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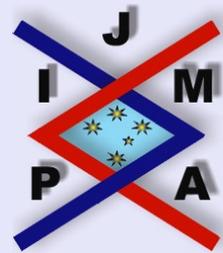
## ON MODULI OF EXPANSION OF THE DUALITY MAPPING OF SMOOTH BANACH SPACES

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Abstract

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## Abstract

Let  $X$  be a Banach space which is uniformly convex and uniformly smooth. We introduce the lower and upper moduli of expansion of the dual mapping  $J$  of the space  $X$ . Some estimation of certain well-known moduli (convexity, smoothness and flatness) and two new moduli introduced in [5] are described with this new moduli of expansion.

Let  $(X, \|\cdot\|)$  be a real normed space,  $X^*$  its conjugate space,  $X^{**}$  the second conjugate of  $X$  and  $S(X)$  the unit sphere in  $X$  ( $S(X) = \{x \in X \mid \|x\| = 1\}$ ).

Moreover, we shall use the following definitions and notations.

The sign  $(S)$  denotes that  $X$  is smooth,  $(R)$  that  $X$  is reflexive,  $(US)$  that  $X$  is uniformly smooth,  $(SC)$  that  $X$  is strictly convex, and  $(UC)$  that  $X$  is uniformly convex.

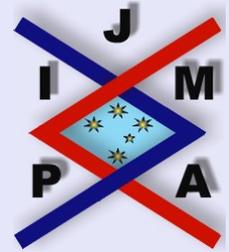
The map  $J : X \rightarrow 2^{X^*}$  is called the dual map if  $J(0) = 0$  and for  $x \in X$ ,  $x \neq 0$ ,

$$J(x) = \{f \in X^* \mid f(x) = \|f\| \|x\|, \|f\| = \|x\|\}.$$

The dual map of  $X^*$  into  $2^{X^{**}}$  we denote by  $J^*$ . The map  $\tau$  is canonical linear isometry of  $X$  into  $X^{**}$ .

It is well known that functional

$$(1) \quad g(x, y) := \frac{\|x\|}{2} \left( \lim_{t \rightarrow -0} \frac{\|x + ty\| - \|x\|}{t} + \lim_{t \rightarrow +0} \frac{\|x + ty\| - \|x\|}{t} \right)$$



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always exists on  $X^2$ . If  $X$  is  $(S)$ , then (1) reduces to

$$g(x, y) = \|x\| \lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t};$$

the functional  $g$  is linear in the second argument,  $J(x)$  is a singleton and  $g(x, \cdot) \in J(x)$ . In this case we shall write  $J(x) = Jx = f_x$ . Then  $[y, x] := g(x, y)$ , defines a so called semi-inner product  $[\cdot, \cdot]$  (s.i.p) on  $X^2$  which generates the norm of  $X$ ,  $([x, x] = \|x\|^2)$ , (see [1]). If  $X$  is an inner-product space (i.p. space) then  $g(x, y)$  is the usual i.p. of the vector  $x$  and the vector  $y$ .

By the use of functional  $g$  we define the angle between vector  $x$  and vector  $y$  ( $x \neq 0, y \neq 0$ ) as

$$(2) \quad \cos(x, y) := \frac{g(x, y) + g(y, x)}{2 \|x\| \|y\|}$$

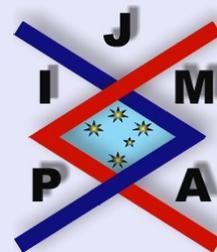
(see [3]). If  $(X, (\cdot, \cdot))$  is an i.p. space, then (2) reduces to

$$\cos(x, y) = \frac{(x, y)}{\|x\| \|y\|}.$$

We say that  $X$  is a quasi-inner product space (q.i.p space) if the following equality holds

$$(3) \quad \|x + y\|^4 - \|x - y\|^4 = 8 [\|x\|^2 g(x, y) + \|y\|^2 g(y, x)], \quad (x, y \in X)^1$$

<sup>1</sup>If  $(\cdot, \cdot)$  is an i.p. on  $X^2$  then  $g(x, y) = (x, y)$  and the equality (3) is the parallelogram equality.



