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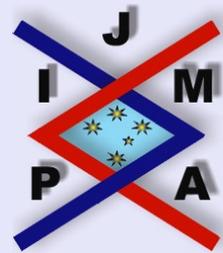
## ON AN INEQUALITY RELATED TO THE LEGENDRE TOTIENT FUNCTION

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## Abstract

Let  $\Delta(x, n) = \varphi(x, n) - x\varphi(n)/n$ , where  $\varphi(x, n)$  is the Legendre totient function and  $\varphi(n)$  is the Euler totient function. An inequality for  $\Delta(x, n)$  is known. In this paper we give a unitary analogue of this inequality, and more generally we give this inequality in the setting of regular convolutions.

*2000 Mathematics Subject Classification:* 11A25

*Key words:* Legendre totient function; Inequality; Regular convolution; Unitary convolution.

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# 1. Introduction

The Legendre totient function  $\varphi(x, n)$  is defined as the number of positive integers  $\leq x$  which are prime to  $n$ . The Euler totient function  $\varphi(n)$  is a special case of  $\varphi(x, n)$ . Namely,  $\varphi(n) = \varphi(n, n)$ . It is well known that

$$(1.1) \quad \varphi(x, n) = \sum_{d|n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor,$$

where  $\mu$  is the Möbius function. A direct consequence of (1.1) is that

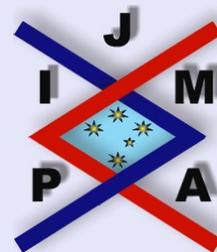
$$(1.2) \quad \varphi(x, n) = \frac{x\varphi(n)}{n} + O(\theta(n)),$$

where  $\theta(n)$  denotes the number of square-free divisors of  $n$  with  $\theta(1) = 1$ . This gives rise to the function  $\Delta(x, n)$  defined as  $\Delta(x, n) = \varphi(x, n) - \frac{x\varphi(n)}{n}$ . Suryanarayana [8] obtains two inequalities for the function  $\Delta(x, n)$ . Sivaramasarma [7] establishes an inequality which sharpens the first inequality and contains as a special case the second inequality of Suryanarayana [8]. The inequality of Sivaramasarma [7] states that if  $x \geq 1$ ,  $n \geq 2$  and  $m = (n, [x])$ , then

$$(1.3) \quad \left| \Delta(x, n) + \{x\} \frac{\varphi(n)}{n} - \frac{1}{2} \left\lfloor \frac{1}{m} \right\rfloor \right| \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m\theta(m)\psi(n)}{n\psi(m)},$$

where  $\{x\} = x - [x]$  and  $\psi$  is the Dedekind totient function. See also [4, §I.32].

In this paper we give (1.3) in the setting of Narkiewicz's regular convolution and the  $k$ th power greatest common divisor. As special cases we obtain (1.3) and its unitary analogue. The proof is adapted from that given by Sivaramasarma [7].



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## 2. Preliminaries

For each  $n$  let  $A(n)$  be a subset of the set of positive divisors of  $n$ . The elements of  $A(n)$  are said to be the  $A$ -divisors of  $n$ . The  $A$ -convolution of two arithmetical functions  $f$  and  $g$  is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d)g\left(\frac{n}{d}\right).$$

Narkiewicz [5] (see also [3]) defines an  $A$ -convolution to be regular if

- (a) the set of arithmetical functions forms a commutative ring with unity with respect to the ordinary addition and the  $A$ -convolution,
- (b) the  $A$ -convolution of multiplicative functions is multiplicative,
- (c) the constant function 1 has an inverse  $\mu_A$  with respect to the  $A$ -convolution, and  $\mu_A(n) = 0$  or  $-1$  whenever  $n$  is a prime power.

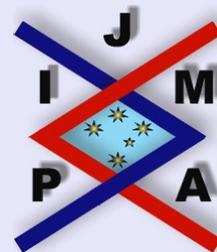
It can be proved [5] that an  $A$ -convolution is regular if and only if

- (i)  $A(mn) = \{de : d \in A(m), e \in A(n)\}$  whenever  $(m, n) = 1$ ,
- (ii) for each prime power  $p^a (> 1)$  there exists a divisor  $t = \tau_A(p^a)$  of  $a$  such that

$$A(p^a) = \{1, p^t, p^{2t}, \dots, p^{rt}\},$$

where  $rt = a$ , and

$$A(p^{it}) = \{1, p^t, p^{2t}, \dots, p^{it}\}, 0 \leq i < r.$$



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The positive integer  $t = \tau_A(p^a)$  in part (ii) is said to be the  $A$ -type of  $p^a$ . A positive integer  $n$  is said to be  $A$ -primitive if  $A(n) = \{1, n\}$ . The  $A$ -primitive numbers are 1 and  $p^t$ , where  $p$  runs through the primes and  $t$  runs through the  $A$ -types of the prime powers  $p^a$  with  $a \geq 1$ .

