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ON GENERALIZATIONS OF L. YANG'S INEQUALITY

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ABSTRACT. A geometric inequality due to L. Yang involving cosine and sine functions is generalized.

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1. INTRODUCTION

The well-known inequality due to L. Yang [1, pp. 116–118] can be stated as follows:
If $A > 0$, $B > 0$, $A + B \leq \pi$ and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A + \cos^2 \lambda B - 2 \cos \lambda A \cdot \cos \lambda B \cdot \cos \lambda\pi \geq \sin^2 \lambda\pi. \tag{1.1}$$

The equality in (1.1) holds if and only if $\lambda = 0$ or $A + B = \pi$.

L. Yang's inequality plays an important role in the theory of distribution of values of functions. See [1].

In this paper we will generalize inequality (1.1).

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2. MAIN RESULTS

In this paper, we use the following notations and abbreviations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!};$$
$$x_{ij}^{[1]} = \frac{\lambda}{2}(\pi + A_i + A_j), \quad x_{ij}^{[2]} = \frac{\lambda}{2}(\pi - A_i - A_j);$$
$$x_{ij}^{[3]} = \frac{\lambda}{2}(\pi + A_i - A_j), \quad x_{ij}^{[4]} = \frac{\lambda}{2}(\pi - A_i + A_j);$$
$$H_{ij} = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 2\cos \lambda A_i \cdot \cos \lambda A_j \cdot \cos \lambda\pi.$$

Lemma 2.1 ([2])**.** *If* $A_i > 0$*,* $A_j > 0$*,* $A_i + A_j \le \pi$ *for* $1 \le i, j \le n$*, and* $-1 \le \lambda \le 1$*, then*

$$H_{ij} - \sin^2 \lambda\pi = 4\prod_{k=1}^{4} \sin x_{ij}^{[k]}.$$

Proof. By using the following two identities

$$\cos\lambda\,(A_i + A_j)\cos\lambda\,(A_i - A_j) = \cos^2 \lambda A_i + \cos^2 \lambda A_j - 1,$$
$$\cos\lambda\,(A_i + A_j) + \cos\lambda\,(A_i - A_j) = 2\cos \lambda A_i \cos \lambda A_j,$$

it is easy to observe that

$$\begin{aligned} H_{ij} - \sin^2 \lambda\pi &= \cos^2 \lambda A_i + \cos^2 \lambda A_j - 1 - 2\cos\lambda A_i \cos\lambda A_j \cos\lambda\pi + \cos^2 \lambda\pi \\ &= \cos^2 \lambda\pi + \cos\lambda\,(A_i + A_j)\cos\lambda\,(A_i - A_j) \\ &\quad - [\cos\lambda\,(A_i + A_j) + \cos\lambda\,(A_i - A_j)]\cos\lambda\pi \\ &= [\cos\lambda\pi - \cos\lambda\,(A_i + A_j)]\,[\cos\lambda\pi - \cos\lambda\,(A_i - A_j)] \\ &= 4\sin\frac{\lambda}{2}\,(\pi + A_i + A_j)\sin\frac{\lambda}{2}\,(\pi - A_i - A_j) \\ &\quad \times \sin\frac{\lambda}{2}\,(\pi + A_i - A_j)\sin\frac{\lambda}{2}\,(\pi - A_i + A_j)\,. \end{aligned}$$

The proof is complete. ☐

Theorem 2.2. *If* $A_i > 0$*,* $\lambda_k > 0$ $(i = 1, 2, \ldots, n$*;* $k = 1, 2, 3, 4)$*,* $\sum_{i=1}^{n} A_i \le \pi$*,* $n \ge 2$ *being a natural number, and* $-1 \le \lambda \le 1$*, then*

$$\begin{aligned} \binom{n}{2}\sin^2\lambda\pi &\le (n - 1 + \cos\lambda\pi)\sum_{k=1}^{n}\cos^2\lambda A_k - \cos\lambda\pi\left[\sum_{i=1}^{n}\cos\lambda A_i\right]^2 \\ &\le \binom{n}{2}\sin^2\lambda\pi + \frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{64\prod_{k=1}^{4}\lambda_k}\sum_{1\le i<j\le n}\sin^4\theta_{ij}, \end{aligned} \tag{2.1}$$

where

$$\theta_{ij} = \frac{\sum_{k=1}^{4}\lambda_k x_{ij}^{[k]}}{\sum_{k=1}^{4}\lambda_k}. \tag{2.2}$$

The equalities in (2.1) *hold if and only if* $\lambda = 0$.

