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# ON MODULI OF EXPANSION OF THE DUALITY MAPPING OF SMOOTH BANACH SPACES

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## Abstract

Let $X$ be a Banach space which is uniformly convex and uniformly smooth. We introduce the lower and upper moduli of expansion of the dual mapping $J$ of the space $X$. Some estimation of certain well-known moduli (convexity, smoothness and flatness) and two new moduli introduced in [5] are described with this new moduli of expansion.

Let $(X, \|\cdot\|)$ be a real normed space, $X^*$ its conjugate space, $X^{**}$ the second conjugate of $X$ and $S(X)$ the unit sphere in $X$ $(S(X) = \{x \in X | \, \|x\| = 1\})$.

Moreover, we shall use the following definitions and notations.

The sign $(S)$ denotes that $X$ is smooth, $(R)$ that $X$ is reflexive, $(US)$ that $X$ is uniformly smooth, $(SC)$ that $X$ is strictly convex, and $(UC)$ that $X$ is uniformly convex.

The map $J : X \to 2^{X^*}$ is called the dual map if $J(0) = 0$ and for $x \in X$, $x \neq 0$,

$$J(x) = \{f \in X^* | f(x) = \|f\| \|x\|, \|f\| = \|x\|\}.$$

The dual map of $X^*$ into $2^{X^{**}}$ we denote by $J^*$. The map $\tau$ is canonical linear isometry of $X$ into $X^{**}$.

It is well known that functional

$$(1) \qquad g(x,y) := \frac{\|x\|}{2} \left( \lim_{t \to -0} \frac{\|x+ty\| - \|x\|}{t} + \lim_{t \to +0} \frac{\|x+ty\| - \|x\|}{t} \right)$$

always exists on $X^2$. If $X$ is $(S)$, then (1) reduces to

$$g(x,y) = \|x\| \lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t};$$

the functional $g$ is linear in the second argument, $J(x)$ is a singleton and $g(x,\cdot) \in J(x)$. In this case we shall write $J(x) = Jx = f_x$. Then $[y,x] := g(x,y)$, defines a so called semi-inner product $[\cdot,\cdot]$ (s.i.p) on $X^2$ which generates the norm of $X$, $\left([x,x] = \|x\|^2\right)$, (see [1]). If $X$ is an inner-product space (i.p. space) then $g(x,y)$ is the usual i.p. of the vector $x$ and the vector $y$.

By the use of functional $g$ we define the angle between vector $x$ and vector $y$ $(x \neq 0,\ y \neq 0)$ as

$$\cos(x,y) := \frac{g(x,y) + g(y,x)}{2\|x\|\|y\|} \tag{2}$$

(see [3]). If $(X,(\cdot,\cdot))$ is an i.p. space, then (2) reduces to

$$\cos(x,y) = \frac{(x,y)}{\|x\|\|y\|}.$$

We say that $X$ is a quasi-inner product space (q.i.p space) if the following equality holds

$$\|x+y\|^4 - \|x-y\|^4 = 8\left[\|x\|^2 g(x,y) + \|y\|^2 g(y,x)\right], \quad (x,y \in X)^{1} \tag{3}$$

[1]If $(\cdot,\cdot)$ is an i.p. on $X^2$ then $g(x,y) = (x,y)$ and the equality (3) is the parallelogram equality.

The equality (3) holds in the space $l^4$, but does not hold in the space $l^1$. A q.i.p. space $X$ is $(SC)$ and $(US)$ (see [6] and [4]).

Alongside the modulus of convexity of $X$, $\delta_X$, and the modulus of smoothness of $X$, $\rho_X$, defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| \,\middle|\, x,y \in S(X);\ \|x-y\| \geq \varepsilon\right\};$$

$$\rho_X(\varepsilon) = \sup\left\{1 - \left\|\frac{x+y}{2}\right\| \,\middle|\, x,y \in S(X);\ \|x-y\| \leq \varepsilon\right\};$$

we have defined in [5] the angle modulus of convexity of $X$, $\delta'_X$, and the angle modulus of smoothness of $X$, $\rho'_X$ by:

$$\delta'_X(\varepsilon) = \inf\left\{\frac{1-\cos(x,y)}{2} \,\middle|\, x,y \in S(X);\ \|x-y\| \geq \varepsilon\right\};$$

$$\rho'_X(\varepsilon) = \sup\left\{\frac{1-\cos(x,y)}{2} \,\middle|\, x,y \in S(X);\ \|x-y\| \leq \varepsilon\right\}.$$

We also recall the known definition of modulus of flatness of $X$, $\eta_X$ (Day's modulus):

$$\eta_X(\varepsilon) = \sup\left\{\frac{2-\|x+y\|}{\|x-y\|} \,\middle|\, x,y \in S(X);\ \|x-y\| \leq \varepsilon\right\}.$$

We now quote three known results.

