} \succeq E_{2a,a} \succeq E_{p,q},$$

if $p+q \leq 3a$ and $\min\{p,q\} \leq a$.

Suppose conversely that $G_{s,t} \succeq E_{p,q}$ holds for all $s, t \geq 0$ with $s+t=a$. Then it holds in particular for $s=a$ and $t=0$ and so

$$G_{a,0} = E_{2a,a} \succeq E_{p,q}.$$

It then follows from Theorem 1.1 that $p+q \le 2a+a$ and $\min\{p,q\} \le \min\{2a,a\} = a$, as claimed. □

5.2. **The Difficult Case – when there are no strong inequalities between Gini means.** We now turn to deriving strong inequalities between Gini means and extended mean values that are not mediated by power means. Since it was shown in Example 5.1 that there is not much possibility of deriving auxiliary inequalities between Gini means and since the author has had no success in direct derivation of inequalities between extended mean values and Gini means, another scheme of mediation is developed. It consists of using a Gini mean as an intermediary for a small value of x and the fact that most Gini means grow asymptotically faster than extended mean values to take care of large values of x.

We start by considering a certain monotonicity property of $g_{s,t}$. This lemma corresponds to Lemma 4.2 for the extended mean value.

Lemma 5.2. *The quantity $g_{1+d,1-d}(x)$ is decreasing in $0 \le d \le 1$ for fixed $x \in [1, 49^{1/2}]$.*

Proof. Let us assume that $d > 0$; the case $d = 0$ follows by continuity. A simple calculation gives

$$f(d) := d\frac{\partial g_{1+d,1-d}}{\partial d} = -\frac{1}{(x/z+1)d} + \frac{x/z \log\{x/z\}}{(x/z+1)^2} + \frac{1}{(xz+1)d} + \frac{xz\log\{xz\}}{(xz+1)^2},$$

where we have denoted $x^d =: z$. Let us multiply the inequality $f(d) \le 0$, which is equivalent with the claim of the lemma, by $(xz+1)(x/z+1)$ and use $d = \log z/\log x$:

$$\begin{aligned}(x/z - xz)&\frac{\log x}{\log z} + \frac{(x^2 + x/z)\log\{x/z\}}{x/z+1} + \frac{(x^2+xz)\log\{xz\}}{xz+1}\\ &= (x/z - xz)\frac{\log x}{\log z} + \left(\frac{\log\{x/z\}}{x/z+1} + \frac{\log\{xz\}}{xz+1}\right)(x^2-1) + 2\log x \le 0.\end{aligned}$$

Let us divide this inequality by $x \log x$ and rearrange

$$\left(\frac{\log\{x/z\}}{x/z+1} + \frac{\log\{xz\}}{xz+1}\right)\frac{x-1/x}{\log x} + \frac{2}{x} \le \frac{z-1/z}{\log z}. \tag{5.1}$$

We will show that the left hand side is decreasing in $z \in [1, x]$ and that the right hand side is increasing in z. Now the latter claim is equivalent with

$$\frac{d}{dz}\frac{z-1/z}{\log z} = \frac{(z^2+1)\log z - (z^2-1)}{z^2\log^2 z} \ge 0,$$

which is clear, since $\log z - (z^2-1)/(z^2+1)$ is increasing in z and hence positive. It remains to prove that

$$g(z) := \frac{\log\{x/z\}}{x/z+1} + \frac{\log\{xz\}}{xz+1}$$

is decreasing in z. A calculation gives

$$zg'(z) = \frac{xz+1-xz\log\{xz\}}{(xz+1)^2} - \frac{x/z+1-(x/z)\log\{x/z\}}{(x/z+1)^2} = h(xz) - h(x/z),$$

where $h(y) := (y+1-y\log y)/(y+1)^2$. The function h is sketched in Figure 5.1 and has the following pertinent characteristics: its only zero is at $y_0 = 3.591...$, its only minimum at $y_1 = 11.016...$ and it is then increasing, but negative.