Proof. Since $y = \sin x$ is a continuous and convex (or concave, resp.) function on $[-\pi, 0]$ (or $[0, \pi]$, resp.), and $x_{ij}^{[k]} \in [-\pi, 0]$ (or $[0, \pi]$, resp.) for $-1 \le \lambda \le 0$ (or $0 \le \lambda \le 1$, resp.), and using Jensen's inequality (see [3]), we observe that

$$\sin\theta_{ij} \le (\text{or } \ge \text{, resp.}) \frac{\sum_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}}{\sum_{k=1}^{4} \lambda_k}.$$

Consequently

$$\left(\sum_{k=1}^{4} \lambda_k\right)^4 (\sin\theta_{ij})^4 \ge \left(\sum_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}\right)^4. \tag{2.3}$$

On the other hand, since $\lambda_k \sin\left(-x_{ij}^{[k]}\right) \ge 0$ $\left(\text{or } \lambda_k \sin\left(x_{ij}^{[k]}\right) \ge 0, \text{ resp.}\right)$, then

$$\begin{gathered} \frac{1}{4}\sum_{k=1}^{4} \lambda_k \sin\left(-x_{ij}^{[k]}\right) \ge \sqrt[4]{\prod_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}}. \\ \left(\text{or } \frac{1}{4}\sum_{k=1}^{4} \lambda_k \sin\left(x_{ij}^{[k]}\right) \ge \sqrt[4]{\prod_{k=1}^{4} \lambda_k \sin x_{ij}^{[k]}}\text{, resp.}\right). \end{gathered} \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\prod_{k=1}^{4} \sin x_{ij}^{[k]} \le \frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{4^4 \prod_{k=1}^{4} \lambda_k} \sin^4\theta_{ij}. \tag{2.5}$$

By using Lemma 2.1, we have

$$\sin^2\lambda\pi \le H_{ij} \le \frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sin^4\theta_{ij} + \sin^2\lambda\pi. \tag{2.6}$$

Summing both sides of (2.6) for $1 \le i < j \le n$ yields

$$\sum_{1\le i<j\le n} \sin^2\lambda\pi \le \sum_{1\le i<j\le n} H_{ij} \le \sum_{1\le i<j\le n} \left(\frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sin^4\theta_{ij} + \sin^2\lambda\pi\right). \tag{2.7}$$

It is not difficult to see that

$$\sum_{1\le i<j\le n} \sin^2\lambda\pi = \binom{n}{2} \sin^2\lambda\pi. \tag{2.8}$$

Direct computing yields

$$\begin{aligned} \sum_{1\le i<j\le n} &\left(\frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sin^4\theta_{ij} + \sin^2\lambda\pi\right) \\ &= \frac{\left(\sum_{k=1}^{4} \lambda_k\right)^4}{64 \prod_{k=1}^{4} \lambda_k} \sum_{1\le i<j\le n} \sin^4\theta_{ij} + \binom{n}{2} \sin^2\lambda\pi, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\sum_{1\le i<j\le n} H_{ij} &= \sum_{1\le i<j\le n} (\cos^2\lambda A_i + \cos^2\lambda A_j - 2\cos\lambda A_i\cos\lambda A_j\cos\lambda\pi) \\ &= \sum_{1\le i<j\le n} (\cos^2\lambda A_i + \cos^2\lambda A_j) - \sum_{1\le i<j\le n} 2\cos\lambda A_i\cos\lambda A_j\cos\lambda\pi \\ &= (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - \cos\lambda\pi\left[\left(\sum_{i=1}^{n}\cos\lambda A_i\right)^2 - \sum_{i=1}^{n}\cos^2\lambda A_i\right] \\ &= (n-1+\cos\lambda\pi)\sum_{k=1}^{n}\cos^2\lambda A_k - \cos\lambda\pi\left(\sum_{i=1}^{n}\cos\lambda A_i\right)^2. \end{aligned} \tag{2.10}$$

Putting (2.8), (2.9) and (2.10) into (2.7), inequality (2.7) reduces to inequality (2.1). The proof is complete. □

Remark 2.3. If taking $\lambda_i = \frac{1}{4}$ $(i = 1,2,3,4)$ in the right-hand of (2.7), we find that

$$\begin{aligned} &\sum_{1\le i<j\le n}\left(\frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{64\prod_{k=1}^{4}\lambda_k}\sin^4\theta_{ij} + \sin^2\lambda\pi\right) \\ &= \sum_{1\le i<j\le n}\left(\frac{\left(\sum_{k=1}^{4}\lambda_k\right)^4}{64\prod_{k=1}^{4}\lambda_k}\left(\sin\frac{\sum_{k=1}^{4}\lambda_k x_{ij}^{[k]}}{\sum_{k=1}^{4}\lambda_k}\right)^4 + \sin^2\lambda\pi\right) \\ &= \sum_{1\le i<j\le n}\left(4\left(\sin\left(\frac{1}{4}\sum_{k=1}^{4}x_{ij}^{[k]}\right)\right)^4 + \sin^2\lambda\pi\right) \\ &= \sum_{1\le i<j\le n}\left(4\sin^4\frac{\lambda}{2}\pi + \sin^2\lambda\pi\right) \\ &= 4\binom{n}{2}\sin^2\frac{\lambda\pi}{2}. \end{aligned} \tag{2.11}$$