**Lemma 1.** *(Theorem 6 in [7] and Theorem 6 in [1]). Let $X$ be a real normed space which is $(S)$, $(SC)$ and $(R)$. Then for all $f \in X^*$ there exists a unique $x \in X$ such that*

$$f(y) = g(x,y), \quad (y \in X).$$

**Lemma 2.** *(Theorem 7 in [1]). Let $X$ be a Banach space which is $(US)$ and $(UC)$ and let $[\cdot,\cdot]$ be an s.i.p. on $X^2$ which generates the norm on $X$ (see [1]). Then the dual space $X^*$ is $(US)$ and $(UC)$ and the functional*

$$\langle Jx, Jy\rangle := [y, x], \quad (x, y \in X),$$

*is an s.i.p on $(X^*)^2$.*

**Lemma 3.** *(Proposition 3 in [2]). Let $X$ be a real normed space. Then for $J, J^*$ and $\tau$ on their respective domains we have*

$$J^{-1} = \tau^{-1}J^* \text{ and } J = J^{*-1}\tau.$$

**Remark 1.** *Under the hypothesis of Lemma 2, the mappings $J, J^*$ and $\tau$ are bijective mappings. Then, by Lemma 3, Lemma 2 and Lemma 1, in this case, we have*

$$\langle Jx, Jy\rangle = g(x, y) = g(f_y, f_x), \quad (x, y \in X).$$

**Lemma 4.** *Let $X$ be a real normed space which is $(S)$, $(SC)$ and $(R)$. Then for $x, y \in S(X)$ we have*

$$1 - \left\|\frac{x+y}{2}\right\| \le \frac{1-\cos(x,y)}{2} \le \frac{\|x-y\|\,\|f_x - f_y\|}{4}. \tag{4}$$

*Proof.* Under the hypothesis of Lemma 4, using Lemma 1, we have $f_x = g(x, \cdot)$ $(x \in X)$. Consequently,

$$\begin{aligned}\|f_x - f_y\| &= \sup\{|g(x,t) - g(y,t)| \mid t \in S(X)\}\\ &\ge g(x,t) - g(y,t) \quad (t \in S(X)).\end{aligned}$$

For $t = \frac{x-y}{\|x-y\|}, (x \neq y)$, we obtain

$$g\left(x, \frac{x-y}{\|x-y\|}\right) - g\left(y, \frac{x-y}{\|x-y\|}\right) \leq \|f_x - f_y\|. \tag{5}$$

Since $X$ is $(S)$, the functional $g$ is linear in the second argument. Hence, from (5) we get

$$1 - g(x,y) - g(y,x) + 1 \leq \|x-y\| \|f_x - f_y\|. \tag{6}$$

Using the inequality

$$1 - \left\|\frac{x+y}{2}\right\| \leq \frac{1-\cos(x,y)}{2} \leq \frac{\|x-y\|}{2}$$

(see Lemma 1 in [5]) and the inequality (6) we obtain the inequality (4). □

**Lemma 5.** *Let $X$ be a Banach space which is $(US)$ and $(UC)$. Let $\delta_{X^*}$ be the modulus of convexity of $X^*$. Then for each $\varepsilon > 0$ and for all $x, y \in S(X)$ the following implications hold*

$$\|x-y\| \leq 2\delta_{X^*}(\varepsilon) \Longrightarrow \|f_x - f_y\| \leq \varepsilon, \tag{7}$$

$$\|f_x - f_y\| \geq \varepsilon \Longrightarrow \|x-y\| \geq 2\delta_{X^*}(\varepsilon). \tag{8}$$

*Proof.* By Lemma 2, $X^*$ is a Banach space which is $(UC)$ and $(US)$. Since $X^*$ is $(UC)$, for each $\varepsilon > 0$, we have $\delta_{X^*}(\varepsilon) > 0$ and, for all $x, y \in S(X)$,

$$\|f_x + f_y\| > 2 - 2\delta_{X^*}(\varepsilon) \Longrightarrow \|f_x - f_y\| < \varepsilon. \tag{9}$$

