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AN INEQUALITY WHICH ARISES IN THE ABSENCE OF THE MOUNTAIN PASS GEOMETRY

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ABSTRACT. An integral inequality is deduced from the negation of the geometrical condition in the bounded mountain pass theorem of Schechter, in a situation where this theorem does not apply. Also two localization results of non-zero solutions to a superlinear boundary value problem are established.

Key words and phrases: Integral inequality, Mountain pass theorem, Laplacean, Boundary value problem, Sobolev space.

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1. INTRODUCTION AND PRELIMINARIES

Let $p \in [2, \infty)$, Ω be a bounded domain of $\mathbb{R}^n$, and let

$$C_0^1\left(\overline{\Omega}\right) = \left\{u \in C^1\left(\overline{\Omega}\right) : \, u = 0 \text{ on } \partial\Omega\right\}.$$

We consider the quantity

$$\lambda_{p-1} = \inf \left\{ \frac{\int_\Omega |u|^{p-2} |\nabla u|^2 \, dx}{\left(\int_\Omega |u|^{\frac{p^2}{2}} \, dx\right)^{\frac{2}{p}}} : \, u \in C_0^1\left(\overline{\Omega}\right) \setminus \{0\} \right\}. \tag{1.1}$$

For $p = 2$, λ_1 is the first eigenvalue of the Laplacean $-\Delta$ under the Dirichlet boundary condition, and $\frac{1}{\lambda_1}$ represents the best constant in the Wirtinger-Poincaré inequality (see [7] for the elementary Wirtinger's inequality, [8] for its extension to functions with values in an arbitrary Banach space, and [11] for Poincaré's inequality). For $p > 2$ and $n = 1$ this quantity arises in the study of compactness properties for integral operators on spaces of vector-valued functions

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The author is indebted to Professor L. Gajek for bringing reference [3] to his attention, for giving the exact value of λ_{p-1} for $n = 1$ as shows Remark 1.1, and for stimulating an improved and much more general version of this paper.

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(see [13]). Let us also note that quantities alike (1.1) arising from physics were studied by Pólya [9] and Pólya and Szegö [10] (see also [6] and its references for more recent advances).

Remark 1.1. For $n = 1$ and $\Omega = (0, T)$, where $0 < T < \infty$, the exact value of λ_{p-1} can be obtained from a result of Gajek, Kałuszka and Lenic [3] in the following way. First by change of the integration variable one has

$$\begin{aligned}
\lambda_{p-1} &= \inf\left\{\frac{\int_0^T |u|^{p-2} u'^2 dx}{\left(\int_0^T |u|^{\frac{p^2}{2}} dx\right)^{\frac{2}{p}}} : u \in C_0^1[0,T] \setminus \{0\}\right\} \\
&= T^{-\left(1+\frac{2}{p}\right)} \inf\left\{\frac{\int_0^1 |u|^{p-2} u'^2 ds}{\left(\int_0^1 |u|^{\frac{p^2}{2}} ds\right)^{\frac{2}{p}}} : u \in C_0^1[0,1] \setminus \{0\}\right\} \\
&= \left(T^{1+\frac{2}{p}} \sup\left\{\left(\int_0^1 |u|^{\frac{p^2}{2}} ds\right)^{\frac{2}{p}} : \ u \in C_0^1[0,1], \int_0^1 |u|^{p-2} u'^2 ds = 1\right\}\right)^{-1}.
\end{aligned}$$

After substituting $v = \left(\frac{2}{p}\right) |u|^{\frac{p}{2}}$, we obtain

$$\lambda_{p-1} = \left(T^{1+\frac{2}{p}} \left(\frac{p}{2}\right)^2 \sup\left\{\left(\int_0^1 |v|^p ds\right)^{\frac{2}{p}} : \ v \in C_0^1[0,1], \int_0^1 v'^2 ds = 1\right\}\right)^{-1}.$$

Notice that

$$\begin{aligned}
&\sup\left\{\int_0^1 |v|^p ds : v \in C_0^1[0,1], \int_0^1 v'^2 ds = 1\right\} \\
&\qquad = \sup\left\{\int_0^1 |v|^p ds : v \in C_0^1[0,1], \int_0^1 v'^2 ds \le 1\right\}.
\end{aligned}$$

