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ON AN INEQUALITY RELATED TO THE LEGENDRE TOTIENT FUNCTION

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ABSTRACT. Let $\Delta(x,n) = \varphi(x,n) - x\varphi(n)/n$, where $\varphi(x,n)$ is the Legendre totient function and $\varphi(n)$ is the Euler totient function. An inequality for $\Delta(x,n)$ is known. In this paper we give a unitary analogue of this inequality, and more generally we give this inequality in the setting of regular convolutions.

Key words and phrases: Legendre totient function; Inequality; Regular convolution; Unitary convolution.

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1. INTRODUCTION

The Legendre totient function $\varphi(x,n)$ is defined as the number of positive integers $\leq x$ which are prime to n. The Euler totient function $\varphi(n)$ is a special case of $\varphi(x,n)$. Namely, $\varphi(n) = \varphi(n,n)$. It is well known that

$$\varphi(x,n) = \sum_{d|n} \mu(d) \left[\frac{x}{d}\right], \tag{1.1}$$

where μ is the Möbius function. A direct consequence of (1.1) is that

$$\varphi(x,n) = \frac{x\varphi(n)}{n} + O(\theta(n)), \tag{1.2}$$

where $\theta(n)$ denotes the number of square-free divisors of n with $\theta(1) = 1$. This gives rise to the function $\Delta(x,n)$ defined as $\Delta(x,n) = \varphi(x,n) - \frac{x\varphi(n)}{n}$. Suryanarayana [8] obtains two inequalities for the function $\Delta(x,n)$. Sivaramasarma [7] establishes an inequality which sharpens the first inequality and contains as a special case the second inequality of Suryanarayana [8]. The inequality of Sivaramasarma [7] states that if $x \geq 1$, $n \geq 2$ and $m = (n,[x])$, then

$$\left|\Delta(x,n) + \{x\}\frac{\varphi(n)}{n} - \frac{1}{2}\left[\frac{1}{m}\right]\right| \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m\theta(m)\psi(n)}{n\psi(m)}, \tag{1.3}$$

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where $\{x\} = x - [x]$ and ψ is the Dedekind totient function. See also [4, §I.32].

In this paper we give (1.3) in the setting of Narkiewicz's regular convolution and the kth power greatest common divisor. As special cases we obtain (1.3) and its unitary analogue. The proof is adapted from that given by Sivaramasarma [7].

2. PRELIMINARIES

For each n let $A(n)$ be a subset of the set of positive divisors of n. The elements of $A(n)$ are said to be the A-divisors of n. The A-convolution of two arithmetical functions f and g is defined by

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right).$$

Narkiewicz [5] (see also [3]) defines an A-convolution to be regular if

(a) the set of arithmetical functions forms a commutative ring with unity with respect to the ordinary addition and the A-convolution,
(b) the A-convolution of multiplicative functions is multiplicative,
(c) the constant function 1 has an inverse μ_A with respect to the A-convolution, and $\mu_A(n) = 0$ or -1 whenever n is a prime power.

It can be proved [5] that an A-convolution is regular if and only if

(i) $A(mn) = \{de : d \in A(m), e \in A(n)\}$ whenever $(m, n) = 1$,
(ii) for each prime power p^a (> 1) there exists a divisor $t = \tau_A(p^a)$ of a such that

$$A(p^a) = \left\{1, p^t, p^{2t}, \ldots, p^{rt}\right\},$$

where $rt = a$, and

$$A(p^{it}) = \left\{1,\ p^t, p^{2t}, \ldots, p^{it}\right\}, 0 \le i < r.$$

The positive integer $t = \tau_A(p^a)$ in part (ii) is said to be the A-type of p^a. A positive integer n is said to be A-primitive if $A(n) = \{1, n\}$. The A-primitive numbers are 1 and p^t, where p runs through the primes and t runs through the A-types of the prime powers p^a with $a \ge 1$.

