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A MONOTONICITY PROPERTY OF POWER MEANS

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ABSTRACT. If A and G are the arithmetic and geometric means of the numbers $x_j \in (a, b)$, a family of inequalities is derived of which $a + b - A > ab/G$ is a special case. These inequalities demonstrate a new monotonicity property for power means.

Key words and phrases: Arithmetic mean, Geometric mean, Monotonicity, Power means.

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1. INTRODUCTION

Let A, G and H be the arithmetic, geometric and harmonic means of the positive numbers $x_1 \le x_2 \le ... \le x_n$ formed with the positive weights w_k whose sum is unity. Then the following inequalities were proved in [5].

If

$$P(x) \equiv \frac{x - G}{2x[x - A]} \text{ and } Q(x) \equiv \frac{x - G}{2x[x - G] - 2G[A - G]},$$

then

$$P(x_1) \sum_1^n w_k(x_k - A)^2 > A - G > P(x_n) \sum_1^n w_k(x_k - A)^2 \tag{1.1}$$

and

$$Q(x_1) \sum_1^n w_k(x_k - G)^2 > A - G > Q(x_n) \sum_1^n w_k(x_k - G)^2, \tag{1.2}$$

provided that at least two of the x_k are distinct.

These inequalities improved similar ones which were proved in [1], [3] and [6]. Instead of the multipliers P and Q appearing here, the earlier results had $(2x_1)^{-1}$and $(2x_n)^{-1}$ appearing in the upper and lower bounds respectively, in each of (1.1) and (1.2). Now these inequalities

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imply that $P(x_1) > P(x_n)$ and $Q(x_1) > Q(x_n)$ and it is a simple matter to see that these in turn are each equivalent to the inequality

$$x_1 + x_n - A > \frac{x_1 x_n}{G}. \tag{1.3}$$

It would seem to be of interest to give a proof of this in its own right but as well as doing that we shall also introduce a family of inequalities of which (1.3) is a special case. That is the purpose of this note.

Note.

In all that follows we shall suppose that $0 < a < b$ and that $a \le x_1 \le x_2 \le \cdots \le x_n \le b$ with at least one of the x_k satisfying $a < x_k < b$.

We now prove a slight generalization of (1.3).

Lemma 1.1.

$$a + b - A > \frac{ab}{G}. \tag{1.4}$$

Proof. Since $(b-t)(t-a)$ is non-negative for $a \le t \le b$, division by t gives

$$a + b - t \ge \frac{ab}{t} \qquad \text{(with equality only if } t = a \text{ or } t = b).$$

Put $t = x_k$ for $k = 1, 2, \dots, n$. Forming the arithmetic mean on the left and the geometric mean on the right completes the proof of the lemma. □

We next look at some consequences of this inequality. Making the substitutions $a \to a^{-1}, b \to b^{-1}, x_k \to x_k^{-1}$ in it and taking inverses extends (1.4) to

$$a + b - A > \frac{ab}{G} > \left(a^{-1} + b^{-1} - H^{-1}\right)^{-1}.$$

With $r > 0$, we substitute $a \to a^r$, $b \to b^r$, $x_k \to x_k^r$ in this and then raise all three members to the power $\frac{1}{r}$. We get

$$\left(a^r + b^r - \sum w_k x_k^r\right)^{\frac{1}{r}} > \frac{ab}{G} > \left(a^{-r} + b^{-r} - \sum w_k x_k^{-r}\right)^{-\frac{1}{r}}.$$

Now introducing the notation

$$Q_r(a, b, x) \equiv \left(a^r + b^r - \sum w_k x_k^r\right)^{\frac{1}{r}} \quad \text{for all real } r \neq 0, \tag{1.5}$$

these last inequalities read

$$Q_r(a, b, x) > Q_0(a, b, x) > Q_{-r}(a, b, x) \text{ for } r > 0, \tag{1.6}$$

where we have written $Q_0(a, b, x)$ for the limit $\lim_{r \to 0} Q_r(a, b, x)$ whose value is easily seen to be

$$\frac{ab}{G}.$$

2. The Main Result

The considerations of the previous section lead us to formulate the following theorem.

Theorem 2.1. *Let* $+\infty > r > s > -\infty$. *Then*

$$b > Q_r(a, b, x) > Q_s(a, b, x) > a. \tag{2.1}$$

Note.

From (1.5) we see that $Q_r(a,b,x)$ can be written as

$$Q_r(a,b,x) \equiv (a^r + b^r - M_r{}^r(x,w))^{\frac{1}{r}},$$

where $M_r(x,w)$ is the 'power mean' of the numbers x_k with weights w_k defined by

$$M_r(x,w) = \left(\sum w_k x_k^r\right)^{\frac{1}{r}} \ (r \neq 0) \text{ and } M_0(x,w) = \lim_{r\to 0} M_r(x,w).$$

For the various properties of these means we refer the reader to [2] or [4]. In particular, it is well-known that they have the monotonicity property:

$$x_n > M_r(x,w) > M_s(x,w) > x_1 \ (+\infty > r > s > -\infty)$$

and so writing (2.1) as

$$b > (a^r + b^r - M_r^r(x,w))^{\frac{1}{r}} > (a^s + b^s - M_s(x,w))^{\frac{1}{s}} > a,$$

we see that this is another monotonicity property of power means.

Proof of the theorem. There are three cases which remain to be considered:

(a) $r > s > 0$,
(b) $0 > r > s$,
(c) $r > 0 > s$.

Once these are proved it is a simple matter to verify that

$$\lim_{r\to+\infty} Q_r(a,b,x) = b \quad \text{and} \quad \lim_{r\to-\infty} Q_r(a,b,x) = a,$$

giving the upper and lower bounds in the theorem.