Now the sup in the right hand side is the quantity denoted by b in [3] and is given by

$$b = \left(\frac{p(p+2)}{\pi}\right)^{\frac{p}{2}} \frac{2^{1-p}}{p+2} \left(\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^p.$$

As a result

$$\lambda_{p-1} = \left\{T^{1+\frac{2}{p}} p^2 \left(\frac{p(p+2)}{\pi}\right)^{\frac{p}{2}} \frac{2^{-1-p}}{p+2} \left(\frac{\Gamma\left(\frac{1}{p}+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^p\right\}^{-1}.$$

In this paper we are interested in finding upper and lower estimations for λ_{p-1}. An upper bound is obtained from the negation of the geometrical condition in Schechter's mountain pass theorem, in a situation where this theorem does not apply. To our knowledge, this is the first time that a bounded mountain pass theorem is used in order to obtain inequalities. Two localization results of non-zero solutions to a superlinear elliptic boundary value problem are also established in terms of λ_{p-1}.

1.1. **Basic Results from the Theory of Linear Elliptic Equations.** Here we recall some well-known results from the theory of linear elliptic boundary value problems.

(P1) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. The Laplacean $-\Delta$ is a self-adjoint operator on $L^2(\Omega)$ with domain $H^2(\Omega) \cap H_0^1(\Omega)$ (see [11, Theorem 3.33], or [4]). It can be regarded as a continuous operator from $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ to $L^q(\Omega)$ for each $q \in (1,\infty)$. Moreover, $-\Delta$ is invertible and $K := (-\Delta)^{-1}$ is a continuous operator from $L^q(\Omega)$ into $W^{2,q}(\Omega)$ (see [2, Theorem 9.32]). Also, K considered in $L^2(\Omega)$ is a positive self-adjoint operator.

(P2) (Sobolev embedding theorem) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $k \in \mathbb{N}$, $1 \le q < \infty$. Then the following holds:

(1^0) If $kq < n$, we have

$$W^{k,q}(\Omega) \subset L^r(\Omega) \tag{1.2}$$

and the embedding is continuous for $r \in \left[1, \frac{nq}{n-kq}\right]$; the embedding is compact if $r \in \left[1, \frac{nq}{n-kq}\right)$.

(2^0) If $kq = n$, then (1.2) holds with compact embedding for $r \in [1,\infty)$.

(3^0) If $0 \le m < k - \frac{n}{q} < m+1$, we have

$$W^{k,q}(\Omega) \subset C^{m,\alpha}\left(\overline{\Omega}\right) \tag{1.3}$$

and the embedding is continuous for $0 \le \alpha \le k-m-\frac{n}{q}$; the embedding is compact if $\alpha < k-m-\frac{n}{q}$.

The above results are valid for $W_0^{k,q}(\Omega)$-spaces on arbitrary bounded domains Ω (see [1], [15, p 213] or [2, pp. 168-169]).

(P3) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Let $p_0 = \frac{2n}{n-2}$ if $n \ge 3$ and p_0 be any number of $(2,\infty)$ if $n = 1$ or $n = 2$. Let q_0 be the conjugate number of p_0. Clearly, $p_0 \in (2,\infty)$ and $q_0 \in (1,2)$. From (P1), (P2) we have that K has the following properties:

(a) $K : L^q(\Omega) \to L^p(\Omega)$ for every $q \in [q_0, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$;
(b) K is continuous from $L^q(\Omega)$ to $L^p(\Omega)$ for every $q \in [q_0, 2]$, $\frac{1}{p} + \frac{1}{q} = 1$;
(c) the operator K considered in $L^2(\Omega)$ is a positive self-adjoint operator.

Indeed, K is continuous from $L^q(\Omega)$ into $W^{2,q}(\Omega)$. On the other hand $W^{2,q}(\Omega) \subset L^p(\Omega)$ with continuous embedding. This is clear if $q \ge \frac{n}{2}$. For $q < \frac{n}{2}$ and $\frac{1}{p} + \frac{1}{q} = 1$, observe that

$$p \le \frac{2n}{n-2} \Longleftrightarrow p \le \frac{nq}{n-2q}.$$

According to [5, pp. 51-56], the properties (a)-(c) are sufficient for that the operator K considered from $L^q(\Omega)$ to $L^p(\Omega)$, where $p \in (2, p_0)$ and $\frac{1}{p} + \frac{1}{q} = 1$, admits a representation in the form

$$K = AA^*,$$

where

$$A : L^2(\Omega) \to L^p(\Omega), \quad Av = K^{\frac{1}{2}} v$$

and

$$A^* : L^q(\Omega) \to L^2(\Omega)$$

is the adjoint of A. Here $K^{\frac{1}{2}}$ is the square root of K considered as an operator acting from $L^2(\Omega)$ into $L^2(\Omega)$.