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The equality (3) holds in the space  $l^4$ , but does not hold in the space  $l^1$ . A q.i.p. space  $X$  is (SC) and (US) (see [6] and [4]).

Alongside the modulus of convexity of  $X$ ,  $\delta_X$ , and the modulus of smoothness of  $X$ ,  $\rho_X$ , defined by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| \mid x, y \in S(X); \|x-y\| \geq \varepsilon \right\};$$

$$\rho_X(\varepsilon) = \sup \left\{ 1 - \left\| \frac{x+y}{2} \right\| \mid x, y \in S(X); \|x-y\| \leq \varepsilon \right\};$$

we have defined in [5] the angle modulus of convexity of  $X$ ,  $\delta'_X$ , and the angle modulus of smoothness of  $X$ ,  $\rho'_X$  by:

$$\delta'_X(\varepsilon) = \inf \left\{ \frac{1 - \cos(x, y)}{2} \mid x, y \in S(X); \|x-y\| \geq \varepsilon \right\};$$

$$\rho'_X(\varepsilon) = \sup \left\{ \frac{1 - \cos(x, y)}{2} \mid x, y \in S(X); \|x-y\| \leq \varepsilon \right\}.$$

We also recall the known definition of modulus of flatness of  $X$ ,  $\eta_X$  (Day's modulus):

$$\eta_X(\varepsilon) = \sup \left\{ \frac{2 - \|x+y\|}{\|x-y\|} \mid x, y \in S(X); \|x-y\| \leq \varepsilon \right\}.$$

We now quote three known results.

**Lemma 1.** (Theorem 6 in [7] and Theorem 6 in [1]). Let  $X$  be a real normed space which is (S), (SC) and (R). Then for all  $f \in X^*$  there exists a unique  $x \in X$  such that

$$f(y) = g(x, y), \quad (y \in X).$$



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**Lemma 2.** (Theorem 7 in [1]). Let  $X$  be a Banach space which is (US) and (UC) and let  $[\cdot, \cdot]$  be an s.i.p. on  $X^2$  which generates the norm on  $X$  (see [1]). Then the dual space  $X^*$  is (US) and (UC) and the functional

$$\langle Jx, Jy \rangle := [y, x], \quad (x, y \in X),$$

is an s.i.p. on  $(X^*)^2$ .

**Lemma 3.** (Proposition 3 in [2]). Let  $X$  be a real normed space. Then for  $J, J^*$  and  $\tau$  on their respective domains we have

$$J^{-1} = \tau^{-1}J^* \quad \text{and} \quad J = J^{*-1}\tau.$$

**Remark 1.** Under the hypothesis of Lemma 2, the mappings  $J, J^*$  and  $\tau$  are bijective mappings. Then, by Lemma 3, Lemma 2 and Lemma 1, in this case, we have

$$\langle Jx, Jy \rangle = g(x, y) = g(f_y, f_x), \quad (x, y \in X).$$

**Lemma 4.** Let  $X$  be a real normed space which is (S), (SC) and (R). Then for  $x, y \in S(X)$  we have

$$(4) \quad 1 - \left\| \frac{x+y}{2} \right\| \leq \frac{1 - \cos(x, y)}{2} \leq \frac{\|x-y\| \|f_x - f_y\|}{4}.$$

*Proof.* Under the hypothesis of Lemma 4, using Lemma 1, we have  $f_x = g(x, \cdot)$  ( $x \in X$ ). Consequently,

$$\begin{aligned} \|f_x - f_y\| &= \sup \{ |g(x, t) - g(y, t)| \mid t \in S(X) \} \\ &\geq g(x, t) - g(y, t) \quad (t \in S(X)). \end{aligned}$$



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For  $t = \frac{x-y}{\|x-y\|}$ , ( $x \neq y$ ), we obtain

$$(5) \quad g\left(x, \frac{x-y}{\|x-y\|}\right) - g\left(y, \frac{x-y}{\|x-y\|}\right) \leq \|f_x - f_y\|.$$