Under the hypothesis of Lemma 5, by Remark 1, we have $g(x,y) = g(f_y, f_x)$. Hence, by inequality

$$1 - \|x-y\| \leq g(x,y) \leq \|x+y\| - 1$$

(see Lemma 1 in [6]), we obtain

$$1 - \|x-y\| \leq g(x,y) = g(f_y, f_x) \leq \|f_x + f_y\| - 1, \tag{10}$$

so that we have

$$\|x-y\| + \|f_x + f_y\| \geq 2. \tag{11}$$

Now, let $x, y \in S(X)$ and $\|x-y\| < 2\delta_{X^*}(\varepsilon)$. Then, by (11) we obtain

$$\|f_x + f_y\| > 2 - 2\delta_{X^*}(\varepsilon).$$

Thus, by (9), we conclude that

$$\|x-y\| < 2\delta_{X^*}(\varepsilon) \Longrightarrow \|f_x - f_y\| < \varepsilon. \tag{12}$$

On the other hand if $\|x-y\| = 2\delta_{X^*}(\varepsilon)$ and $\|f_x - f_y\| > \varepsilon$, by (9), it follows

$$\|x-y\| + \|f_x + f_y\| \leq 2.$$

So, by (11), we get

$$\|x-y\| + \|f_x + f_y\| = 2.$$

Hence, using (10), we conclude that $g(x,y) = 1 - \|x-y\|$, i.e., $g(x, x-y) = \|x\| \|x-y\|$. Thus, since $X$ is $(SC)$, using Lemma 5 in [1], we get $x = x - y$, which is impossible. So, the implication (7) is correct. The implication (8) follows from the implication (12). □

We now introduce a new definition.

According to the inequality (4), to make further progress in the estimates of the moduli $\delta_X, \delta'_X, \rho_X, \rho'_X$, it is convenient to introduce

**Definition 1.** *Let* $X$ *be* $(S)$ *and* $x, y \in S(X)$. *The function* $\underline{e_J}\colon [0,2] \to [0,2]$, *defined by*

$$\underline{e_J}(\varepsilon) := \inf\{\|f_x - f_y\| \mid \|x - y\| \geq \varepsilon\}$$

*will be called the lower modulus of expansion of the dual mapping* $J$.

*The function* $\overline{e_J} : [0,2] \to [0,2]$, *defined as*

$$\overline{e_J}(\varepsilon) := \sup\{\|f_x - f_y\| \mid \|x - y\| \leq \varepsilon\}$$

*is the upper modulus of expansion of the dual mapping* $J$.

Now, we quote our new results. Firstly, we note some elementary properties of the moduli $\underline{e_J}$ and $\overline{e_J}$.

**Theorem 6.** *Let* $X$ *be* $(S)$. *Then the following assertions are valid.*

a) *The function* $\underline{e_J}$ *is nondecreasing on* $[0,2]$.

b) *The function* $\overline{e_J}$ *is nondecreasing on* $[0,2]$.

c) $\underline{e_J}(\varepsilon) \leq \overline{e_J}(\varepsilon) \quad (\varepsilon \in [0,2])$.

d) *If* $X$ *is a Hilbert space, then* $\underline{e_J}(\varepsilon) = \overline{e_J}(\varepsilon)$.

*Proof.* The assertions a) and b) follow from the implications

$$\varepsilon_1 < \varepsilon_2 \Longrightarrow \{(x,y) \mid \|x-y\| \geq \varepsilon_1\} \supset \{(x,y) \mid \|x-y\| \geq \varepsilon_2\} \qquad (x,y \in S(X)),$$

$$\varepsilon_1 < \varepsilon_2 \Longrightarrow \{(x,y) \mid \|x-y\| \leq \varepsilon_1\} \subset \{(x,y) \mid \|x-y\| \leq \varepsilon_2\} \qquad (x,y \in S(X)).$$

c) Assume, to the contrary, i.e., that there is an $\varepsilon \in [0,2]$ such that $\underline{e_J}(\varepsilon) > \overline{e_J}(\varepsilon)$. Then

$$\begin{aligned} \inf\{\|f_x - f_y\| \mid \|x-y\| = \varepsilon\} &\geq \inf\{\|f_x - f_y\| \mid \|x-y\| \geq \varepsilon\} \\ &> \sup\{\|f_x - f_y\| \mid \|x-y\| \leq \varepsilon\} \\ &\geq \sup\{\|f_x - f_y\| \mid \|x-y\| = \varepsilon\}, \end{aligned}$$

which is not possible.

d) In a Hilbert space, we have

$$\|f_x - f_y\| = \sup\{|(x,t)-(y,t)| \mid t \in S(X)\} \leq \|x-y\|.$$

On the other hand, the functional $f_x - f_y$ attains its maximum in $t = \frac{x-y}{\|x-y\|} \in S(X)$.