For all  $n$  let  $D(n)$  denote the set of all positive divisors of  $n$  and let  $U(n)$  denote the set of all unitary divisors of  $n$ , that is,

$$U(n) = \left\{ d > 0 : d \mid n, \left( d, \frac{n}{d} \right) = 1 \right\} = \{d > 0 : d \parallel n\}.$$

The  $D$ -convolution is the classical Dirichlet convolution and the  $U$ -convolution is the unitary convolution [1]. These convolutions are regular with  $\tau_D(p^a) = 1$  and  $\tau_U(p^a) = a$  for all prime powers  $p^a (> 1)$ .

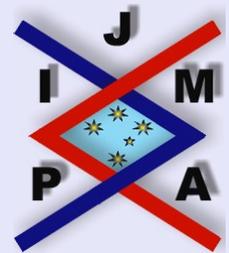
Let  $k$  be a positive integer. We denote  $A_k(n) = \{d > 0 : d^k \in A(n^k)\}$ . It is known [6] that the  $A_k$ -convolution is regular whenever the  $A$ -convolution is regular. The symbol  $(m, n)_{A,k}$  denotes the greatest  $k$ th power divisor of  $m$  which belongs to  $A(n)$ . In particular,  $(m, n)_{D,1}$  is the usual greatest common divisor  $(m, n)$  of  $m$  and  $n$ , and  $(m, n)_{U,1}$ , usually written as  $(m, n)^*$ , is the greatest unitary divisor of  $n$  which is a divisor of  $m$ .

*Throughout the rest of the paper  $A$  will be an arbitrary but fixed regular convolution and  $k$  is a positive integer.*

The  $A$ -analogue of the Möbius function  $\mu_A$  is the multiplicative function given by

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } p^a (> 1) \text{ is } A\text{-primitive,} \\ 0 & \text{if } p^a \text{ is non-} A\text{-primitive.} \end{cases}$$

In particular,  $\mu_D = \mu$ , the classical Möbius function, and  $\mu_U = \mu^*$ , the unitary analogue of the Möbius function [1].




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The generalized Legendre totient function  $\varphi_{A,k}(x, n)$  is defined as the number of positive integers  $a \leq x$  such that  $(a, n^k)_{A,k} = 1$ . It is known [2] that

$$(2.1) \quad \varphi_{A,k}(x, n) = \sum_{d \in A_k(n)} \mu_{A_k}(d) \left[ \frac{x}{d^k} \right].$$

In particular,  $\varphi_{A,k}(n) = \varphi_{A,k}(n^k, n)$ . We recall that

$$(2.2) \quad \varphi_{A,k}(n) = n^k \prod_{p|n} (1 - p^{-tk}),$$

where  $n = \prod_p p^{n(p)}$  is the canonical factorization of  $n$  and  $t = \tau_{A_k}(p^{n(p)})$ , and we define the generalized Dedekind totient function  $\psi_{A,k}$  as

$$(2.3) \quad \psi_{A,k}(n) = n^k \prod_{p|n} (1 + p^{-tk}).$$

If  $A$  is the Dirichlet convolution and  $k = 1$ , then  $\varphi_{A,k}(x, n)$ ,  $\varphi_{A,k}(n)$  and  $\psi_{A,k}(n)$ , respectively, reduce to the Legendre totient function, the Euler totient function and the Dedekind totient function.

It follows from (2.1) that

$$\begin{aligned} \varphi_{A,k}(x, n) &= \sum_{d \in A_k(n)} \mu_{A_k}(d) \left( \frac{x}{d^k} + O(1) \right) \\ &= \frac{x \varphi_{A,k}(n)}{n^k} + O \left( \sum_{d \in A_k(n)} \mu_{A_k}^2(d) \right) \\ &= \frac{x \varphi_{A,k}(n)}{n^k} + O(\theta(n)). \end{aligned}$$



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This suggests we define

$$(2.4) \quad \Delta_{A,k}(x, n) = \varphi_{A,k}(x, n) - \frac{x\varphi_{A,k}(n)}{n^k}.$$

We next present four lemmas which are needed in the proof of our inequality for the function  $\Delta_{A,k}(x, n)$ .

**Lemma 2.1.** *If  $f(x, n^k) = \left\{ \frac{[x]}{n^k} \right\}$  and  $m^k = ([x], n^k)_{A,k}$ , then*

(i)  $f(x, n^k) = 0$  if  $m = n$ ,

(ii)  $\frac{m^k}{n^k} \leq f(x, n^k) \leq 1 - \frac{m^k}{n^k}$  if  $m < n$ .