Since  $X$  is  $(S)$ , the functional  $g$  is linear in the second argument. Hence, from (5) we get

$$(6) \quad 1 - g(x, y) - g(y, x) + 1 \leq \|x - y\| \|f_x - f_y\|.$$

Using the inequality

$$1 - \left\| \frac{x+y}{2} \right\| \leq \frac{1 - \cos(x, y)}{2} \leq \frac{\|x - y\|}{2}$$

(see Lemma 1 in [5]) and the inequality (6) we obtain the inequality (4).  $\square$

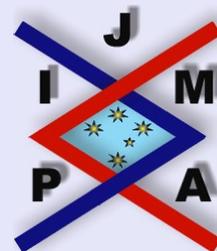
**Lemma 5.** *Let  $X$  be a Banach space which is  $(US)$  and  $(UC)$ . Let  $\delta_{X^*}$  be the modulus of convexity of  $X^*$ . Then for each  $\varepsilon > 0$  and for all  $x, y \in S(X)$  the following implications hold*

$$(7) \quad \|x - y\| \leq 2\delta_{X^*}(\varepsilon) \implies \|f_x - f_y\| \leq \varepsilon,$$

$$(8) \quad \|f_x - f_y\| \geq \varepsilon \implies \|x - y\| \geq 2\delta_{X^*}(\varepsilon).$$

*Proof.* By Lemma 2,  $X^*$  is a Banach space which is  $(UC)$  and  $(US)$ . Since  $X^*$  is  $(UC)$ , for each  $\varepsilon > 0$ , we have  $\delta_{X^*}(\varepsilon) > 0$  and, for all  $x, y \in S(X)$ ,

$$(9) \quad \|f_x + f_y\| > 2 - 2\delta_{X^*}(\varepsilon) \implies \|f_x - f_y\| < \varepsilon.$$



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Under the hypothesis of Lemma 5, by Remark 1, we have  $g(x, y) = g(f_y, f_x)$ . Hence, by inequality

$$1 - \|x - y\| \leq g(x, y) \leq \|x + y\| - 1$$

(see Lemma 1 in [6]), we obtain

$$(10) \quad 1 - \|x - y\| \leq g(x, y) = g(f_y, f_x) \leq \|f_x + f_y\| - 1,$$

so that we have

$$(11) \quad \|x - y\| + \|f_x + f_y\| \geq 2.$$

Now, let  $x, y \in S(X)$  and  $\|x - y\| < 2\delta_{X^*}(\varepsilon)$ . Then, by (11) we obtain

$$\|f_x + f_y\| > 2 - 2\delta_{X^*}(\varepsilon).$$

Thus, by (9), we conclude that

$$(12) \quad \|x - y\| < 2\delta_{X^*}(\varepsilon) \implies \|f_x - f_y\| < \varepsilon.$$

On the other hand if  $\|x - y\| = 2\delta_{X^*}(\varepsilon)$  and  $\|f_x - f_y\| > \varepsilon$ , by (9), it follows

$$\|x - y\| + \|f_x + f_y\| \leq 2.$$

So, by (11), we get

$$\|x - y\| + \|f_x + f_y\| = 2.$$

Hence, using (10), we conclude that  $g(x, y) = 1 - \|x - y\|$ , i.e.,  $g(x, x - y) = \|x\| \|x - y\|$ . Thus, since  $X$  is  $(SC)$ , using Lemma 5 in [1], we get  $x = x - y$ , which is impossible. So, the implication (7) is correct. The implication (8) follows from the implication (12).  $\square$



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We now introduce a new definition.

According to the inequality (4), to make further progress in the estimates of the moduli  $\delta_X, \delta'_X, \rho_X, \rho'_X$ , it is convenient to introduce

**Definition 1.** Let  $X$  be  $(S)$  and  $x, y \in S(X)$ . The function  $e_J: [0, 2] \rightarrow [0, 2]$ , defined by

$$e_J(\varepsilon) := \inf \{ \|f_x - f_y\| \mid \|x - y\| \geq \varepsilon \}$$

will be called the lower modulus of expansion of the dual mapping  $J$ .

