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AN INEQUALITY IMPROVING THE FIRST HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS DEFINED ON LINEAR SPACES AND APPLICATIONS FOR SEMI-INNER PRODUCTS

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ABSTRACT. An integral inequality for convex functions defined on linear spaces is obtained which contains in a particular case a refinement for the first part of the celebrated Hermite-Hadamard inequality. Applications for semi-inner products on normed linear spaces are also provided.

Key words and phrases: Hermite-Hadamard integral inequality, Convex functions, Semi-Inner Products.

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1. INTRODUCTION

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = (\nabla_{\pm} f[(1-s)a + sb])(b-a)$, $s \in (0,1)$
(ii) $g'_{+}(a,b)(0) = (\nabla_{+} f(a))(b-a)$
(iii) $g'_{-}(a,b)(1) = (\nabla_{-} f(b))(b-a)$

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where $(\bigtriangledown_{\pm} f(x))(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$(\bigtriangledown_{+} f(x))(y) := \lim_{h\to 0+}\left[\frac{f(x+hy)-f(x)}{h}\right],$$
$$(\bigtriangledown_{-} f(x))(y) := \lim_{k\to 0-}\left[\frac{f(x+ky)-f(x)}{k}\right], \quad x,y\in X.$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b]\subset X$:

(HH) $$f\left(\frac{a+b}{2}\right)\le \int_0^1 f[(1-t)a+tb]\,dt \le \frac{f(a)+f(b)}{2},$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b):[0,1]\to\mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right)\le \int_0^1 g(a,b)(t)\,dt \le \frac{g(a,b)(0)+g(a,b)(1)}{2}.$$

For other related results see the monograph on line [1].

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := (\bigtriangledown_{+} f_0(y))(x) = \lim_{t\to 0+}\left[\frac{\|y+tx\|^2-\|y\|^2}{2t}\right]$;

(v) $\langle x,y\rangle_i := (\bigtriangledown_{-} f_0(y))(x) = \lim_{s\to 0-}\left[\frac{\|y+sx\|^2-\|y\|^2}{2s}\right]$;

for any $x,y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [2]), assuming that $p,q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;

(aa) $\langle \alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if $\alpha,\beta\ge 0$ and $x,y\in X$;

(aaa) $\left|\langle x,y\rangle_p\right|\le\|x\|\|y\|$ for all $x,y\in X$;

(av) $\langle \alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if $x,y\in X$ and $\alpha\in\mathbb{R}$;

(v) $\langle -x,y\rangle_p=-\langle x,y\rangle_q$ for all $x,y\in X$;

(va) $\langle x+y,z\rangle_p\le\|x\|\|z\|+\langle y,z\rangle_p$ for all $x,y,z\in X$;

(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);

(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s=\langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i\le\langle y,x\rangle_s$ for all $x,y\in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i=\langle y,x\rangle=\langle y,x\rangle_s$ for all $x,y\in X$.

Applying inequality (HH) for the convex function $f_0(x)=\frac{1}{2}\|x\|^2$, one may deduce the inequality

(1.1) $$\left\|\frac{x+y}{2}\right\|^2\le\int_0^1\|(1-t)x+ty\|^2\,dt\le\frac{\|x\|^2+\|y\|^2}{2}$$

for any $x,y\in X$. The same (HH) inequality applied for $f_1(x)=\|x\|$, will give the following refinement of the triangle inequality:

(1.2) $$\left\|\frac{x+y}{2}\right\|\le\int_0^1\|(1-t)x+ty\|\,dt\le\frac{\|x\|+\|y\|}{2},\quad x,y\in X.$$

In this paper we point out an integral inequality for convex functions which is related to the first Hermite-Hadamard inequality in (HH) and investigate its applications for semi-inner products in normed linear spaces.

2. THE RESULTS

We start with the following lemma which is also of interest in itself.

Lemma 2.1. *Let $h : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then for any $\gamma \in [\alpha, \beta]$ one has the inequality*

$$\frac{1}{2}\left[(\beta-\gamma)^2 h'_+(\gamma) - (\gamma-\alpha)^2 h'_-(\gamma)\right] \le \int_\alpha^\beta h(t)\,dt - (\beta-\alpha)h(\gamma) \le \frac{1}{2}\left[(\beta-\gamma)^2 h'_-(\beta) - (\gamma-\alpha)^2 h'_+(\alpha)\right]. \tag{2.1}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $\gamma = \alpha$ or $\gamma = \beta$.