Hence $\|x-y\| = \|f_x - f_y\|$. Because of that, we have $\underline{e_J}(\varepsilon) = \overline{e_J}(\varepsilon) = \varepsilon$. □

In the next theorems some relation between moduli $\delta'_X, \rho'_X, \underline{e_J}, \overline{e_J}$ are given.

**Theorem 7.** *Let $X$ be $(S)$, $(SC)$ and $(R)$. Then, for $\varepsilon \in (0,2]$ we have*

*a)* $\delta'_X(\varepsilon) \leq \frac{1}{2}\underline{e_J}(\varepsilon)$

*b)* $\rho'_X(\varepsilon) \leq \frac{\varepsilon}{4}\overline{e_J}(\varepsilon)$,

*c)* $\frac{2}{\varepsilon}\rho_X(\varepsilon) \leq \eta_X(\varepsilon) \leq \frac{1}{2}\overline{e_J}(\varepsilon)$.

*Proof.* The proof of the assertions a) and b) follows immediately using the definitions of the functions $\delta'_X$ and $\rho'_X$ and the inequality (4).

c) Let $x, y \in S(X)$, $x \neq y$. By Lemma 4, we have

$$\begin{aligned} \frac{2-\|x+y\|}{\|x-y\|} &= \frac{2}{\|x-y\|}\left(1-\frac{\|x+y\|}{2}\right) \\ &\leq \frac{1-\cos(x,y)}{\|x-y\|} \\ &\leq \frac{\|x-y\|\,\|f_x-f_y\|}{2\,\|x-y\|} \\ &= \frac{\|f_x-f_y\|}{2}. \end{aligned}$$

So

$$\frac{2-\|x+y\|}{\|x-y\|} \leq \frac{\|f_x-f_y\|}{2}.$$

Using the definition of $\eta_X$ and $\overline{e_J}$, we obtain

$$\eta_X(\varepsilon) \leq \frac{1}{2}\overline{e_J}(\varepsilon).$$

On the other hand

$$(0<\|x-y\|\leq\varepsilon)\Longrightarrow\left(\frac{1}{\|x-y\|}\geq\frac{1}{\varepsilon}\right)$$
$$\Longrightarrow\frac{2-\|x+y\|}{\|x-y\|}\geq\frac{2}{\varepsilon}\left(1-\frac{\|x+y\|}{2}\right).$$

Because of that we have

$$\eta_X(\varepsilon)\geq\frac{2}{\varepsilon}\rho_X(\varepsilon).$$

□

**Remark 2.** *The last inequality is true for an arbitrary space $X$.*

**Corollary 8.** *For a q.i.p. space, it holds that*

$$\underline{e_J}(\varepsilon)\geq\left(\frac{\varepsilon}{2}\right)^4\quad(\varepsilon\in[0,2]). \tag{13}$$

*Proof.* By a) of Theorem 7 and the inequality $\frac{\varepsilon^4}{32}\leq\delta'_X(\varepsilon)$ (see Corollary 2 in [5]), we get (13). □

**Corollary 9.** *If $X$ is $(S)$, $(SC)$ and $(R)$ then*

*a)* $\delta'_{X^*}(\varepsilon)\leq\frac{1}{2}\underline{e_J}(\varepsilon)$,

*b)* $\rho'_{X^*}\leq\frac{1}{2}\overline{e_{J^*}}(\varepsilon)$,

c) $\frac{2}{3}\rho_{X^*}(\varepsilon) \leq \eta_{X^*}(\varepsilon) \leq \frac{1}{2}\overline{e_{J^*}}(\varepsilon)$.