The function  $\bar{e}_J: [0, 2] \rightarrow [0, 2]$ , defined as

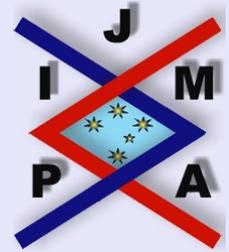
$$\bar{e}_J(\varepsilon) := \sup \{ \|f_x - f_y\| \mid \|x - y\| \leq \varepsilon \}$$

is the upper modulus of expansion of the dual mapping  $J$ .

Now, we quote our new results. Firstly, we note some elementary properties of the moduli  $e_J$  and  $\bar{e}_J$ .

**Theorem 6.** Let  $X$  be  $(S)$ . Then the following assertions are valid.

- a) The function  $e_J$  is nondecreasing on  $[0, 2]$ .
- b) The function  $\bar{e}_J$  is nondecreasing on  $[0, 2]$ .
- c)  $e_J(\varepsilon) \leq \bar{e}_J(\varepsilon)$  ( $\varepsilon \in [0, 2]$ ).
- d) If  $X$  is a Hilbert space, then  $e_J(\varepsilon) = \bar{e}_J(\varepsilon)$ .




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*Proof.* The assertions a) and b) follow from the implications

$$\varepsilon_1 < \varepsilon_2 \implies \{(x, y) \mid \|x - y\| \geq \varepsilon_1\} \supset \{(x, y) \mid \|x - y\| \geq \varepsilon_2\} \\ (x, y \in S(X)),$$

$$\varepsilon_1 < \varepsilon_2 \implies \{(x, y) \mid \|x - y\| \leq \varepsilon_1\} \subset \{(x, y) \mid \|x - y\| \leq \varepsilon_2\} \\ (x, y \in S(X)).$$

c) Assume, to the contrary, i.e., that there is an  $\varepsilon \in [0, 2]$  such that  $\underline{e}_J(\varepsilon) > \overline{e}_J(\varepsilon)$ . Then

$$\inf \{\|f_x - f_y\| \mid \|x - y\| = \varepsilon\} \geq \inf \{\|f_x - f_y\| \mid \|x - y\| \geq \varepsilon\} \\ > \sup \{\|f_x - f_y\| \mid \|x - y\| \leq \varepsilon\} \\ \geq \sup \{\|f_x - f_y\| \mid \|x - y\| = \varepsilon\},$$

which is not possible.

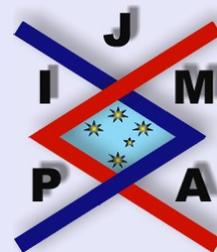
d) In a Hilbert space, we have

$$\|f_x - f_y\| = \sup \{|(x, t) - (y, t)| \mid t \in S(X)\} \leq \|x - y\|.$$

On the other hand, the functional  $f_x - f_y$  attains its maximum in  $t = \frac{x-y}{\|x-y\|} \in S(X)$ .

Hence  $\|x - y\| = \|f_x - f_y\|$ . Because of that, we have  $\underline{e}_J(\varepsilon) = \overline{e}_J(\varepsilon) = \varepsilon$ .  $\square$

In the next theorems between moduli  $\delta'_X, \rho'_X, \underline{e}_J, \overline{e}_J$  are given.



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**Theorem 7.** Let  $X$  be  $(S)$ ,  $(SC)$  and  $(R)$ . Then, for  $\varepsilon \in (0, 2]$  we have

$$a) \delta'_X(\varepsilon) \leq \frac{1}{2} \underline{e}_J(\varepsilon)$$

$$b) \rho'_X(\varepsilon) \leq \frac{\varepsilon}{4} \overline{e}_J(\varepsilon),$$

$$c) \frac{2}{\varepsilon} \rho_X(\varepsilon) \leq \eta_X(\varepsilon) \leq \frac{1}{2} \overline{e}_J(\varepsilon).$$

*Proof.* The proof of the assertions a) and b) follows immediately using the definitions of the functions  $\delta'_X$  and  $\rho'_X$  and the inequality (4).