Proof. It is easy to see that for any locally absolutely continuous function $h : (\alpha, \beta) \to \mathbb{R}$, we have the identity

$$\int_\alpha^\gamma (t-\alpha)h'(t)\,dt + \int_\gamma^\beta (t-\beta)h'(t)\,dt = h(\gamma) - \int_\alpha^\beta h(t)\,dt \tag{2.2}$$

for any $\gamma \in (\alpha, \beta)$, where h' is the derivative of h which exists a.e. on (α, β).

Since h is convex, then it is locally Lipschitzian and thus (2.2) holds. Moreover, for any $\gamma \in (\alpha, \beta)$, we have the inequalities

$$h'(t) \le h'_-(\gamma) \text{ for a.e. } t \in [\alpha, \gamma] \tag{2.3}$$

and

$$h'(t) \ge h'_+(\gamma) \text{ for a.e. } t \in [\gamma, \beta]. \tag{2.4}$$

If we multiply (2.3) by $t - \alpha \ge 0$, $t \in [\alpha, \gamma]$ and integrate on $[\alpha, \gamma]$, we get

$$\int_\alpha^\gamma (t-\alpha)h'(t)\,dt \le \frac{1}{2}(\gamma-\alpha)^2 h'_-(\gamma) \tag{2.5}$$

and if we multiply (2.4) by $\beta - t \ge 0$, $t \in [\gamma, \beta]$, and integrate on $[\gamma, \beta]$, we also have

$$\int_\gamma^\beta (\beta-t)h'(t)\,dt \ge \frac{1}{2}(\beta-\gamma)^2 h'_+(\gamma). \tag{2.6}$$

If we subtract (2.6) from (2.5) and use the representation (2.2), we deduce the first inequality in (2.1).

Now, assume that the first inequality (2.1) holds with $C > 0$ instead of $\frac{1}{2}$, i.e.,

$$C\left[(\beta-\gamma)^2 h'_+(\gamma) - (\gamma-\alpha)^2 h'_-(\gamma)\right] \le \int_\alpha^\beta h(t)\,dt - (\beta-\alpha)h(\gamma). \tag{2.7}$$

Consider the convex function $h_0(t) := k\left|t - \frac{\alpha+\beta}{2}\right|$, $k > 0$, $t \in [\alpha, \beta]$. Then

$$h'_{0+}\left(\frac{\alpha+\beta}{2}\right) = k, \quad h'_{0-}\left(\frac{\alpha+\beta}{2}\right) = -k, \quad h_0\left(\frac{\alpha+\beta}{2}\right) = 0$$

and

$$\int_\alpha^\beta h_0(t)\,dt = \frac{1}{4}k(\beta-\alpha)^2.$$

If in (2.7) we choose $h=h_0$, $\gamma=\frac{\alpha+\beta}{2}$, then we get

$$C\left[\frac{1}{4}(\beta-\alpha)^2 k+\frac{1}{4}(\beta-\alpha)^2 k\right]\leq\frac{1}{4}k(\beta-\alpha)^2$$

which gives $C\leq\frac{1}{2}$ and the sharpness of the constant in the first part of (2.1) is proved.

If either $h'_+(\alpha)=-\infty$ or $h'_-(\beta)=-\infty$, then the second inequality in (2.1) holds true.