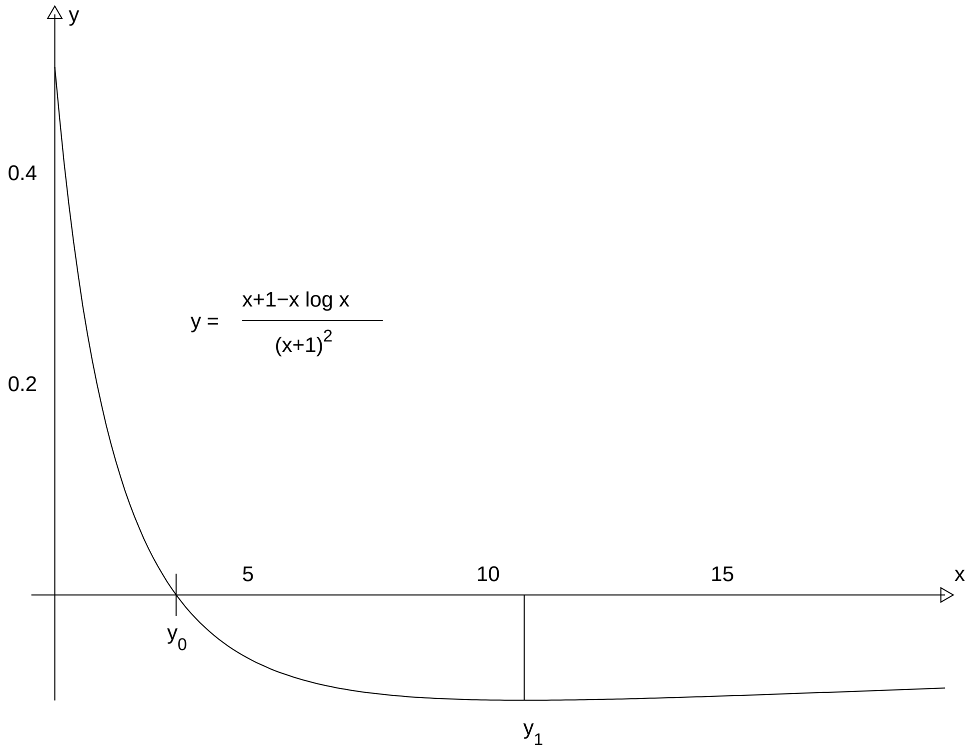

Figure 5.1: The function h.

Suppose now that x is such that the condition

$$(x/z \leq y_0) \vee (xz \leq 14 \wedge x/z \leq 7) \tag{5.2}$$

holds for all $z \in [1, x]$. We then claim that $h(x/z) \leq h(xz)$ holds: because, for a given z, one of the following conditions holds:

(1) $y_1 \geq xz$,
(2) $y_1 < xz$ and $x/z \leq y_0$ or
(3) $y_1 < xz \leq 14$ and $x/z \leq 7$.

If (1) holds then $h(x/z) \geq h(xz)$ since h is decreasing on $[1, y_1]$ and $xz \geq x/z$. If (2) holds then $h(x/z) \geq 0 \geq h(xz)$. If (3) holds then we have

$$h(x/z) \geq h(7) > -0.088 > -0.097 > h(14) \geq h(xz).$$

If $x \leq 7$ then the condition (5.2) holds. For if $x \not\leq y_0 z$ then $x/z \leq x \leq 7$ and $xz \leq x^2/y_0 < 49/3.6 < 13.7$ so that the second condition holds. We have shown, then, that for $x \leq 7$ we have $zg'(z) = h(xz) - h(x/z) \leq 0$ for all $z \in [1, x]$ and so we see that g is decreasing in the same range.

Let us now return to inequality (5.1). Since the left hand side is decreasing in z and the right hand side is increasing in the same, it clearly suffices to show that the inequality holds for $z = 1+$. Calculating, we see we have to show that

$$\frac{2\log x}{x+1}\frac{x-1/x}{\log x} + \frac{2}{x} \leq 2,$$

which is actually an equality and hence the claim is clear. □

Remark 5.3. The restriction on x in the previous lemma is not superfluous, for the claim does not hold for large x and all d. However, numerical evidence does suggest that $\partial g_d(x)/\partial d$ has character $-$ or $-|+$, hence we would have a certain monotonicity property for large x also. Unfortunately the author has not been able to prove this fact.

We now proceed to the second phase of the scheme presented, showing that for large x, $G_{s,t}$ has a large derivative. Note that the constant $11/189$ is chosen to suffice for Remark 5.10.

Lemma 5.4. *If* $11/189 \le s/t \le 189/11$ *and* $s+t=1$ *then* $g_{s,t}(x) \ge 0$ *for* $x \ge 47$.

Proof. Assume without loss of generality that $s > t$. We have to prove that

$$f(x) := (s-t)(x^s+1)(x^t+1)g_{s,t}(x) = t(x^s+1) - s(x^t+1) \ge 0$$

for $x \ge 47$. Since

$$xf'(x) = ts(x^s - x^t) \ge 0$$

it suffices to show that $f(47) \ge 0$. Let us divide $f(47)$ by s and denote $v := t/s$. The inequality becomes

$$g(v) := v(47^{1/(1+v)}+1) - 47^{v/(1+v)} - 1 \ge 0.$$

Clearly $g(1) = 0$ and we also find that $g(11/189) > 0.035$. Hence it suffices to show that $g'(v)$ has characteristic $+|-$ for $v \in [11/189, 1]$. A calculation gives

$$g'(v) := 47^{1/(1+v)} + 1 - \frac{\log 47}{(1+v)^2}\left(47^{v/(1+v)} + v47^{1/(1+v)}\right).$$

Let us write the inequality $g'(v) \ge 0$ in terms of the original variable, $s = 1/(1+v)$, divide by $\log\{47^s\}$ and rearrange some:

$$\frac{47^s+1}{\log 47^s} \ge s47^{1-s} + (1-s)47^s.$$

We will show that the left hand side is increasing in s and that the right hand side is decreasing in s. From this it follows, on checking the boundary values $s = 1/2$ and $s = 189/200$, that g' has characteristic $-|+$, which completes the proof.