c) Let  $x, y \in S(X)$ ,  $x \neq y$ . By Lemma 4, we have

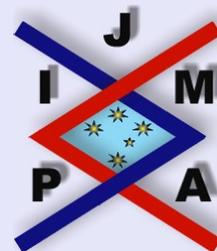
$$\begin{aligned} \frac{2 - \|x + y\|}{\|x - y\|} &= \frac{2}{\|x - y\|} \left( 1 - \frac{\|x + y\|}{2} \right) \\ &\leq \frac{1 - \cos(x, y)}{\|x - y\|} \\ &\leq \frac{\|x - y\| \|f_x - f_y\|}{2 \|x - y\|} \\ &= \frac{\|f_x - f_y\|}{2}. \end{aligned}$$

So

$$\frac{2 - \|x + y\|}{\|x - y\|} \leq \frac{\|f_x - f_y\|}{2}.$$

Using the definition of  $\eta_X$  and  $\overline{e}_J$ , we obtain

$$\eta_X(\varepsilon) \leq \frac{1}{2} \overline{e}_J(\varepsilon).$$



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On the other hand

$$\begin{aligned} (0 < \|x - y\| \leq \varepsilon) &\implies \left( \frac{1}{\|x - y\|} \geq \frac{1}{\varepsilon} \right) \\ &\implies \frac{2 - \|x + y\|}{\|x - y\|} \geq \frac{2}{\varepsilon} \left( 1 - \frac{\|x + y\|}{2} \right). \end{aligned}$$

Because of that we have

$$\eta_X(\varepsilon) \geq \frac{2}{\varepsilon} \rho_X(\varepsilon).$$

□

**Remark 2.** *The last inequality is true for an arbitrary space  $X$ .*

**Corollary 8.** *For a q.i.p. space, it holds that*

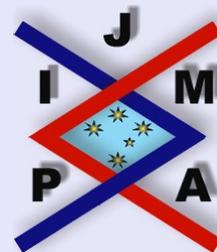
$$(13) \quad \underline{e}_J(\varepsilon) \geq \left( \frac{\varepsilon}{2} \right)^4 \quad (\varepsilon \in [0, 2]).$$

*Proof.* By a) of Theorem 7 and the inequality  $\frac{\varepsilon^4}{32} \leq \delta'_{X^*}(\varepsilon)$  (see Corollary 2 in [5]), we get (13). □

**Corollary 9.** *If  $X$  is (S), (SC) and (R) then*

$$a) \delta'_{X^*}(\varepsilon) \leq \frac{1}{2} \underline{e}_J(\varepsilon),$$

$$b) \rho'_{X^*} \leq \frac{1}{2} \underline{e}_{J^*}(\varepsilon),$$



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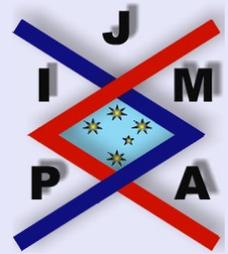


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$$c) \frac{2}{3} \rho_{X^*}(\varepsilon) \leq \eta_{X^*}(\varepsilon) \leq \frac{1}{2} \overline{e_{J^*}}(\varepsilon).$$

*Proof.* It is well-known that if  $X$  is  $(S)$ ,  $(SC)$  and  $(R)$  then  $X^*$  is  $(S)$ ,  $(SC)$  and  $(R)$ . Hence Theorem 7 is valid for  $X^*$ .  $\square$

**Theorem 10.** Let  $X$  be a Banach space which is  $(UC)$  and  $(US)$ . Then, for all  $\varepsilon > 0$ , we have the following estimations:

$$a) \rho'_X(2\delta_{X^*}(\varepsilon)) \leq \frac{\varepsilon \delta_{X^*}(\varepsilon)}{2},$$

$$b) \rho'_{X^*}(2\delta_X(\varepsilon)) \leq \frac{\varepsilon \delta_X(\varepsilon)}{2},$$

$$c) \overline{e_{J^*}}(\varepsilon) \geq 2\delta_{X^*}(\varepsilon),$$

$$d) \overline{e_J}(2\delta_{X^*}(\varepsilon)) \leq \varepsilon, \quad (\overline{e_{J^*}}(2\delta_X(\varepsilon)) \leq \varepsilon).$$