Throughout, by $|\cdot|_p$ we shall mean the usual norm on $L^p(\Omega)$ and by $|A|$ we shall mean

$$|A| = \sup\left\{|Av|_p : \, v \in L^2(\Omega), \, |v|_2 = 1\right\}.$$

1.2. **Schechter's Mountain Pass Theorem.** Now we present the main tool in this paper, Schechter's mountain pass theorem [14]. Let X be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, $B_R = \{v \in X : \, |v| \leq R\}$ the closed ball of X of radius R, $E : X \to \mathbb{R}$ a C^1-functional on X, $v_0, \, v_1 \in X$ and $r > 0$ with

$$|v_0| < r < |v_1| \leq R.$$

Let

$$\Phi = \{\varphi \in C([0,1]; B_R) : \, \varphi(0) = v_0, \, \varphi(1) = v_1\},$$

$$c_R = \inf_{\varphi \in \Phi} \max_{t \in [0,1]} E(\varphi(t))$$

and let

$$\mathcal{K}_{c_R} = \{v \in B_R : \, E(v) = c_R, \, E'(v) = 0\}$$

be the set of critical points of E in B_R at level c_R.

We say that E satisfies the Schechter-Palais-Smale condition on B_R ((S-P-S)$_R$-condition) if

$$(v_k) \subset B_R, \, E(v_k) \text{ - bounded, } (E'(v_k), v_k) \to \nu \leq 0,$$

$$E'(v_k) - \frac{(E'(v_k), \, v_k)}{|v_k|^2} v_k \to 0$$

$$\Longrightarrow (v_k) \text{ has a convergent subsequence.}$$

Theorem 1.2 (Schechter)**.** *Suppose*

- **(i):** E *satisfies* $(S\text{-}P\text{-}S)_R$*-condition;*
- **(ii):** *there exists a constant* C *with* $-(E'(v), v) \leq C$ *for* $|v| = R$*;*
- **(iii):** $v \neq \lambda(v - E'(v))$ *for* $|v| = R$ *and* $\lambda \in (0,1)$*;*
- **(iv):** $\max\{E(v_0), E(v_1)\} \leq \inf\{E(v) : \, |v| = r\}$.

Then $\mathcal{K}_{c_R} \setminus \{v_0, v_1\} \neq \emptyset$.

We note that by the mountain pass geometry of a functional E we mean the geometrical condition (iv) in Theorem 1.2.

2. MAIN RESULTS

We first obtain a lower bound for all non-zero solutions of the superlinear problem

$$\begin{cases} -\Delta u = |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Theorem 2.1. *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *with* C^2*-boundary, let* $p \in \left(2, \frac{2n}{n-2}\right)$ *if* $n \geq 3$ *and* $p \in (2, \infty)$ *if* $n = 1$ *or* $n = 2$*, and let* $\frac{1}{p} + \frac{1}{q} = 1$*. If* $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ *is a non-zero solution of the problem (2.1), then the function* $v = A^*\left(|u|^{p-2} u\right) = A^{-1}u$ *satisfies the inequality*

$$|v|_2 \geq |A|^{-1} \left[(p-1)\lambda_{p-1}\right]^{\frac{1}{p-2}}. \tag{2.2}$$