Since 47^s is obviously increasing in s we have first to show that $h(y) := (y+1)/\log y$ is increasing for $y \in [47^{1/2}, 47^{0.945}]$. We have

$$(\log y)^2 h'(y) = \log y - 1 - 1/y.$$

Since $\log y - 1 - 1/y$ is increasing in y, it is clear that

$$h'(y) \ge \frac{\log\sqrt{47} - 1 - 47^{-1/2}}{\log^2 47} \approx 0.058 > 0.$$

Next we want to show that $m(s) := s47^{1-s} + (1-s)47^s$ is decreasing in s for $s \in [1/2, 189/200]$. Let us differentiate:

$$m'(s) = 47^{1-s} - 47^s + \left((1-s)47^s - s47^{1-s}\right)\log 47.$$

Then $m'(s) \le 0$ if and only if

$$n(47^{1-s}) = \frac{\log 47^{1-s} - 1}{47^{1-s}} \le \frac{\log 47^s - 1}{47^s} = n(47^s),$$

where we have denoted $n(z) := (\log z - 1)/z$. This function has the following relevant characteristics: only zero at e and only maximum at e^2. In what follows we will essentially approximate $n(z)$ by a step function which allows us to arrive at the desired conclusion.

Since $47^s \geq 47^{1-s}$ by assumption on s, we see that $n(47^{1-s}) \leq n(47^s)$ if $47^s \leq e^2$ or, equivalently, $s \leq 0.5194$, since $n(z)$ is increasing for $z \leq e^2$. If $s > 0.5194$ then $47^{1-s} < 6.363$ and $n(47^{1-s}) < 0.1336$. Since $n(8.7) > 0.1337$ it follows that

$$n(47^s) \geq \min\{n(47^{0.5194}), n(8.7)\} > n(47^{0.4806}) \geq n(47^{1-s})$$

for $0.5194 \leq s \leq 0.5618 < \log 8.7/\log 47$. Making a second iteration, we find that for $s \geq 0.5618$ we have $n(47^{1-s}) < 0.1272$, and $n(10.8) > 0.1277$. Hence

$$n(47^s) \geq n(10.8) > n(47^{0.4382}) \geq n(47^{1-s})$$

for $0.5618 \leq s \leq 0.6180 < \log 10.8/\log 47$. Continuing with a third and a fourth iteration we find that

$$n(47^s) \geq n(16) > n(47^{0.382}) \geq n(47^{1-s})$$

for $0.6180 \leq s \leq 0.72 < \log 16/\log 47$ and that

$$n(47^s) \geq n(47) > n(47^{0.28}) \geq n(47^{1-s})$$

for $0.72 \leq s \leq 1 = \log 47/\log 47$ and so we are done. □

Using the previous two lemmas we will be able to derive strong inequalities for many Gini means by proving just a few simple inequalities, which effectively amount to solving polynomial inequalities.

Lemma 5.5. *Let $r > 0$. Then $G_{3r,r} \succeq E_{p,q}$ if and only if $p + q \leq 12r$.*

Proof. Assume first that $p+q \leq 12r$. Since $E_{p,q} \preceq E_{u,u}$, where $u \geq (p+q)/2$, by Theorem 1.1, it suffices to prove that $G_{3r,r} \succeq E_{u,u}$ with $u = 6r$. This is equivalent with

$$\frac{x^r - 2}{x^{2r} - x^r + 1} = \frac{1}{2r}\left(\frac{r}{x^r + 1} - \frac{3r}{x^{3r} + 1}\right) = g_{3r,r} \geq e_{u,u} = \frac{1}{x^u - 1} - \frac{ux^u \log x}{(x^u - 1)^2}.$$

Let us set $y := x^r$ and multiply by $(x^u - 1)^2/x^u$:

$$\frac{(y^6 - 1)^2}{2y^6}\frac{y - 2}{y^2 - y + 1} \geq 1 - y^{-6} - 6\log y.$$

This inequality surely holds for $y = 1$, hence it suffices to show that the left hand side has a greater derivative than the right hand side for $y > 1$:

$$3(y^5 - y^{-7})\frac{y - 2}{y^2 - y + 1} - \frac{(y^6 - 1)^2}{2y^7}\frac{y^2 - 4y + 1}{(y^2 - y + 1)^2} \geq 6y^{-7} - 6/y.$$

Let us multiply both sides by $y^7/(y^6 - 1)$:

$$3(y^6 + 1)\frac{y - 2}{y^2 - y + 1} - \frac{y^6 - 1}{2}\frac{y^2 - 4y + 1}{(y^2 - y + 1)^2}y \geq -6.$$

We can then move the two terms with minus signs to the opposite sides, divide by $y(y^2 - 1)$ and multiply by $2(y^2 - y + 1)^2$ to get

$$6(y^4 - 2y^3 + y^2 - 2y + 1)(y^2 - y + 1) \geq (y^4 + y^2 + 1)(y^2 - 4y + 1).$$

Multiplying out and rearranging gives the inequality

$$5y^6 - 14y^5 + 22y^4 - 26y^3 + 22y^2 - 14y + 5 \geq 0.$$

Dividing by $(y - 1)^2$ gives

$$5y^4 - 4y^3 + 9y^2 - 4y + 5 \geq 0,$$

which holds since $5y^4 \geq 4y^3$ and $9y^2 \geq 4y$ for $y \geq 1$.