*Proof.* a) Using, in succession, the definition of the function  $\rho'_X$ , the inequality (4) in Lemma 2 and the implication (7), we obtain:

$$\begin{aligned} \rho'_X(2\delta_{X^*}(\varepsilon)) &= \sup \left\{ \frac{1 - \cos(x, y)}{2} \mid \|x - y\| \leq 2\delta_{X^*}(\varepsilon) \right\} \\ &\leq \frac{1}{4} \sup \{ \|x - y\| \|f_x - f_y\| \mid \|x - y\| \leq 2\delta_{X^*}(\varepsilon) \} \\ &\leq \frac{1}{4} 2\varepsilon \delta_{X^*}(\varepsilon) \\ &= \frac{\varepsilon \delta_{X^*}(\varepsilon)}{2}. \end{aligned}$$

b) If, in a), we set  $X^*$  instead of  $X$  ( $X^{**}$  instead of  $X^*$ ), we get

$$(14) \quad \rho'_{X^*} (2\delta_{X^{**}} (\varepsilon)) \leq \frac{\varepsilon \delta_{X^{**}} (\varepsilon)}{2}.$$

Let  $F, G \in S (X^{**})$ . Under the hypothesis of Theorem 10, we have

$$\begin{aligned} \delta_{X^{**}} (\varepsilon) &= \inf \left\{ 1 - \frac{\|F + G\|}{2} \mid \|F - G\| \geq \varepsilon \right\} \\ &= \inf \left\{ 1 - \frac{\|\tau x + \tau y\|}{2} \mid \|\tau x - \tau y\| \geq \varepsilon \right\} \\ &= \inf \left\{ 1 - \frac{\|\tau (x + y)\|}{2} \mid \|\tau (x - y)\| \geq \varepsilon \right\} \\ &= \inf \left\{ 1 - \frac{\|x + y\|}{2} \mid \|x - y\| \geq \varepsilon \right\} \\ &= \delta_X (\varepsilon). \end{aligned}$$

Consequently the inequality (14) is equivalent to the inequality b).

c) Using, in succession, the definition of  $\underline{e}_J$ , Lemma 3, and the implication (8), we get

$$\begin{aligned} \underline{e}_{J^*} (\varepsilon) &= \inf \{ \|J^* f_x - J^* f_y\| \mid \|f_x - f_y\| \geq \varepsilon \} \\ &= \inf \{ \|\tau x - \tau y\| \mid \|f_x - f_y\| \geq \varepsilon \} \\ &\geq 2\delta_{X^*} (\varepsilon). \end{aligned}$$



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d) Using the definition of  $\bar{e}_J$  and the implication (7), we get

$$\bar{e}_J(2\delta_{X^*}(\varepsilon)) = \sup \{ \|f_x - f_y\| \mid \|x - y\| \leq 2\delta_{X^*}(\varepsilon) \} \leq \varepsilon.$$

Replacing, here,  $X^*$  with  $X^{**}$  and  $J$  with  $J^*$ , we get the second inequality.  $\square$

Since in a Banach space  $X$  we have

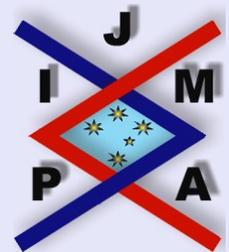
$$\delta_X(\varepsilon) \leq 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \quad \text{and} \quad \delta_X(\varepsilon) \leq \delta'_X(\varepsilon)$$

(see Theorem 1 in [5]), using b) and a) of Theorem 10, we obtain

**Corollary 11.** *Under the hypothesis of Theorem 10, we have*

$$a) \quad \frac{2}{\varepsilon} \rho'_{X^*}(2\delta_X(\varepsilon)) \leq \delta_X(\varepsilon) \leq \frac{2}{\varepsilon} \delta'_X(\varepsilon),$$

$$b) \quad \rho'_X(2\delta_{X^*}(\varepsilon)) \leq \frac{\varepsilon}{2} \left( 1 - \sqrt{1 - \frac{\varepsilon^2}{4}} \right).$$



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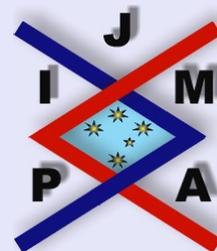
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Banach Spaces

Pavle M. Miličić

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