*Proof.* (i) If  $m = n$ , then  $n^k | [x]$  and thus  $\left\{ \frac{[x]}{n^k} \right\} = 0$ .

(ii) Let  $m < n$ . Then  $n^k \nmid [x]$  and thus  $[x] = an^k + r$ , where  $0 < r < n^k$ .

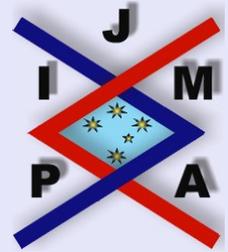
Therefore  $\left\{ \frac{[x]}{n^k} \right\} = \frac{r}{n^k}$ , where  $0 < r < n^k$ , that is,  $\frac{1}{n^k} \leq \left\{ \frac{[x]}{n^k} \right\} \leq 1 - \frac{1}{n^k}$ .

Now, writing  $\frac{[x]}{n^k} = \frac{([x]/m^k)}{(n^k/m^k)}$  we arrive at our result. □

**Lemma 2.2.** *For  $n \geq 2$*

$$\sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) = \frac{\theta(n)}{2},$$

where  $\omega(d)$  is the number of distinct prime divisors of  $d$ .



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*Proof.* It is clear that

$$\sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) = \binom{\omega(n)}{1} + \binom{\omega(n)}{3} + \dots = 2^{\omega(n)-1} = \frac{\theta(n)}{2}.$$

□

**Lemma 2.3.** Let  $m^k = ([x], n^k)_{A,k}$ . Then

$$\sum_{d \in A_k(n)} \frac{\mu_{A_k}^2(d) ([x], d^k)_{A,k}}{d^k} = \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)}.$$

*Proof.* By multiplicativity it is enough to consider the case in which  $n$  is a prime power. For the sake of brevity we do not present the details. □

**Lemma 2.4.** We have

$$\Delta_{A,k}(x, n) + \{x\} \frac{\varphi_{A,k}(n)}{n^k} = - \sum_{d \in A_k(n)} \mu_{A_k}(d) f(x, d^k).$$

*Proof.* Clearly

$$\begin{aligned} \varphi_{A,k}(x, n) &= \sum_{d \in A_k(n)} \mu_{A_k}(d) \left( \frac{x}{d^k} - \left\{ \frac{x}{d^k} \right\} \right) \\ &= \frac{x \varphi_{A,k}(n)}{n^k} - \sum_{d \in A_k(n)} \mu_{A_k}(d) \left\{ \frac{x}{d^k} \right\}. \end{aligned}$$



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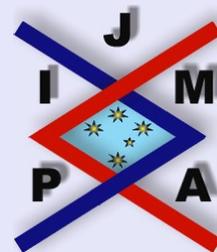
Thus

$$\Delta_{A,k}(x, n) = - \sum_{d \in A_k(n)} \mu_{A_k}(d) \left\{ \frac{x}{d^k} \right\}.$$

It can be verified that  $\left\{ \frac{x}{d^k} \right\} = \frac{\{x\}}{d^k} + \left\{ \frac{[x]}{d^k} \right\}$ . Thus

$$\Delta_{A,k}(x, n) = -\{x\} \frac{\varphi_{A,k}(n)}{n^k} - \sum_{d \in A_k(n)} \mu_{A_k}(d) \left\{ \frac{[x]}{d^k} \right\}.$$

This completes the proof. □



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### 3. Generalization of (1.3)

**Theorem 3.1.** Let  $x \geq 1$ ,  $n \geq 2$  and  $m^k = ([x], n^k)_{A,k}$ . Then

$$(3.1) \quad \left| \Delta_{A,k}(x, n) + \{x\} \frac{\varphi_{A,k}(n)}{n^k} - \frac{1}{2} \left[ \frac{1}{m} \right] \right| \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)}.$$

*Proof.* Firstly, suppose that  $m = n$ , that is,  $n^k \mid [x]$ . Then

$$\varphi_{A,k}(x, n) = \sum_{d \in A_k(n)} \frac{\mu_{A_k}(d)[x]}{d^k} = [x] \frac{\varphi_{A,k}(n)}{n^k}.$$

Thus

$$\Delta_{A,k}(x, n) + \frac{\{x\} \varphi_{A,k}(n)}{n^k} = 0.$$

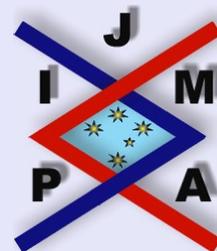
Since  $n \geq 2$ , the left-hand side of (3.1) is = 0. Therefore (3.1) holds.