The converse implication, that $G_{3r,r} \succeq E_{p,q}$ implies $p+q \le 12r$, follows since $r = g_{3r,r}(1+) \ge e_{p,q}(1+) = (p+q)/12$, which concludes the proof. $\square$

Proof of Theorem 1.6. Suppose first that $G_{s,t} \succeq E_{p,q}$.Then

$$(s+t)/4 = g_{s,t}(1+) \succeq e_{p,q} = (p+q)/12,$$

hence $p+q \le 3(s+t)$, which proves one implication.

Suppose conversely that $p+q \le 3(s+t)$ and $1/3 \le s/t \le 3$. It follows from Lemma 5.2 that $g_{s,t}(x) \ge g_{3r/4,r/4}(x)$ for $x \in [1, 49^{1/(s+t)}]$ and $r := s+t$. It follows from Lemma 5.5 that $g_{3r/4,r/4}(x) \ge e_{3r/2,3r/2}(x)$ for the same x. Using $e_{3r/2,3r/2}(x) \ge e_{p,q}(x)$ from Theorem 1.1 completes the proof in the case of small values of x.

If $x > 47^{1/(s+t)}$ we have

$$g_{s,t}(x) \ge 0 \ge e_{p,q}(x),$$

where the first inequality follows from Lemma 5.4 and the second one from Lemma 3.2. Hence the claim is clear in this case as well. $\square$

Let us now give one more specific Gini mean extended mean value inequality (with corollary) before moving on to summarize the results of this section.

Lemma 5.6. *We have* $G_{9,1} \succeq E_{16,14}$.

Proof. We have to show that

$$\frac{1}{8}\left(\frac{1}{x+1} - \frac{9}{x^9+1}\right) \ge \frac{1}{2}\left(\frac{16}{x^{16}-1} - \frac{14}{x^{14}-1}\right).$$

Let us multiply this by $8(x+1)(x^9+1)(x^{16}-1)(x^{14}-1)x^{-20}$ and move all the terms to the same side. We get the equivalent inequality

$$\begin{aligned} f(x) := x^{19} - x^{-19} - 9(x^{11} - x^{-11}) - 8(x^{10} - x^{-10}) + 56(x^6 - x^{-6}) \\ + 55(x^5 - x^{-5}) - 64(x^4 - x^{-4}) - 65(x^3 - x^{-3}) \ge 0. \end{aligned}$$

Since $f(1) = 0$ it suffices to show that $f'(x) \ge 0$ for $x \ge 1$. Let $g(x) := xf'(x)$. We will show that g is increasing in x, from which it follows that $g(x) \ge 0$ for $x \ge 1$, since $g(1) = f'(1) = 0$. Since g is positive if and only if f' is (for $x > 0$), it follows that $f'(x) \ge 0$. Now

$$\begin{aligned} h(x) &:= xg'(x) \\ &= 361(x^{19} - x^{-19}) - 1089(x^{11} - x^{-11}) - 800(x^{10} - x^{-10}) + 2016(x^6 - x^{-6}) \\ &\quad + 1375(x^5 - x^{-5}) - 1024(x^4 - x^{-4}) - 585(x^3 - x^{-3}), \end{aligned}$$

and g is increasing if and only if $h(x) \ge 0$. Since $h(1) = 0$, it suffices to show that h is increasing and since $h'(1) = 0$, that $m(x) := xh'(x)$ is increasing. We have

$$\begin{aligned} m'(x) = 130123(x^{19} - x^{-19}) - 131769(x^{11} - x^{-11}) - 80000(x^{10} - x^{-10}) \\ + 72576(x^6 - x^{-6}) + 34375(x^5 - x^{-5}) - 16384(x^4 - x^{-4}) - 5265(x^3 - x^{-3}). \end{aligned}$$

Since

$$72576(x^6 - x^{-6}) + 34375(x^5 - x^{-5}) \ge 16384(x^4 - x^{-4}) + 5265(x^3 - x^{-3})$$

we may drop the last four terms in the expression of $m'(x)$. It then suffices to show that (we have divided by 10000 and rounded suitably)

$$n(x) := 13(x^{19} - x^{-19}) - 14(x^{11} - x^{-11}) - 8(x^{10} - x^{-10}) \ge 0$$

for $x \geq 1$. Differentiating one last time we find

$$xn'(x) = 247(x^{19} + x^{-19}) - 154(x^{11} + x^{-11}) - 80(x^{10} + x^{-10}).$$

Since $x^y + x^{-y}$ is increasing in $y > 0$ for fixed $x \geq 1$, we clearly have $n'(x) \geq 0$, hence $n(x) \geq n(1) = 0$ and so we are done. □

Corollary 5.7. *Let $s > t > 0$ and $p > q > 0$ be such that $s/t \leq 9$ and $p/q \geq 8/7$. Then $G_{s,t} \succeq E_{p,q}$ if and only if $p + q \leq 3(s + t)$.*