Assume that $h'_+(\alpha)$ and $h'_-(\beta)$ are finite. Since h is convex on $[\alpha,\beta]$, we have

$$h'(t)\geq h'_+(\alpha)\ \text{ for a.e. } t\in[\alpha,\gamma]\quad(\gamma \text{ may be equal to } \beta) \tag{2.8}$$

and

$$h'(t)\leq h'_-(\beta)\ \text{ for a.e. } t\in[\gamma,\beta]\quad(\gamma \text{ may be equal to } \alpha). \tag{2.9}$$

If we multiply (2.8) by $t-\alpha\geq 0$, $t\in[\alpha,\gamma]$ and integrate on $[\alpha,\gamma]$, then we deduce

$$\int_\alpha^\gamma (t-\alpha)h'(t)\,dt\geq\frac{1}{2}(\gamma-\alpha)^2 h'_+(\alpha) \tag{2.10}$$

and if we multiply (2.9) by $\beta-t\geq 0$, $t\in[\gamma,\beta]$, and integrate on $[\gamma,\beta]$, then we also have

$$\int_\gamma^\beta (\beta-t)h'(t)\,dt\leq\frac{1}{2}(\beta-\gamma)^2 h'_-(\beta). \tag{2.11}$$

Finally, if we subtract (2.10) from (2.11) and use the representation (2.2), we deduce the second inequality in (2.1). Now, assume that the second inequality in (2.1) holds with a constant $D>0$ instead of $\frac{1}{2}$, i.e.,

$$\int_\alpha^\beta h(t)\,dt-(\beta-\alpha)h(\gamma)\leq D\left[(\beta-\gamma)^2 h'_-(\beta)-(\gamma-\alpha)^2 h'_+(\alpha)\right]. \tag{2.12}$$

If we consider the convex function $h_0(t)=k\left|t-\frac{\alpha+\beta}{2}\right|$, $k>0$, $t\in[\alpha,\beta]$, then we have $h'_{0-}(\beta)=k$, $h'_{0+}(\alpha)=-k$ and by (2.12) applied for h_0 in $\gamma=\frac{\alpha+\beta}{2}$ we get

$$\frac{1}{4}k(\beta-\alpha)^2\leq D\left[\frac{1}{4}k(\beta-\alpha)^2+\frac{1}{4}k(\beta-\alpha)^2\right],$$

giving $D\geq\frac{1}{2}$ which proves the sharpness of the constant $\frac{1}{2}$ in the second inequality in (2.1). □

Corollary 2.2. *With the assumptions of Lemma 2.1 and if $\gamma\in(\alpha,\beta)$ is a point of differentiability for h, then*

$$\left(\frac{\alpha+\beta}{2}-\gamma\right)h'(\gamma)\leq\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt-h(\gamma). \tag{2.13}$$

Now, recall that the following inequality, which is well known in the literature as the Hermite-Hadamard inequality for convex functions, holds

$$h\left(\frac{\alpha+\beta}{2}\right)\leq\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt\leq\frac{h(\alpha)+h(\beta)}{2}. \tag{2.14}$$

The following corollary provides both a sharper lower bound for the difference,

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt-h\left(\frac{\alpha+\beta}{2}\right),$$

which we know is nonnegative, and an upper bound.

Corollary 2.3. *Let $h : [\alpha, \beta] \to \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then we have the inequality*

$$0 \leq \frac{1}{8}\left[h'_{+}\left(\frac{\alpha+\beta}{2}\right) - h'_{-}\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \leq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} h(t)\,dt - h\left(\frac{\alpha+\beta}{2}\right) \leq \frac{1}{8}\left[h'_{-}(\beta) - h'_{+}(\alpha)\right](\beta-\alpha). \tag{2.15}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Example 2.1. Assume that $-\infty < \alpha < 0 < \beta < \infty$ and consider the convex function $h : [\alpha, \beta] \to \mathbb{R}$, $h(x) = \exp|x|$. We have

$$h'(x) = \begin{cases} -e^{-x} & \text{if } x < 0, \\ e^{x} & \text{if } x > 0; \end{cases}$$

and $h'_{-}(0) = -1$, $h'_{+}(0) = 1$. Also,

$$\int_{\alpha}^{\beta} h(t)\,dt = \int_{\alpha}^{0} e^{-x}dx + \int_{0}^{\beta} e^{x}dx = \exp(\beta) + \exp(-\alpha) - 2.$$

Now, if $\frac{\alpha+\beta}{2} \neq 0$, then by (2.15) we deduce the elementary inequality

$$0 \leq \frac{\exp(\beta) + \exp(-\alpha) - 2}{\beta - \alpha} - \exp\left|\frac{\alpha+\beta}{2}\right| \leq \frac{1}{8}\left[\exp(\beta) + \exp(-\alpha)\right](\beta-\alpha). \tag{2.16}$$

If $\frac{\alpha+\beta}{2} = 0$ and if we denote $\beta = a$, $a > 0$, thus $\alpha = -a$ and by (2.15) we also have

$$\frac{1}{2}a \leq \frac{\exp(a) - 1}{a} - 1 \leq \frac{1}{2}a\exp(a). \tag{2.17}$$

The reader may produce other elementary inequalities by choosing in an appropriate way the convex function h. We omit the details.