For all n let $D(n)$ denote the set of all positive divisors of n and let $U(n)$ denote the set of all unitary divisors of n, that is,

$$U(n) = \left\{d > 0 : d \middle| n, \left(d, \frac{n}{d}\right) = 1\right\} = \{d > 0 : d \parallel n\}.$$

The D-convolution is the classical Dirichlet convolution and the U-convolution is the unitary convolution [1]. These convolutions are regular with $\tau_D(p^a) = 1$ and $\tau_U(p^a) = a$ for all prime powers p^a (> 1).

Let k be a positive integer. We denote $A_k(n) = \left\{d > 0 : d^k \in A\left(n^k\right)\right\}$. It is known [6] that the A_k-convolution is regular whenever the A-convolution is regular. The symbol $(m, n)_{A,k}$ denotes the greatest kth power divisor of m which belongs to $A(n)$. In particular, $(m, n)_{D,1}$ is the usual greatest common divisor (m, n) of m and n, and $(m, n)_{U,1}$, usually written as $(m, n)^*$, is the greatest unitary divisor of n which is a divisor of m.

Throughout the rest of the paper A will be an arbitaray but fixed regular convolution and k is a positive integer.

The A-analogue of the Möbius function μ_A is the multiplicative function given by

$$\mu_A(p^a) = \begin{cases} -1 & \text{if } p^a (> 1) \text{ is } A\text{-primitive}, \\ 0 & \text{if } p^a \text{ is non-}A\text{-primitive}. \end{cases}$$

In particular, $\mu_D = \mu$, the classical Möbius function, and $\mu_U = \mu^*$, the unitary analogue of the Möbius function [1].

The generalized Legendre totient function $\varphi_{A,k}(x,n)$ is defined as the number of positive integers $a \le x$ such that $(a, n^k)_{A,k} = 1$. It is known [2] that

$$\varphi_{A,k}(x,n) = \sum_{d \in A_k(n)} \mu_{A_k}(d) \left[\frac{x}{d^k}\right]. \tag{2.1}$$

In particular, $\varphi_{A,k}(n) = \varphi_{A,k}(n^k, n)$. We recall that

$$\varphi_{A,k}(n) = n^k \prod_{p|n} \left(1 - p^{-tk}\right), \tag{2.2}$$

where $n = \prod_p p^{n(p)}$ is the canonical factorization of n and $t = \tau_{A_k}(p^{n(p)})$, and we define the generalized Dedekind totient function $\psi_{A,k}$ as

$$\psi_{A,k}(n) = n^k \prod_{p|n} \left(1 + p^{-tk}\right). \tag{2.3}$$

If A is the Dirichlet convolution and $k = 1$, then $\varphi_{A,k}(x,n)$, $\varphi_{A,k}(n)$ and $\psi_{A,k}(n)$, respectively, reduce to the Legendre totient function, the Euler totient function and the Dedekind totient function.

It follows from (2.1) that

$$\begin{aligned} \varphi_{A,k}(x,n) &= \sum_{d \in A_k(n)} \mu_{A_k}(d) \left(\frac{x}{d^k} + O(1)\right) \\ &= \frac{x\varphi_{A,k}(n)}{n^k} + O\left(\sum_{d \in A_k(n)} \mu^2_{A_k}(d)\right) \\ &= \frac{x\varphi_{A,k}(n)}{n^k} + O(\theta(n)). \end{aligned}$$

This suggests we define

$$\Delta_{A,k}(x,n) = \varphi_{A,k}(x,n) - \frac{x\varphi_{A,k}(n)}{n^k}. \tag{2.4}$$

We next present four lemmas which are needed in the proof of our inequality for the function $\Delta_{A,k}(x,n)$.

Lemma 2.1. *If* $f\left(x, n^k\right) = \left\{\frac{[x]}{n^k}\right\}$ *and* $m^k = \left([x], n^k\right)_{A,k}$, *then*

(i) $f\left(x, n^k\right) = 0$ *if* $m = n$,

(ii) $\frac{m^k}{n^k} \le f(x, n^k) \le 1 - \frac{m^k}{n^k}$ *if* $m < n$.