Proof. We have already seen that $G_{s,t} \succeq E_{p,q}$ implies that $p + q \leq 3(s + t)$ so we need only show that $s/t \leq 9$, $p/q \geq 8/7$ and $p + q \leq 3(s + t)$ imply the strong inequality. The proof of this is exactly the same as the proof of Theorem 1.6; use Lemma 5.2 and Corollary 5.7 and finish up by Theorem 1.1 for small values of x and use Lemmas 5.4 and 3.2 for large values of x. □

5.3. **Summary of Results on Gini Means.** Let us now summarize the results from Theorems 1.5 and 1.6 and Corollary 5.7 in pictorial form. Since the inequality

$$G_{s,t} \succeq E_{p,q}$$

has one degree of homogeneity in the parameters (by Lemma 2.1) we are left with a three dimensional graph. On this graph we will show only the case $p + q = 3(s + t)$, which is the critical case in the sense that the inequality does not hold for smaller $s + t$.

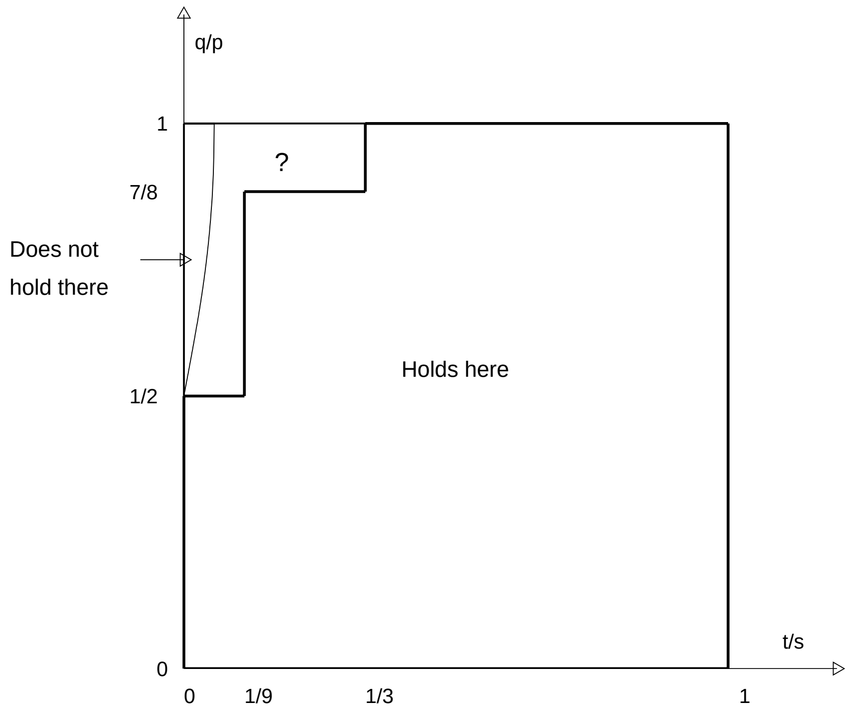

Figure 5.2: When does $G_{s,t} \succeq E_{p,q}$ hold?

We next give a result which shows that the inequality does not hold for certain values of s, t, p and q.

Lemma 5.8. *Let $s \geq t \geq 0$ and $p \geq q \geq 0$ with $p+q = 3(s+t) > 0$. Then $G_{s,t} \not\succeq E_{p,q}$ for $x \leq 9-4\sqrt{5}$ if*

$$y > \frac{5(x^2+1) - 3(x+1)\sqrt{x^2-18x+1}}{4x^2+18x+4},$$

where $x := t/s$ and $y := q/p$.

Remark 5.9. The curve determined by the inequality in the lemma is show in the upper left corner of Figure 5.2.

Proof. Assume that $G_{s,t} \succeq E_{p,q}$ so that $g_{s,t}(z) \geq e_{p,q}(z)$ holds for all $z \in [1,\infty)$. We may assume without loss of generality that $p+q = 3 = 3(s+t)$ and that $s,t,p,q > 0$. If we multiply the inequality by $(z^t+1)(z^s+1)(z^p-1)(z^q-1)$ we get the equivalent inequality

$$f(z) := \frac{tz^s - sz^t + t - s}{s-t}(z^p-1)(z^q-1) - \frac{pz^q - qz^p + q - p}{p-q}(z^t+1)(z^s+1) \geq 0.$$

Since $f(1) = 0$ it follows that $f'(1) \geq 0$ (since $f \in C^\infty$). Upon calculating $f'(1)$ we find that it equals zero, as well. Continuing in this manner we find that $f''(1) = f^{(3)}(1) = f^{(4)}(1) = 0$. With the fifth derivative we start getting somewhere, indeed, we find that

$$f^{(5)}(1) = (p-1)(p-2) + 5s(1-s),$$

hence the condition $f^{(5)}(1) \geq 0$ implies that

$$(p-1)(p-2) + 5s(1-s) = \frac{(2-y)(1-2y)}{(1+y)^2} + \frac{5x}{(1+x)^2} \geq 0,$$

where we have solved p from the system of equations $p+q = 3$ and $q/p = y$ and x from $s+t = 1$ and $t/s = x$. Solving this second degree equation in y gives the desired result. $\square$