Proof. Let us first prove that any solution of (2.1) belongs to $C^1\left(\overline{\Omega}\right)$. For $n=1$ this follows from (1.3) (choose $\alpha=0$, $m=1$ and $k=2$). Suppose $n\geq 2$ and fix any number $q_0>n\left(p-1\right)$. If $q\geq\frac{n}{2}$, then (P2) guarantees $u\in L^{q_0}\left(\Omega\right)$. Assume $q<\frac{n}{2}$ and denote $q_1=q$. Since $u\in W^{2,q_1}\left(\Omega\right)$ and $q_1<\frac{n}{2}$, from (1.2) we have $u\in L^{q_1^*}\left(\Omega\right)$, where $q_1^*=\frac{nq_1}{n-2q_1}$. Then $|u|^{p-2}u\in L^{\frac{q_1^*}{p-1}}\left(\Omega\right)$. Let $q_2=\frac{q_1^*}{p-1}$. Since $u=K\left(|u|^{p-2}u\right)$ and $|u|^{p-2}u\in L^{q_2}\left(\Omega\right)$, from (P1), we have that $u\in W^{2,q_2}\left(\Omega\right)$. If $q_2\geq\frac{n}{2}$, as above $u\in L^{q_0}\left(\Omega\right)$; otherwise we continue this way. At the step j we find that

$$u\in W^{2,q_j}\left(\Omega\right),\ q_j=\frac{q_{j-1}^*}{p-1},\ q_{j-1}^*=\frac{nq_{j-1}}{n-2q_{j-1}}, \tag{2.3}$$

where $q_1,q_2,...,q_{j-1}<\frac{n}{2}$ $(j\geq 2)$. We claim that there exists a j with $q_j\geq\frac{n}{2}$. To prove this, suppose the contrary, that is $q_j<\frac{n}{2}$ for every $j\geq 1$. Using $p<\frac{2n}{n-2}$ we can show by induction that the sequence (q_j) is increasing. Consequently, $q_j\to\bar{q}\in\left[q,\frac{n}{2}\right]$ as $j\to\infty$. Next, from (2.3) we obtain

$$q_j\left(n-2q_{j-1}\right)\left(p-1\right)=nq_{j-1}.$$

Letting $j\to\infty$ this yields $\bar{q}\left(n-2\bar{q}\right)\left(p-1\right)=n\bar{q}$ and so

$$\bar{q}=\frac{n\left(p-2\right)}{2\left(p-1\right)}\geq q=\frac{p}{p-1}.$$

This implies $p\geq\frac{2n}{n-2}$, a contradiction. Thus our claim is proved. Therefore, $u\in L^{q_0}\left(\Omega\right)$. Furthermore $|u|^{p-2}u\in L^{q_0/(p-1)}\left(\Omega\right)$ and since $u=K\left(|u|^{p-2}u\right)$, we have $u\in W^{2,q_0/(p-1)}\left(\Omega\right)$. Since $\frac{q_0}{p-1}>n$, by (1.3) one has $W^{2,\frac{q_0}{p-1}}\left(\Omega\right)\subset C^1\left(\overline{\Omega}\right)$ (choose $\alpha=0$, $k=2$, $m=1$). Hence $u\in C^1\left(\overline{\Omega}\right)$.

Let $\overline{u}=K\left(|u|^{p-1}\right)$. Clearly, like u, $\overline{u}\in C^1\left(\overline{\Omega}\right)$ and $\overline{u}=0$ on $\partial\Omega$. By the weak maximum principle, we have

$$|u|\leq\overline{u}\ \text{ on } \overline{\Omega}. \tag{2.4}$$

Hence

$$-\Delta\overline{u}=|u|^{p-1}\leq|u|^{p-2}\,\overline{u}.$$

If we "multiply" by $\overline{u}^{p-1}$ and "integrate" on Ω, we obtain

$$\left(p-1\right)\int_\Omega\overline{u}^{p-2}\left|\nabla\overline{u}\right|^2dx\leq\int_\Omega|u|^{p-2}\,\overline{u}^pdx. \tag{2.5}$$

Now Hölder's inequality yields

$$\begin{aligned}\int_\Omega|u|^{p-2}\,\overline{u}^pdx &\leq \left(\int_\Omega\overline{u}^{\frac{p^2}{2}}dx\right)^{\frac{2}{p}}\left(\int_\Omega|u|^p\,dx\right)^{\frac{p-2}{p}}\\ &= |A\left(v\right)|_p^{p-2}\left(\int_\Omega\overline{u}^{\frac{p^2}{2}}dx\right)^{\frac{2}{p}}.\end{aligned} \tag{2.6}$$