*Proof.* It is well-known that if $X$ is $(S)$, $(SC)$ and $(R)$ then $X^*$ is $(S)$, $(SC)$ and $(R)$. Hence Theorem 7 is valid for $X^*$. □

**Theorem 10.** *Let $X$ be a Banach space which is $(UC)$ and $(US)$. Then, for all $\varepsilon > 0$, we have the following estimations:*

a) $\rho'_X(2\delta_{X^*}(\varepsilon)) \leq \dfrac{\varepsilon\delta_{X^*}(\varepsilon)}{2}$,

b) $\rho'_{X^*}(2\delta_X(\varepsilon)) \leq \dfrac{\varepsilon\delta_X(\varepsilon)}{2}$,

c) $\underline{e_{J^*}}(\varepsilon) \geq 2\delta_{X^*}(\varepsilon)$,

d) $\overline{e_J}(2\delta_{X^*}(\varepsilon)) \leq \varepsilon, \quad (\overline{e_{J^*}}(2\delta_X(\varepsilon)) \leq \varepsilon)$.

*Proof.* a) Using, in succession, the definition of the function $\rho'_X$, the inequality (4) in Lemma 2 and the implication (7), we obtain:

$$\begin{aligned}
\rho'_X(2\delta_{X^*}(\varepsilon)) &= \sup\left\{ \frac{1-\cos(x,y)}{2} \,\middle|\, \|x-y\| \leq 2\delta_{X^*}(\varepsilon) \right\} \\
&\leq \frac{1}{4}\sup\left\{ \|x-y\|\,\|f_x - f_y\| \mid \|x-y\| \leq 2\delta_{X^*}(\varepsilon) \right\} \\
&\leq \frac{1}{4}2\varepsilon\delta_{X^*}(\varepsilon) \\
&= \frac{\varepsilon\delta_{X^*}(\varepsilon)}{2}.
\end{aligned}$$

b) If, in a), we set $X^*$ instead of $X$ ($X^{**}$ instead of $X^*$), we get

$$\rho'_{X^*}\left(2\delta_{X^{**}}(\varepsilon)\right) \leq \frac{\varepsilon\delta_{X^{**}}(\varepsilon)}{2}. \tag{14}$$

Let $F, G \in S\left(X^{**}\right)$. Under the hypothesis of Theorem 10, we have

$$\begin{aligned}
\delta_{X^{**}}(\varepsilon) &= \inf\left\{1 - \frac{\|F+G\|}{2} \,\middle|\, \|F-G\| \geq \varepsilon\right\} \\
&= \inf\left\{1 - \frac{\|\tau x + \tau y\|}{2} \,\middle|\, \|\tau x - \tau y\| \geq \varepsilon\right\} \\
&= \inf\left\{1 - \frac{\|\tau(x+y)\|}{2} \,\middle|\, \|\tau(x-y)\| \geq \varepsilon\right\} \\
&= \inf\left\{1 - \frac{\|x+y\|}{2} \,\middle|\, \|x-y\| \geq \varepsilon\right\} \\
&= \delta_X(\varepsilon).
\end{aligned}$$

Consequently the inequality (14) is equivalent to the inequality b).

c) Using, in succession, the definition of $\underline{e}_J$, Lemma 3, and the implication (8), we get

$$\begin{aligned}
\underline{e}_{J^*}(\varepsilon) &= \inf\left\{\|J^* f_x - J^* f_y\| \mid \|f_x - f_y\| \geq \varepsilon\right\} \\
&= \inf\left\{\|\tau x - \tau y\| \mid \|f_x - f_y\| \geq \varepsilon\right\} \\
&\geq 2\delta_{X^*}(\varepsilon).
\end{aligned}$$

d) Using the definition of $\overline{e_J}$ and the implication (7), we get

$$\overline{e_J}(2\delta_{X^*}(\varepsilon)) = \sup\{\|f_x - f_y\| \mid \|x-y\| \le 2\delta_{X^*}(\varepsilon)\} \le \varepsilon.$$

Replacing, here, $X^*$ with $X^{**}$ and $J$ with $J^*$, we get the second inequality.

□

Since in a Banach space $X$ we have

$$\delta_X(\varepsilon) \le 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \quad \text{and} \quad \delta_X(\varepsilon) \le \delta'_X(\varepsilon)$$

(see Theorem 1 in [5]), using b) and a) of Theorem 10, we obtain

**Corollary 11.** *Under the hypothesis of Theorem 10, we have*

*a)* $\frac{2}{\varepsilon}\rho'_{X^*}(2\delta_X(\varepsilon)) \le \delta_X(\varepsilon) \le \frac{2}{\varepsilon}\delta'_X(\varepsilon)$,

*b)* $\rho'_X(2\delta_{X^*}(\varepsilon)) \le \frac{\varepsilon}{2}\left(1 - \sqrt{1 - \frac{\varepsilon^2}{4}}\right)$.

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