Remark 5.10. It follows from the previous lemma that $G_{s,t} \succeq E_{p,q}$ does not hold for every $p,q \in \mathbb{R}^+$ with $p+q = 3(s+t)$ unless

$$\frac{\sqrt{5}-2}{4} = \frac{9-4\sqrt{5}}{4\sqrt{5}-8} \leq s/t \leq 4(\sqrt{5}+2). \tag{5.3}$$

Moreover, numerical evidence suggests that this bound is also sharp, that is to say that $G_{s,t} \succeq E_{p,q}$ would hold if and only if s and t satisfy (5.3). Since $11/189 < (\sqrt{5}-2)/4$, it would suffice to show that

$$G_{4\sqrt{5}-8,9-4\sqrt{5}} \succeq E_{3/2,3/2}$$

in order to prove this claim, using Lemma 5.2.

6. SEIFFERT'S MEAN

In this section we derive exact bounds on when Stolarsky's mean is strongly less than or equal to the Seiffert mean, $P(x,y)$, defined in Section 3.4. We also give an example of an extended mean value which is strongly greater than the Seiffert mean.

Proof of Theorem 1.9. Assume first that $P \succeq S_\alpha$, or, equivalently, $p(x) \geq s_\alpha(x)$, where $s_\alpha := e_{1,1-\alpha}$. We know from Section 3 that $p(1+) = s_\alpha(1+) = -1/2$ and we see that $p(x) \geq s_\alpha(x)$ implies that the derivative of p is greater than that of s_α at $1+$. Now the condition $p'(1+) \geq s'_\alpha(1+)$ is equivalent to $1/6 \geq (2-\alpha)/12$ or $\alpha \geq 0$, again using results from Section 3.

We see that as $x \to \infty$ we have $p(x) \sim -(2/\pi)x^{-1/2}$ and $s_\alpha(x) \sim (1-1/\alpha)x^{\alpha-1}$ if $\alpha > 0$ and $s_\alpha(x) \sim -x^{-1}\log\{x\}$ for $\alpha = 0$, and so $p \geq s_\alpha$ implies that $\alpha - 1/2 \geq 0$.

Suppose conversely then that $\alpha \geq 1/2$. Since $S_\beta \succeq S_\alpha$ if and only if $\alpha \geq \beta$ by Theorem 1.1, it suffices to show that $P \succeq S_{1/2}$, or equivalently

$$\frac{1}{x-1} - \frac{2\sqrt{x}}{x+1}\frac{1}{4\arctan(\sqrt{x})-\pi} \geq \frac{2}{x-1} - \frac{1}{x^{1/2}-1},$$

which can be written as

$$\frac{1}{y-1} - \frac{1}{y^2-1} = \frac{y}{y^2-1} \geq \frac{2y}{y^2+1}\frac{1}{4\arctan y-\pi},$$

where we used the substitution $y = \sqrt{x}$. This is equivalent to

$$f(y) := 4\arctan y - \pi - 2(y^2-1)/(y^2+1) \geq 0.$$

Clearly $f(1) = 0$. Since

$$(y^2+1)^2 f'(y) = 4(y^2+1) - 8y = 4(y-1)^2 \geq 0$$

it is clear that $f(y) \geq f(1) = 0$, which concludes the proof. □

Although it does not have any relevance to the question of relative metrics, we will now give a reverse type inequality, which in turn gives a better ordinary inequality that the previous result, as is seen in Corollary 6.4. This proposition is the strong version of the inequality $P \leq A_{2/3}$ proved by A. A. Jager in [15]. Recall that A_p denotes the power mean $E_{2p,p}$.

Proposition 6.1. *Let $p \in \mathbb{R}$. Then $A_p \succeq P$ if and only if $p \geq 2/3$.*

Proof. Suppose first that $A_p \succeq P$. Then $e'_{2p,p}(1+) = (2p+p)/12 \geq 1/6 = p'(1+)$, by the formulae derived in Section 3, hence $p \geq 2/3$.

Suppose conversely that $p \geq 2/3$. Since $A_p \succeq A_q$ if and only if $p \geq q$ by Theorem 1.1, we see that it suffices to check the claim for $p = 2/3$. The condition $A_{2/3} \succeq P$ is equivalent with

$$\frac{1}{x-1} - \frac{2\sqrt{x}}{x+1}\frac{1}{4\arctan(\sqrt{x})-\pi} \leq -\frac{1}{x^{2/3}+1}.$$

Let $x =: y^6$ and rearrange to get

$$2\frac{(y^6-1)(y^4+1)}{(y^6+1)(y^2+1)y} \geq 4\arctan(y^3) - \pi.$$

Since this equation holds for $y = 1$, it suffices to check that the left hand side has a greater derivative than the right hand side. Let us differentiate both sides of the inequality and multiply by $(y^6+1)^2(y^2+1)^2y^2$:

$$(10y^{10}+6y^6-4y^5)(y^6+1)(y^2+1)-(y^6-1)(y^4+1)(9y^8+7y^6+3y^2+1) \geq 6(y^6+1)(y^2+1)^2y^4.$$

This eighteenth degree polynomial can be written as

$$(y^6-1)(y^4-1)(y^2-1)^2[y^4+5y^2+1] \geq 0,$$

which clearly holds. □

Corollary 6.2. *Let $p, q \in \mathbb{R}^>$ with $1/2 \leq p/q \leq 2$. Then $P \preceq E_{p,q}$ if and only if $p + q \geq 2$.*

Proof. A trivial modification of the first paragraph of the previous proof shows that $E_{p,q} \succeq P$ implies that $p + q \geq 2$.