Since $|Av|_p\leq|A|\,|v|_2$ and by (2.4) one has $\overline{u}\neq 0$, from (2.5) and (2.6) we deduce that

$$\left(p-1\right)\lambda_{p-1}\leq|A|^{p-2}\,|v|_2^{p-2}$$

that is (2.2). □

Our next result is the following inequality.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2-boundary. Then for every $p > 2$ one has the inequality*

$$\lambda_{p-1} \leq \frac{1}{(p-1)|A|^2}\,. \tag{2.7}$$

Proof. We consider the functional $E : L^2(\Omega) \to \mathbb{R}$, given by

$$E(v) = \int_\Omega \left(\frac{1}{2}|v(x)|^2 - \frac{1}{p}|(Av)(x)|^p \right) dx. \tag{2.8}$$

Clearly, we have

$$E(v) = \frac{|v|_2^2}{2} - \frac{|Av|_p^p}{p}.$$

For every $v,\ w \in L^2(\Omega)$, it is easy to compute

$$(E'(v), w) = \lim_{\lambda \to 0} \lambda^{-1} (E(v + \lambda w) - E(v))$$

and find

$$(E'(v), w) = (v - A^* F A v, w)\,,$$

where

$$F : L^p(\Omega) \to L^q(\Omega)\,,\ \ F(u) = |u|^{p-2}\, u.$$

Hence

$$E'(v) = v - A^* F A v.$$

Notice if u is a solution of (2.1) then $v = A^*\left(|u|^{p-2}\, u\right) = A^{-1}u$ is a critical point of the functional (2.8). Conversely, if v is a critical point of the functional (2.8), then $u = Av$ is a solution of (2.1).

Our plan is as follows: we show that for every $R < R_0$, where

$$R_0 = |A|^{-1}\left[(p-1)\lambda_{p-1}\right]^{\frac{1}{p-2}} \tag{2.9}$$

(of course here we assume $\lambda_{p-1} > 0$, (2.7) being trivial if $\lambda_{p-1} = 0$), $v_0 = 0$ is the unique critical point of E in $B_R = \{v \in L^2(\Omega) : |v|_2 \leq R\}$ and that the hypotheses (i)-(iii) in Theorem 1.2 hold. Consequently, there exist no v_1 and r with $0 < r < |v_1|_2 \leq R$ such that the geometrical condition (iv) is satisfied. As a result we obtain (2.7).

(a) The (S-P-S)$_R$-condition is satisfied for every $R > 0$. Indeed, let (v_k) be any sequence of functions in B_R with

$$(E'(v_k), v_k) \to \nu \leq 0, \quad E'(v_k) - \beta(v_k)\, v_k \to 0, \tag{2.10}$$

where $\beta(v_k) = \frac{(E'(v_k), v_k)}{|v_k|_2^2}$. Passing if necessarily to a subsequence, we may suppose that $|v_k|_2 \to d$ for some $d \in [0, R]$. If $d = 0$ we are done. So assume $d > 0$. Denote $w_k = E'(v_k) - \beta(v_k)\, v_k$. We have $w_k = (1 - \beta(v_k))\, v_k - A^* F A v_k$. Hence

$$v_k = (1 - \beta(v_k))^{-1}(w_k - A^* F A v_k) \tag{2.11}$$

and so

$$A v_k = (1 - \beta(v_k))^{-1}(A w_k - K F A v_k)\,. \tag{2.12}$$

Notice $K(L^q(\Omega)) \subset W^{2,q}(\Omega)$ and the embedding of $W^{2,q}(\Omega)$ into $L^p(\Omega)$ is compact. Indeed, from $p \in \left(2, \frac{2n}{n-2}\right)$ and $\frac{1}{p} + \frac{1}{q} = 1$, we easily see that $p \in \left(2, \frac{nq}{n-2q}\right)$ when $q < \frac{n}{2}$.

Hence the compact embedding is guaranteed by (P2). As a result, we may suppose that (at least for a subsequence) $(KFAv_k)$ is convergent. In addition, by (2.10), we have

$$Aw_k \to 0, \quad (1-\beta(v_k))^{-1} \to \left(1-\frac{\nu}{d^2}\right)^{-1} \in (0,1].$$

Then, from (2.12), we find that (at least for a subsequence) (Av_k) is convergent. Finally (2.11) guarantees that the corresponding subsequence of (v_k) is convergent.