We are now able to state the corresponding result for convex functions defined on linear spaces.

Theorem 2.4. *Let X be a linear space, $a, b \in X$, $a \neq b$ and $f : [a, b] \subset X \to \mathbb{R}$ be a convex function on the segment $[a, b]$. Then for any $s \in (0, 1)$ one has the inequality*

$$\frac{1}{2}\left[(1-s)^2\left(\triangledown_{+}f\left[(1-s)a+sb\right]\right)(b-a) - s^2\left(\triangledown_{-}f\left[(1-s)a+sb\right]\right)(b-a)\right] \leq \int_0^1 f\left[(1-t)a+tb\right]dt - f\left[(1-s)a+sb\right] \leq \frac{1}{2}\left[(1-s)^2\left(\triangledown_{-}f(b)\right)(b-a) - s^2\left(\triangledown_{+}f(a)\right)(b-a)\right]. \tag{2.18}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $s = 0$ or $s = 1$.

Proof. Follows by Lemma 2.1 applied for the convex function $h(t) = g(a,b)(t) = f[(1-t)a+tb]$, $t \in [0,1]$, and the choices $\alpha = 0$, $\beta = 1$, and $\gamma = s$. □

Corollary 2.5. *If $f : [a,b] \to \mathbb{R}$ is as in Theorem 2.4 and Gâteaux differentiable in $c := (1-\lambda)a + \lambda b$, $\lambda \in (0,1)$ along the direction $(b-a)$, then we have the inequality:*

$$\left(\frac{1}{2}-\lambda\right)(\bigtriangledown f(c))(b-a) \le \int_0^1 f[(1-t)a+tb]\,dt - f(c). \tag{2.19}$$

The following result related to the first Hermite-Hadamard inequality for functions defined on linear spaces also holds.

Corollary 2.6. *If f is as in Theorem 2.4, then*

$$\begin{aligned} 0 &\le \frac{1}{8}\left[\bigtriangledown_+ f\left(\frac{a+b}{2}\right)(b-a) - \bigtriangledown_- f\left(\frac{a+b}{2}\right)(b-a)\right] \\ &\le \int_0^1 f[(1-t)a+tb]\,dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{1}{8}\left[(\bigtriangledown_- f(b))(b-a) - (\bigtriangledown_+ f(a))(b-a)\right]. \end{aligned} \tag{2.20}$$

The constant $\frac{1}{8}$ is sharp in both inequalities.

Now, let $\Omega \subset \mathbb{R}^n$ be an open and convex set in $\mathbb{R}^n$.

If $F : \Omega \to \mathbb{R}$ is a differentiable convex function on Ω, then, obviously, for any $\bar{c} \in \Omega$ we have

$$\nabla F(\bar{c})(\bar{y}) = \sum_{i=1}^n \frac{\partial F(\bar{c})}{\partial x_i} \cdot y_i, \qquad \bar{y} \in \mathbb{R}^n,$$

where $\frac{\partial F}{\partial x_i}$ are the partial derivatives of F with respect to the variable x_i $(i = 1, \dots, n)$.

Using (2.18), we may state that

$$\begin{aligned} &\left(\frac{1}{2}-\lambda\right)\sum_{i=1}^n \frac{\partial F\left(\lambda\bar{a}+(1-\lambda)\bar{b}\right)}{\partial x_i}\cdot(b_i-a_i) \\ &\qquad \le \int_0^1 F\left[(1-t)\bar{a}+t\bar{b}\right]dt - F\left((1-\lambda)\bar{a}+\lambda\bar{b}\right) \\ &\qquad \le (1-\lambda)^2\sum_{i=1}^n \frac{\partial F\left(\bar{b}\right)}{\partial x_i}\cdot(b_i-a_i) - \lambda^2\sum_{i=1}^n \frac{\partial F(\bar{a})}{\partial x_i}\cdot(b_i-a_i) \end{aligned} \tag{2.21}$$

for any $\bar{a}, \bar{b} \in \Omega$ and $\lambda \in (0,1)$.