Proof. (i) If $m = n$, then $n^k | [x]$ and thus $\left\{\frac{[x]}{n^k}\right\} = 0$.

(ii) Let $m < n$. Then $n^k \nmid [x]$ and thus $[x] = an^k + r$, where $0 < r < n^k$. Therefore $\left\{\frac{[x]}{n^k}\right\} = \frac{r}{n^k}$, where $0 < r < n^k$, that is, $\frac{1}{n^k} \le \left\{\frac{[x]}{n^k}\right\} \le 1 - \frac{1}{n^k}$. Now, writing $\frac{[x]}{n^k} = \frac{([x]/m^k)}{(n^k/m^k)}$ we arrive at our result. □

Lemma 2.2. *For* $n \ge 2$

$$\sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu^2_{A_k}(d) = \frac{\theta(n)}{2},$$

where $\omega(d)$ *is the number of distinct prime divisors of* d.

Proof. It is clear that

$$\sum_{\substack{d\in A_k(n)\\ \omega(d)\text{ is odd}}} \mu^2_{A_k}(d) = \binom{\omega(n)}{1} + \binom{\omega(n)}{3} + \cdots = 2^{\omega(n)-1} = \frac{\theta(n)}{2}.$$

□

Lemma 2.3. *Let* $m^k = ([x], n^k)_{A,k}$*. Then*

$$\sum_{d\in A_k(n)} \frac{\mu^2_{A_k}(d)([x], d^k)_{A,k}}{d^k} = \frac{m^k\theta(m)\psi_{A,k}(n)}{n^k\psi_{A,k}(m)}.$$

Proof. By multiplicativity it is enough to consider the case in which n is a prime power. For the sake of brevity we do not present the details. □

Lemma 2.4. *We have*

$$\Delta_{A,k}(x,n) + \{x\}\frac{\varphi_{A,k}(n)}{n^k} = -\sum_{d\in A_k(n)} \mu_{A_k}(d) f(x, d^k).$$

Proof. Clearly

$$\begin{aligned}\varphi_{A,k}(x,n) &= \sum_{d\in A_k(n)} \mu_{A_k}(d)\left(\frac{x}{d^k} - \left\{\frac{x}{d^k}\right\}\right)\\ &= \frac{x\varphi_{A,k}(n)}{n^k} - \sum_{d\in A_k(n)} \mu_{A_k}(d)\left\{\frac{x}{d^k}\right\}.\end{aligned}$$

Thus

$$\Delta_{A,k}(x,n) = -\sum_{d\in A_k(n)} \mu_{A_k}(d)\left\{\frac{x}{d^k}\right\}.$$

It can be verified that $\left\{\frac{x}{d^k}\right\} = \frac{\{x\}}{d^k} + \left\{\frac{[x]}{d^k}\right\}$. Thus

$$\Delta_{A,k}(x,n) = -\{x\}\frac{\varphi_{A,k}(n)}{n^k} - \sum_{d\in A_k(n)} \mu_{A_k}(d)\left\{\frac{[x]}{d^k}\right\}.$$

This completes the proof. □

3. GENERALIZATION OF (1.3)

Theorem 3.1. *Let* $x \geq 1$*,* $n \geq 2$ *and* $m^k = ([x], n^k)_{A,k}$*. Then*

$$\left|\Delta_{A,k}(x,n) + \{x\}\frac{\varphi_{A,k}(n)}{n^k} - \frac{1}{2}\left[\frac{1}{m}\right]\right| \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m^k\theta(m)\psi_{A,k}(n)}{n^k\psi_{A,k}(m)}. \tag{3.1}$$

Proof. Firstly, suppose that $m = n$, that is, $n^k \mid [x]$. Then

$$\varphi_{A,k}(x,n) = \sum_{d\in A_k(n)} \frac{\mu_{A_k}(d)[x]}{d^k} = [x]\frac{\varphi_{A,k}(n)}{n^k}.$$

Thus

$$\Delta_{A,k}(x,n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} = 0.$$

Since $n \geq 2$, the left-hand side of (3.1) is $= 0$. Therefore (3.1) holds.