Assume conversely that $1/2 \leq p/q \leq 2$ and $a := p + q \geq 2$. Then

$$E_{p,q} \succeq E_{2a/3,a/3} \succeq E_{4/3,2/3} = A_{2/3} \succeq P,$$

where the first inequality follows from Lemma 4.2 since $p + q = 2a/3 + a/3$ and $a/3 \leq p, q \leq 2a/3$ and the second inequality follows from Lemma 4.1 as $a \geq 2$. □

Remark 6.3. It is not clear how far the condition $1/2 \leq p/q \leq 2$ in the previous corollary can be relaxed. By considering $x \to \infty$, as was done in the proof of Theorem 1.9, we see that the claim does not hold for $p + q = 2$ with $p < 1/2$, i.e. $p/q < 1/3$.

We also have the following corollary of ordinary inequalities, which follows by the method presented in Section 2.2.

Corollary 6.4. *Let $x, y \in \mathbb{R}^>$ Then*

$$\frac{2^{3/2}}{\pi} A_{2/3}(x,y) \leq P(x,y) \leq A_{2/3}(x,y).$$

Both inequalities are sharp.

Remark 6.5. The estimate of P in Corollary 6.4 is better than the one in Corollary 1.11 in the sense that the former has the ratio $\pi/2^{3/2} \approx 1.1107$ between the upper and lower bounds, whereas the latter has a ratio of at least $4/\pi \approx 1.2732$. Note also that it is probably possible to find an extended mean value which has a smaller such ratio but satisfies neither $E \succeq P$ nor $P \succeq E$.

Let us end this section by proving the following strong version of the inequality $A \leq T$, where T denotes the second Seiffert mean. In fact, the proof is so simple, that it would not be worth giving, were it not for the fact that we will be able to put the lemma to good use in Section 7.

Lemma 6.6. *Let $p \in \mathbb{R}$. Then $A_p \preceq T$ if $p \leq 1$ and also $T \succeq S_\alpha$ for all $\alpha \in (0, 1]$.*

Proof. Clearly it suffices to prove the claim for $p = 1$. Using the formulae for $e_{2,1}(x)$ and $t(x)$ we find that it suffices to show that

$$\frac{1}{x-1} - \frac{x}{x^2+1}\left(\arctan\frac{x-1}{x+1}\right)^{-1} \geq -\frac{1}{x+1}.$$

This becomes

$$\arctan\frac{x-1}{x+1} \geq \frac{1}{2}\frac{x^2-1}{x^2+1}.$$

There is equality for $x = 1$, so we differentiate to find the sufficient condition

$$\frac{1}{x^2+1} \geq \frac{2x}{(x^2+1)^2},$$

which is immediately clear. Since $A \succeq S_\alpha$ for all $\alpha \in (0, 1]$ by Theorem 1.1 the second claim follows by the transitivity of $\succeq$. $\square$

7. NEW RELATIVE METRICS

In this section we show how the results of this paper relate to the so-called M–relative metric, which has been recently studied by the author in [7], [8] and [9]. Let us remind the reader that by a Stolarsky mean we understand a extended mean value with parameters 1 and $1 - \alpha$, hence $S_\alpha = E_{1,1-\alpha}$.

Let us denote by $X := \mathbb{R}^n \setminus \{0\}$ for the rest of this section. Let $M\colon \mathbb{R}^> \times \mathbb{R}^> \to \mathbb{R}^>$ be a symmetric function and let $\rho_M\colon X \times X \to \mathbb{R}^>$ be defined by

$$\rho_M(x,y) := \frac{|x-y|}{M(|x|,|y|)}$$

for all $x, y \in X$. The function ρ_M is called the M–relative distance, and, when it is a metric, the *M–relative metric*. The following result gives the connection between strong inequalities and M–relative metric that has been alluded to previously in this paper.

Theorem 7.1. [7, Lemma 3.1] *Let* $0 < \alpha \leq 1$ *and* $M\colon \mathbb{R}^> \times \mathbb{R}^> \to \mathbb{R}^>$ *be a symmetric homogeneous increasing mean.*

(1) *If* $M \succeq S_\alpha$ *then* ρ_{M^α} *is a metric in* X.
(2) *If* ρ_{M^α} *is a metric in* X *then* $M \geq S_\alpha$.
(3) *If* ρ_{M^α} *is a metric in* X *then* $t_M(x^2)/t_M(x) \geq t_{S_\alpha}(x^2)/t_{S_\alpha}(x)$ *for all* $x > 1$.