Therefore, inequality (2.1) reduces to the following inequality

$$\begin{aligned}\binom{n}{2}\sin^2\lambda\pi &\le (n-1+\cos\lambda\pi)\sum_{k=1}^{n}\cos^2\lambda A_k - \cos\lambda\pi\left(\sum_{i=1}^{n}\cos\lambda A_i\right)^2 \\ &\le 4\binom{n}{2}\sin^2\frac{\lambda}{2}\pi. \end{aligned} \tag{2.12}$$

The equalities in (2.12) hold if and only if $\lambda = 0$.

Letting $n = 2$ in (2.12), we have the following result: If $A > 0$, $B > 0$, $A + B \le \pi$, and $-1 \le \lambda \le 1$, then

$$\sin^2\lambda\pi \le \cos^2\lambda A + \cos^2\lambda B - 2\cos\lambda A\cos\lambda B\cos\lambda\pi \le 4\sin^2\frac{\lambda}{2}\pi. \tag{2.13}$$

The equalities in (2.13) holds if and only if $\lambda = 0$.

It is obvious that inequality (2.13) is the same as L. Yang's inequality (1.1).

Theorem 2.4. *If $A_i > 0$ and $\lambda_k > 0$ for $i = 1, 2, \dots, n$ and $k = 1, 2, 3, 4$, $\sum_{i=1}^n A_i \le \pi$, $n \ge 2$, $-1 \le \lambda \le 1$, then*

$$0 \le (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - \binom{n}{2}\sin^2\lambda\pi - \cos\lambda\pi \sum_{\substack{1\le i,j\le n\\ i\ne j}}\cos\lambda A_i\cos\lambda A_j \le \frac{\left(\sum_{k=1}^4\lambda_k\right)^4}{128\prod_{k=1}^4\lambda_k}\sum_{\substack{1\le i,j\le n\\ i\ne j}}\sin^4\theta_{ij}, \tag{2.14}$$

where

$$\theta_{ij} = \frac{\sum_{k=1}^4 \lambda_k x_{ij}^{[k]}}{\sum_{k=1}^4\lambda_k}.$$

The equalities in (2.14) *hold if and only if* $\lambda = 0$.

Proof. It follows from inequality (2.6) that

$$0 \le H_{ij} - \sin^2\lambda\pi \le \frac{\left(\sum_{k=1}^4\lambda_k\right)^4}{64\prod_{k=1}^4\lambda_k}\sin^4\theta_{ij}. \tag{2.15}$$

Summing both sides of (2.15) for $i \ne j$, first over j from 1 to n and then over i from 1 to n of the resulting inequality, then

$$0 \le \sum_{\substack{1\le i,j\le n\\ i\ne j}} H_{ij} - 2\binom{n}{2}\sin^2\lambda\pi \le \frac{\left(\sum_{k=1}^4\lambda_k\right)^4}{64\prod_{k=1}^4\lambda_k}\sum_{\substack{1\le i,j\le n\\ i\ne j}}\sin^4\theta_{ij}. \tag{2.16}$$

On the other hand

$$\sum_{\substack{1\le i,j\le n\\ i\ne j}} H_{ij} = 2(n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - 2\cos\lambda\pi\sum_{\substack{1\le i,j\le n\\ i\ne j}}\cos\lambda A_i\cos\lambda A_j. \tag{2.17}$$

Putting (2.17) into (2.16), we get inequality (2.14). □

Remark 2.5. In (2.14), if $\lambda_k = \frac{1}{4}$ for $k = 1, 2, 3, 4$, then

$$0 \le (n-1)\sum_{k=1}^{n}\cos^2\lambda A_k - \binom{n}{2}\sin^2\lambda\pi - \cos\lambda\pi \sum_{\substack{1\le i,j\le n\\ i\ne j}}\cos\lambda A_i\cos\lambda A_j \le \binom{n}{2}\sin^4\frac{\lambda}{2}\pi. \tag{2.18}$$

The equalities in (2.18) hold if and only if $\lambda = 0$.

Letting $n = 2$, then (2.18) reduces to inequality (2.13).

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