(b) For each $R > 0$, there exists a constant C_R such that

$$-(E'(v), v) \leq C_R \text{ for all} v \in L^2(\Omega) \text{ with } |v|_2 = R.$$

Indeed, if $|v|_2 = R$, then

$$\begin{aligned} -(E'(v), v) &= -|v|_2^2 + (A^* F A v, v) \\ &= -|v|_2^2 + (FAv, Av) \\ &= -|v|_2^2 + |Av|_p^p \\ &\leq -|v|_2^2 + |A|^p |v|_2^p \\ &= -R^2 + |A|^p R^p \\ &= : C_R. \end{aligned}$$

(c) Zero is the unique critical point of E with $|v|_2 < R_0$ (here R_0 is given by (2.9)). Indeed, if $v \in L^2(\Omega)$ is a non-zero critical point of E, then $v = A^* F A v$ and so $Av = KFAv$. Hence $u = Av$ is a non-zero solution of problem (2.1). Therefore, according to Theorem 2.1, $|v|_2 \geq R_0$.

(d) The Leray-Schauder boundary condition (iii) holds for every $R < R_0$. To prove this suppose the contrary. Then there exists a $v \in L^2(\Omega)$ with $|v|_2 = R$ and a $\lambda \in (0,1)$ with $v = \lambda(v - E'(v))$, i.e. $v = \lambda A^* F A v$. It is easily seen that the function $\overline{v} = \lambda^{1/(p-2)} v$ satisfies $\overline{v} = A^* F A \overline{v}$, i.e. $\overline{v}$ is a critical point of E with $|\overline{v}|_2 < R_0$. According to the conclusion of step (c), $\overline{v} = 0$ and so $v = 0$, a contradiction.

(e) Proof of (2.7). Let

$$r = |A|^{-\frac{p}{p-2}}.$$

Obviously, (2.7) can be written as $r \geq R_0$. To prove it, we shall assume the contrary, i.e. $r < R_0$. Choose any $R \in (r, R_0)$, $\lambda \in (r, R]$ and $\varepsilon > 0$ sufficiently small so that

$$\phi(\lambda) + p^{-1}\lambda^p \varepsilon \leq \phi(r), \tag{2.13}$$

where

$$\phi(\sigma) = \frac{\sigma^2}{2} - p^{-1}\sigma^p |A|^p \quad (\sigma \geq 0).$$

Notice r is the maximum point of ϕ, ϕ is increasing on $[0, r]$ and decreasing on $[r, \infty)$. Now we choose a function $v_2 \in L^2(\Omega)$ with

$$|v_2|_2 = 1 \text{ and } |Av_2|_p^p \geq |A|^p - \varepsilon.$$

We claim that condition (iv) in Theorem 1.2 holds for $v_0 = 0$ and $v_1 = \lambda v_2$. Indeed

$$\begin{aligned} E(v_1) &= E(\lambda v_2) \\ &= \frac{\lambda^2}{2} - p^{-1}\lambda^p |Av_2|_p^p \\ &\leq \frac{\lambda^2}{2} - p^{-1}\lambda^p |A|^p + p^{-1}\lambda^p \varepsilon \\ &= \phi(\lambda) + p^{-1}\lambda^p \varepsilon. \end{aligned} \tag{2.14}$$

Also, for every $v \in L^2(\Omega)$ with $|v|_2 = r$, we have

$$E(v) = \frac{r^2}{2} - p^{-1} r^p \left|A\left(r^{-1}v\right)\right|_p^p \geq \frac{r^2}{2} - p^{-1} r^p |A|^p = \phi(r). \tag{2.15}$$

Now (2.13), (2.14) and (2.15) guarantee (iv). From Theorem 1.2 it follows that E has a non-zero critical point in the closed ball B_R of $L^2(\Omega)$. This contradiction to the conclusion at step (c) proves (2.7).

□

We note that Theorems 2.1-2.2 were previously announced in [12].

The next inequality of Poincaré type shows that $\lambda_{p-1} > 0$ for $p \in \left[2, \frac{2n}{n-2}\right]$ if $n \geq 3$ and for $p \in [2, \infty)$ if $n = 2$. Moreover, its proof connects λ_{p-1} to the embedding constant of $W_0^{1,2}(\Omega)$ into $L^p(\Omega)$.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded open and let $p \in \left[2, \frac{2n}{n-2}\right]$ if $n \geq 3$, $p \in [2, \infty)$ for $n = 2$. Then there exists a constant $c > 0$ depending only on p and Ω, such that*

$$\left(\int_\Omega |u|^{\frac{p^2}{2}} dx\right)^{\frac{2}{p}} \leq c \int_\Omega |u|^{p-2} |\nabla u|^2 dx \tag{2.16}$$

for all $u \in C_0^1\left(\overline{\Omega}\right)$.