Remark 7.2. The second condition of the previous theorem says almost that t_M/t_{S_p} is increasing in a neighborhood of 1 and the third almost that it is increasing in a neighborhood of ∞. It turns out that all the means studied in this paper are sufficiently regular for this "almost" result becomes a real result.

Combining this result with the theorems from Section 1 gives the following corollaries:

Corollary 7.3. *Let* $0 \leq q \leq p$ *and* $\alpha \in (0,1]$. *Then* $\rho_{E^\alpha_{p,q}}$ *is a metric in* X *if and only if* $p+q \geq 2-\alpha$ *and* $q \geq 1-\alpha$.

Proof. Assume first that $p+q \geq 2-\alpha$ and $q \geq 1-\alpha$. Then by Theorem 1.1 $E_{p,q} \succeq E_{1,1-\alpha}$ and so $\rho_{E^\alpha_{p,q}}$ is a metric in X by Theorem 7.1(1).

If $p+q < 2-\alpha$ then $E_{p,q}(x,1)/E_{1,1-\alpha}(x,1)$ is decreasing for small x, since $e_{p,q} < e_{1,1-\alpha}$ in some neighborhood of x. This means that the inequality $E_{p,q}(x,1) \geq E_{1,1-\alpha}(x,1)$ does not hold, and so $\rho_{E^\alpha_{p,q}}$ is not a metric in X, by Theorem 7.1(2).

If $p = q$ and $q < 1-\alpha$ then $p+q < 2-2\alpha \leq 2-\alpha$ and we proceed as in the previous paragraph to show that $\rho_{E^\alpha_{p,q}}$ is not a metric. If $q < p$ and $q < 1-\alpha$ then $E_{p,q}(x,1)/E_{1,1-\alpha}(x,1)$ is decreasing for large x, since $e_{p,q} \sim -qx^{-q}/(p-q) < (1/\alpha - 1)x^{\alpha-1} \sim e_{1,1-\alpha}$ when $\alpha < 1$ and $e_{p,q} \sim -qx^{-q}/(p-q) < -1/\log x \sim e_{1,0}$ (the case $q = 0$ follows similarly). It follows that the third condition of Theorem 7.1 is not satisfied for large x, which means that $\rho_{E^\alpha_{p,q}}$ is not a metric in X. □

Remark 7.4. If we set $p = q/2$ in the previous corollary we get Theorem 1.1 of [7], which is thus a special case of the previous result. Similarly, in Corollary 7.5 we regain this theorem if we set $q = 0$.

Corollary 7.5. *Let* $p, q \in [0,\infty)$. *If* $p+q \geq \max\{(2-\alpha)/3, 1-\alpha\}$ *then* $\rho_{G^\alpha_{p,q}}$ *is a metric in* X .

Proof. Follows immediately from Theorem 1.5 and Theorem 7.1(1). □

Corollary 7.6. *Let* $p, q \in [0,\infty)$ *and* $p/q \leq 3$. *Then* $\rho_{G^\alpha_{p,q}}$ *is a metric in* X *if and only if* $3(p+q) \geq 2-\alpha$.

Proof. That $\rho_{G^\alpha_{p,q}}$ is a metric in X implies that $3(p+q) \geq 2-\alpha$ follows from the last paragraph of the proof of Theorem 1.5. The other implication follows from Theorem 1.6 and Theorem 7.1(1). □

Corollary 7.7. *If* $\alpha \in (0,1]$ *then* ρ_{P^α} *is a metric in* X *if and only if* $1/2 \leq \alpha \leq 1$.

Proof. If $\alpha \geq 1/2$ then ρ_{P^α} is a metric by Theorem 7.1(1). If $\alpha < 1/2$ then P/S_α is decreasing for large x, as was seen in the proof of Theorem 1.9, hence ρ_{P^α} is not a metric, by Theorem 7.1(3). □

For the Seiffert mean we get particularly simple metrics, which was the principal reason for considering strong inequalities of this mean. For instance

$$\rho_P(x,y) = 2\frac{|x-y|}{|x|-|y|}\arcsin\left(\frac{|x|-|y|}{|x|+|y|}\right),$$

for $|x| \neq |y|$ and $\rho_P(x,y) = |x-y|/|x|$ for $|x| = |y|$. We get an even simpler form for x and y on the same ray originating in the origin:

$$\rho_P(se, te) = 2\arcsin\{(s-t)/(s+t)\}$$

where $s > t > 0$ and e is a vector in X.

Corollary 7.8. *If $0 < \alpha \leq 1$ then ρ_{T^α} is a metric in X.*

Proof. Follows directly from Lemma 6.6 and Theorem 7.1(1). □

As in the previous case we get some very simple metrics from this corollary. For instance if $x > y > 0$ and $0 < \alpha \leq 1$ then

$$\rho_{T^\alpha}(xe, ye) = (x-y)^{1-\alpha}(2\arctan\{(x-y)/(x+y)\})^\alpha,$$

where e is an arbitrary unit vector in X. Again, the case $\alpha = 1$ is particularly simple: $\rho_{T^\alpha}(xe, ye) = 2\arctan\{(x-y)/(x+y)\}$.

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