The cases (b) and (c) follow easily from (a) and (1.5) above. So let us suppose the truth of case (a) for the moment and dispose of these other cases first.

(a) reads

$$\left(a^r + b^r - \sum w_k x_k^r\right)^{\frac{1}{r}} > \left(a^s + b^s - \sum w_k x_k^s\right)^{\frac{1}{s}} \quad \text{when} \quad r > s > 0.$$

If we make the substitutions $a \to a^{-1}, \ b \to b^{-1}$, $x_k \to x_k^{-1}$ in this and then invert both sides it reads

$$\left(a^{-r} + b^{-r} - \sum w_k x_k^{-r}\right)^{-\frac{1}{r}} < \left(a^{-s} + b^{-s} - \sum w_k x_k^{-s}\right)^{-\frac{1}{s}} \quad \text{when} \quad -r < -s < 0.$$

Writing $r = -p$ and $s = -q$ this reads

$$Q_q(a,b,x) > Q_p(a,b,x) \ \text{ when } 0 > q > p$$

which is case (b).

The case (c) where $r > 0 > s$ has two subcases namely $|r| > |s|$ and $|s| > |r|$.

The former follows by noting that $Q_r(a,b,x) > Q_{-s}(a,b,x) > Q_s(a,b,x)$ by virtue of (a) and (1.5).

The latter follows since $Q_r(a,b,x) > Q_{-r}(a,b,x) > Q_s(a,b,x)$ by virtue of (1.5) and (b). So the cases (b) and (c) have been dealt with.

It now remains to give the proof of case (a) and it is sufficient to prove that

$$\left(a^\alpha + b^\alpha - \sum w_k x_k^\alpha\right)^{\frac{1}{\alpha}} > \left(a + b - \sum w_k x_k\right) \quad \text{for} \ \ \alpha > 1. \tag{2.2}$$

For once this is proved (a) follows on putting $\alpha = \frac{r}{s}$, making the substitutions $a \to a^s$, $b \to b^s$, $x_k \to x_k^s$ and then raising each side to the power $\frac{1}{s}$. We now proceed to prove (2.2).

Let

$$V = \left(a^\alpha + b^\alpha - \sum w_k x_k^\alpha\right)^{\frac{1}{\alpha}} - \left(a + b - \sum w_k x_k\right)$$

and differentiate this with respect to some particular x_j which satisfies $a < x_j < b$. We get

$$\frac{dV}{dx_j} = \left[\left(a^\alpha + b^\alpha - \sum w_k x_k^\alpha\right)^{\frac{1-\alpha}{\alpha}}\right]\left(-w_j x_j^{\alpha-1}\right) + w_j.$$

After a few lines the right side reduces to

$$w_j \left[1 - \left(\frac{x_j}{Z}\right)^{\alpha-1}\right]$$

where, for brevity, we have written $Z \equiv (a^\alpha + b^\alpha - \sum w_k x_k^\alpha)^{\frac{1}{\alpha}}$.

So we see that $\frac{dV}{dx_j}$ is positive or negative in case x_j is less than (greater than) Z. So V will decrease if $x_j < Z$ and x_j decreases, or if $x_j > Z$ and x_j increases. Also, by differentiating $|Z - x_j|$ with respect to x_j we find that this derivative is

$$\mp \left[w_j x_j^{\alpha-1}\left(a^\alpha + b^\alpha - \sum w_k x_k^\alpha\right)^{\frac{1-\alpha}{\alpha}} + 1\right]$$

in case x_j is less than (greater than) Z. Hence, if x_j is less than (greater than) Z it will remain so as x_j decreases (increases).

These considerations lead us to proceed as follows:

Taking those x_k which lie strictly between a and b in the order of increasing subscript we let them tend to a or b, one by one, according to the rules:

(i) if x_k is less than the current Z let $x_k \to a$.
(ii) if x_k is greater than or equal to the current Z let $x_k \to b$.

In this way we produce a strictly decreasing sequence whose first member is V and whose last member contains no x_k's at all. We conclude that

$$\begin{aligned} V &= \left(a^\alpha + b^\alpha - \sum w_k x_k^\alpha\right)^{\frac{1}{\alpha}} - \left(a + b - \sum w_k x_k\right) \\ &> [(1-W_1)a^\alpha + (1-W_2)b^\alpha]^{\frac{1}{\alpha}} - [(1-W_1)a + (1-W_2)b], \end{aligned}$$

where W_1 and W_2 are positive numbers with $W_1 + W_2 = 1$. But this last expression is positive by virtue of the classical inequality $M_r(x, w) > M_s(x, w)$, $(r > s)$ for power means already referred to above.

This completes the proof of the theorem. □

References

[1] H. ALZER, A new refinement of the arithmetic mean - geometric mean inequality, *Rocky Mountain J. Math.*, **27**(2) (1997), 663–667.

[2] P.S. BULLEN, D.S. MITRINOVIĆ AND P.M. VASIC, *Means and their Inequalities*, Kluwer Academic Publishers, Dordrecht/Boston/Lancaster/Tokyo, 1988.

[3] D.I. CARTWRIGHT AND M.J. FIELD, A refinement of the arithmetic mean - geometric mean inequality, *Proc. Amer. Math. Soc.*, **71** (1978), 36–38.

[4] G.H. HARDY, J.E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, United Kingdom, 1952.

[5] A.McD. MERCER, Improved upper and lower bounds for the difference $A_n - G_n$, *Rocky Mountain J. Math.*, **31**(2) (2001), 553–560.

[6] A.McD. MERCER, Bounds for $A-G, A-H, G-H$, and a family of inequalities of Ky Fan's type, using a general method, *J. Math. Anal. App.*, **243** (2000), 163–173.