Proof. According to (P2), we have $W_0^{1,2}(\Omega) \subset L^p(\Omega)$ with continuous embedding. Hence there exists a constant $c_0 > 0$ with

$$|v|_p \leq c_0 |v|_{W_0^{1,2}(\Omega)} \text{ for all } v \in W_0^{1,2}(\Omega).$$

Here

$$|v|_{W_0^{1,2}(\Omega)} = \left(\int_\Omega |\nabla v|^2 dx\right)^{\frac{1}{2}}.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$, we may suppose that

$$c_0 = \sup\left\{|v|_p : v \in C_0^\infty(\Omega), |v|_{W_0^{1,2}(\Omega)} = 1\right\}.$$

The space $C_0^\infty(\Omega)$ is also dense in $C_0^1\left(\overline{\Omega}\right)$, and so

$$\lambda_{p-1} = \left(\sup\left\{\left(\int_\Omega |u|^{p^2/2} dx\right)^{\frac{2}{p}} : u \in C_0^\infty(\Omega), \int_\Omega |u|^{p-2} |\nabla u|^2 dx = 1\right\}\right)^{-1}.$$

After substituting $v = \left(\frac{2}{p}\right)|u|^{\frac{p}{2}}$, we obtain

$$\begin{aligned} \lambda_{p-1} &= \left(\frac{2}{p}\right)^2 \left(\sup\left\{|v|_p^2 : v \in C_0^\infty(\Omega), |v|_{W_0^{1,2}(\Omega)} = 1\right\}\right)^{-1} \\ &= \left(\frac{2}{p\,c_0}\right)^2. \end{aligned}$$

Thus (2.16) holds with the smallest constant

$$c = \lambda_{p-1}^{-1} = \left(p\,\frac{c_0}{2}\right)^2.$$

□

Finally we establish a localization result for a non-zero solution to the problem (2.1).

Theorem 2.4. *Let Ω be a bounded domain of $\mathbb{R}^n$ with C^2-boundary and let $p \in \left(2, \frac{2n}{n-2}\right)$ if $n \geq 3$ and $p \in (2, \infty)$ if $n = 1$ or $n = 2$. Then the problem (2.1) has a solution u with*

$$|A|^{-1} \left[(p-1)\, \lambda_{p-1}\right]^{\frac{1}{p-2}} \leq \left|A^{-1}u\right|_2 \leq |A|^{-\frac{p}{p-2}}\,. \tag{2.17}$$

Proof. First notice the left inequality in (2.17) is true for all non-zero solutions of (2.1) according to Theorem 2.1.

Next we prove that for each $R > r = |A|^{-\frac{p}{p-2}}$, (2.1) has a solution u such that

$$\left|A^{-1}u\right|_2 \leq R. \tag{2.18}$$

Indeed, two cases are possible:

(1) The Leray-Schauder boundary condition (iii) in Theorem 1.2 does not hold. Then, there are $v \in L^2(\Omega)$ and $\lambda \in (0,1)$ such that $|v|_2 = R$ and $v = \lambda A^* F A v$. It is easy to see that the function $\overline{v} = \lambda^{1/(p-2)} v$ satisfies $\overline{v} = A^* F A \overline{v}$, i.e. $\overline{v}$ is a critical point of E, and $0 < |\overline{v}|_2 < |v|_2 = R$. Hence $u := A\overline{v}$ is a solution of (2.1) and satisfies (2.18).

(2) Condition (iii) in Theorem 1.2 holds. Then, as follows from the proof of Theorem 2.2, all the assumptions of Theorem 1.2 are satisfied. Now the existence of a solution u of (2.1) satisfying (2.18) is guaranteed by Theorem 1.2.

Finally, for each positive integer k we put $R = r + \frac{1}{k}$ to obtain a solution u_k with $|A^{-1}u_k|_2 \leq r + \frac{1}{k}$, and the result will follow via a limit argument. □

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