Secondly, suppose that $1 \leq m < n$. Then, by Lemma 2.4,

$$\Delta_{A,k}(x,n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} = \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) f(x,d^k) - \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is even}}} \mu_{A_k}^2(d) f(x,d^k).$$

By Lemma 2.1,

$$\begin{aligned} &\Delta_{A,k}(x,n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} \\ &\quad \leq \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd} \\ d^k \nmid [x]}} \mu_{A_k}^2(d) \left(1 - \frac{([x],d^k)_{A,k}}{d^k}\right) - \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is even} \\ d^k \nmid [x]}} \mu_{A_k}^2(d) \frac{([x],d^k)_{A,k}}{d^k} \\ &\quad = \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) - \sum_{\substack{d \in A_k(n) \\ \omega(d) \text{ is odd} \\ d^k \mid [x]}} \mu_{A_k}^2(d) \\ &\qquad - \sum_{d \in A_k(n)} \mu_{A_k}^2(d) \frac{([x],d^k)_{A,k}}{d^k} + \sum_{\substack{d \in A_k(n) \\ d^k \mid [x]}} \mu_{A_k}^2(d) \frac{([x],d^k)_{A,k}}{d^k}. \end{aligned}$$

By Lemmas 2.2 and 2.3 and definition of the number m,

$$\Delta_{A,k}(x,n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} \leq \frac{\theta(n)}{2} - \sum_{\substack{d \in A_k(m) \\ \omega(d) \text{ is odd}}} \mu_{A_k}^2(d) - \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)} + \theta(m).$$

We distinguish the cases $m = 1$ and $m > 1$ and apply Lemma 2.2 in the case $m > 1$ to obtain

$$\Delta_{A,k}(x,n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} - \frac{1}{2}\left[\frac{1}{m}\right] \leq \frac{\theta(n)}{2} + \frac{\theta(m)}{2} - \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)}.$$

In a similar way we can show that

$$\Delta_{A,k}(x,n) + \frac{\{x\}\varphi_{A,k}(n)}{n^k} - \frac{1}{2}\left[\frac{1}{m}\right] \geq -\frac{\theta(n)}{2} - \frac{\theta(m)}{2} + \frac{m^k \theta(m) \psi_{A,k}(n)}{n^k \psi_{A,k}(m)}.$$

This completes the proof. □

Remark 3.2. If A is the Dirichlet convolution and $k = 1$, then (3.1) reduces to (1.3).

4. UNITARY ANALOGUE OF (1.3)

We recall that a positive integer d is said to be a unitary divisor of n (written as $d \| n$) if d is a divisor of n and $\left(d, \frac{n}{d}\right) = 1$. The unitary analogue of the Legendre totient function $\varphi^*(x,n)$ is the number of positive integers $a \leq x$ such that $(a,n)^* = 1$. Its arithmetical expression is

$$\varphi^*(x,n) = \sum_{d \| n} \mu^*(d) \left[\frac{x}{d}\right].$$

In particular, the unitary analogue of the Euler totient function is given by $\varphi^*(n) = \varphi^*(n,n)$. We define the unitary analogue of the Dedekind totient function as

$$\psi^*(n) = n \prod_{p^e \| n} (1 + p^{-e}).$$

It is easy to see that $\psi^*(n) = \sigma^*(n)$, where $\sigma^*(n)$ is the sum of the unitary divisors of n. The function $\Delta^*(x,n)$ is defined as $\Delta^*(x,n) = \varphi^*(x,n) - \frac{x\varphi^*(n)}{n}$.

The unitary analogue of (1.3) is

$$\left|\Delta^*(x,n)+\{x\}\frac{\varphi^*(n)}{n}-\frac{1}{2}\left[\frac{1}{m}\right]\right|\le\frac{\theta(n)}{2}+\frac{\theta(m)}{2}-\frac{m\theta(m)\sigma^*(n)}{n\sigma^*(m)},\tag{4.1}$$

where $m=([x],n)^*$. In fact, if A is the unitary convolution and $k=1$, then (3.1) reduces to (4.1).

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