In particular, for $\lambda = \frac{1}{2}$, we get

$$\begin{aligned} 0 &\le \int_0^1 F\left[(1-t)\bar{a}+t\bar{b}\right]dt - F\left(\frac{\bar{a}+\bar{b}}{2}\right) \\ &\le \frac{1}{8}\sum_{i=1}^n\left(\frac{\partial F\left(\bar{b}\right)}{\partial x_i} - \frac{\partial F(\bar{a})}{\partial x_i}\right)\cdot(b_i-a_i). \end{aligned} \tag{2.22}$$

In (2.22) the constant $\frac{1}{8}$ is sharp.

3. Applications for Semi-Inner Products

Let $(X, \|\cdot\|)$ be a real normed linear space. We may state the following results for the semi-inner products $\langle\cdot,\cdot\rangle_i$ and $\langle\cdot,\cdot\rangle_s$.

Proposition 3.1. *For any* $x, y \in X$ *and* $\sigma \in (0,1)$ *we have the inequalities:*

$$(1-\sigma)^2 \langle y-x,(1-\sigma)x+\sigma y\rangle_s - \sigma^2 \langle y-x,(1-\sigma)x+\sigma y\rangle_i \leq \int_0^1 \|(1-t)x+ty\|^2 dt - \|(1-\sigma)x+\sigma y\|^2 \leq (1-\sigma)^2 \langle y-x,y\rangle_i - \sigma^2 \langle y-x,y\rangle_s . \tag{3.1}$$

The second inequality in (3.1) also holds for $\sigma = 0$ *or* $\sigma = 1$.

The proof is obvious by Theorem 2.4 applied for the convex function $f(x) = \frac{1}{2}\|x\|^2$, $x \in X$.

If the space is *smooth*, then we may put $[x,y] = \langle x,y\rangle_i = \langle x,y\rangle_s$ for each $x, y \in X$ and the first inequality in (3.1) becomes

$$(1-2\sigma)[y-x,(1-\sigma)x+\sigma y] \leq \int_0^1 \|(1-t)x+ty\|^2 dt - \|(1-\sigma)x+\sigma y\|^2 . \tag{3.2}$$

An interesting particular case one can get from (3.1) is the one for $\sigma = \frac{1}{2}$,

$$0 \leq \frac{1}{8}\left[\langle y-x,y+x\rangle_s - \langle y-x,y+x\rangle_i\right] \leq \int_0^1 \|(1-t)x+ty\|^2 dt - \left\|\frac{x+y}{2}\right\|^2 \leq \frac{1}{4}\left[\langle y-x,y\rangle_i - \langle y-x,x\rangle_s\right]. \tag{3.3}$$

The inequality (3.3) provides a refinement and a counterpart for the first inequality (1.1).

If we consider now two linearly independent vectors $x, y \in X$ and apply Theorem 2.4 for $f(x) = \|x\|$, $x \in X$, then we get

Proposition 3.2. *For any linearly independent vectors* $x, y \in X$ *and* $\sigma \in (0,1)$, *one has the inequalities:*

$$\frac{1}{2}\left[(1-\sigma)^2 \frac{\langle y-x,(1-\sigma)x+\sigma y\rangle_\sigma}{\|(1-\sigma)x+\sigma y\|} - \sigma^2 \frac{\langle y-x,(1-\sigma)x+\sigma y\rangle_i}{\|(1-\sigma)x+\sigma y\|}\right] \leq \int_0^1 \|(1-t)x+ty\| dt - \|(1-\sigma)x+\sigma y\| \leq \frac{1}{2}\left[(1-\sigma)^2 \frac{\langle y-x,y\rangle_i}{\|y\|} - \sigma^2 \frac{\langle y-x,x\rangle_s}{\|x\|}\right]. \tag{3.4}$$

The second inequality also holds for $\sigma = 0$ *or* $\sigma = 1$.