Secondly, suppose that  $1 \leq m < n$ . Then, by Lemma 2.4,

$$\Delta_{A,k}(x, n) + \frac{\{x\} \varphi_{A,k}(n)}{n^k} = \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) f(x, d^k) - \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is even}}} \mu_{A_k}^2(d) f(x, d^k).$$

By Lemma 2.1,

$$\Delta_{A,k}(x, n) + \frac{\{x\} \varphi_{A,k}(n)}{n^k}$$



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$$\begin{aligned}
&\leq \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd} \\ d^k \parallel [x]}} \mu_{A_k}^2(d) \left(1 - \frac{([x], d^k)_{A,k}}{d^k}\right) - \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is even} \\ d^k \nmid [x]}} \mu_{A_k}^2(d) \frac{([x], d^k)_{A,k}}{d^k} \\
&= \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) - \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd} \\ d^k \parallel [x]}} \mu_{A_k}^2(d) \\
&\quad - \sum_{d \in A_k(n)} \mu_{A_k}^2(d) \frac{([x], d^k)_{A,k}}{d^k} + \sum_{\substack{d \in A_k(n) \\ d^k \parallel [x]}} \mu_{A_k}^2(d) \frac{([x], d^k)_{A,k}}{d^k}.
\end{aligned}$$

By Lemmas 2.2 and 2.3 and definition of the number  $m$ ,

$$\Delta_{A,k}(x, n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} \leq \frac{\theta(n)}{2} - \sum_{\substack{d \in A_k(m) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) - \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)} + \theta(m).$$

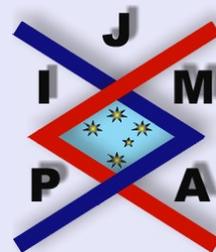
We distinguish the cases  $m = 1$  and  $m > 1$  and apply Lemma 2.2 in the case  $m > 1$  to obtain

$$\Delta_{A,k}(x, n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} - \frac{1}{2} \left[ \frac{1}{m} \right] \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)}.$$

In a similar way we can show that

$$\Delta_{A,k}(x, n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} - \frac{1}{2} \left[ \frac{1}{m} \right] \geq -\frac{\theta(n)}{2} - \frac{\theta(m)}{2} + \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)}.$$

This completes the proof.  $\square$



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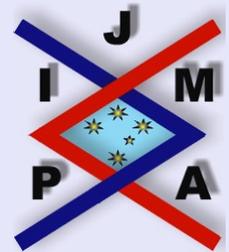
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**Remark 3.1.** If  $A$  is the Dirichlet convolution and  $k = 1$ , then (3.1) reduces to (1.3).



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## 4. Unitary Analogue of (1.3)

We recall that a positive integer  $d$  is said to be a unitary divisor of  $n$  (written as  $d||n$ ) if  $d$  is a divisor of  $n$  and  $(d, \frac{n}{d}) = 1$ . The unitary analogue of the Legendre totient function  $\varphi^*(x, n)$  is the number of positive integers  $a \leq x$  such that  $(a, n)^* = 1$ . Its arithmetical expression is

$$\varphi^*(x, n) = \sum_{d||n} \mu^*(d) \left\lfloor \frac{x}{d} \right\rfloor.$$

In particular, the unitary analogue of the Euler totient function is given by  $\varphi^*(n) = \varphi^*(n, n)$ . We define the unitary analogue of the Dedekind totient function as

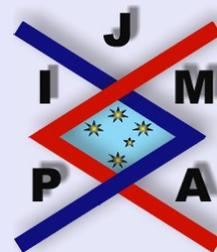
$$\psi^*(n) = n \prod_{p^e || n} (1 + p^{-e}).$$

It is easy to see that  $\psi^*(n) = \sigma^*(n)$ , where  $\sigma^*(n)$  is the sum of the unitary divisors of  $n$ . The function  $\Delta^*(x, n)$  is defined as  $\Delta^*(x, n) = \varphi^*(x, n) - \frac{x\varphi^*(n)}{n}$ .

The unitary analogue of (1.3) is

$$(4.1) \quad \left| \Delta^*(x, n) + \{x\} \frac{\varphi^*(n)}{n} - \frac{1}{2} \left\lfloor \frac{1}{m} \right\rfloor \right| \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m\theta(m)\sigma^*(n)}{n\sigma^*(m)},$$

where  $m = ([x], n)^*$ . In fact, if  $A$  is the unitary convolution and  $k = 1$ , then (3.1) reduces to (4.1).



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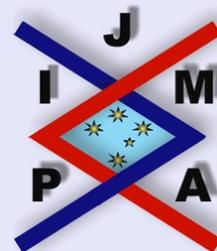
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