We note that if the space is smooth, then we have

$$\left(\frac{1}{2}-\sigma\right)\cdot\frac{[y-x,(1-\sigma)x+\sigma y]}{\|(1-\sigma)x+\sigma y\|} \leq \int_0^1 \|(1-t)x+ty\| dt - \|(1-\sigma)x+\sigma y\| , \tag{3.5}$$

and for $\sigma = \frac{1}{2}$, (3.4) will give the simple inequality

$$0 \leq \frac{1}{8}\left[\left\langle y-x,\frac{\frac{x+y}{2}}{\left\|\frac{x+y}{2}\right\|}\right\rangle_s - \left\langle y-x,\frac{\frac{x+y}{2}}{\left\|\frac{x+y}{2}\right\|}\right\rangle_i\right] \leq \int_0^1 \|(1-t)x+ty\| dt - \left\|\frac{x+y}{2}\right\| \leq \frac{1}{8}\left[\left\langle y-x,\frac{y}{\|y\|}\right\rangle_i - \left\langle y-x,\frac{x}{\|x\|}\right\rangle_s\right]. \tag{3.6}$$

The inequality (3.6) provides a refinement and a counterpart for the first inequality in (1.2).

Moreover, if we assume that $(H, \langle \cdot, \cdot \rangle)$ is an inner product space, then by (3.6) we get for any $x, y \in H$ with $\|x\| = \|y\| = 1$ that

$$0 \leq \int_0^1 \|(1-t)\,x + ty\|\,dt - \left\|\frac{x+y}{2}\right\| \leq \frac{1}{8}\,\|y-x\|^2 . \tag{3.7}$$

The constant $\frac{1}{8}$ is sharp.

Indeed, if $H = \mathbb{R}$, $\langle a, b \rangle = a \cdot b$, then taking $x = -1$, $y = 1$, we obtain equality in (3.7).

We give now some examples.

(1) Let $\ell^2(\mathbb{K})$, $\mathbb{K} = \mathbb{C}, \mathbb{R}$; be the Hilbert space of sequences $x = (x_i)_{i\in\mathbb{N}}$ with $\sum_{i=0}^{\infty} |x_i|^2 < \infty$. Then, by (3.7), we have the inequalities

$$\begin{aligned} 0 &\leq \int_0^1 \left(\sum_{i=0}^{\infty} |(1-t)\,x_i + ty_i|^2\right)^{\frac{1}{2}} dt - \left(\sum_{i=0}^{\infty} \left|\frac{x_i+y_i}{2}\right|^2\right)^{\frac{1}{2}} \\ &\leq \frac{1}{8} \cdot \sum_{i=0}^{\infty} |y_i - x_i|^2 , \end{aligned} \tag{3.8}$$

for any $x, y \in \ell^2(\mathbb{K})$ provided $\sum_{i=0}^{\infty} |x_i|^2 = \sum_{i=0}^{\infty} |y_i|^2 = 1$.

(2) Let μ be a positive measure, $L_2(\Omega)$ the Hilbert space of $\mu-$measurable functions on Ω with complex values that are $2-$integrable on Ω, i.e., $f \in L_2(\Omega)$ iff $\int_\Omega |f(t)|^2\,d\mu(t) < \infty$. Then, by (3.7), we have the inequalities

$$\begin{aligned} 0 &\leq \int_0^1 \left(\int_\Omega |(1-\lambda)\,f(t) + \lambda g(t)|^2\,d\mu(t)\right)^{\frac{1}{2}} d\lambda \\ &\quad - \left(\int_\Omega \left|\frac{f(t)+g(t)}{2}\right|^2 d\mu(t)\right)^{\frac{1}{2}} \\ &\leq \frac{1}{8} \cdot \int_\Omega |f(t) - g(t)|^2\,d\mu(t) \end{aligned} \tag{3.9}$$

for any $f, g \in L_2(\Omega)$ provided $\int_\Omega |f(t)|^2\,d\mu(t) = \int_\Omega |g(t)|^2\,d\mu(t) = 1$.

REFERENCES

[1] I. CIORANESCU, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[2] S.S. DRAGOMIR AND C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: http://rgmia.vu.edu.au/monographs)