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ON A NONCOERCIVE SYSTEM OF QUASI-VARIATIONAL INEQUALITIES RELATED TO STOCHASTIC CONTROL PROBLEMS

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ABSTRACT. This paper deals with a system of quasi-variational inequalities with noncoercive operators. We prove the existence of a unique weak solution using a lower and upper solutions approach. Furthermore, by means of a Banach's fixed point approach, we also prove that the standard finite element approximation applied to this system is quasi-optimally accurate in L^∞.

Key words and phrases: Quasi-variational inequalities, Contraction, Fixed point finite element, Error estimate.

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1. INTRODUCTION

We are interested in the following system of quasi-variational inequalities (QVI's): find a vector $U = (u^1, \dots, u^M) \in (H_0^1(\Omega))^M$ such that

$$
(1.1)\qquad \begin{cases} a^i(u^i, v - u^i) \geqq (f^i, v - u^i) \ \forall v \in H_0^1(\Omega) \\ u^i \le k + u^{i+1},\ v \le k + u^{i+1} \\ u^{M+1} = u^1, \end{cases}
$$

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where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 1$ with boundary Γ, $a^i(u\,,\,v)$ are M variational forms, f^i are regular functions and k is a positive number.

This system arises in stochastic control problems. It also plays a fundamental role in solving the Hamilton-Jacobi-Bellman equation, [1], [2].

Its coercive version is well understood from both the mathematical and numerical analysis viewpoints (cf. eg., [1], [2], [6]).

In this paper we shall be concerned with the noncoercive case, that is, where the bilinear forms $a^i(u,v)$ do not satisfy the usual coercivity condition.

To handle this new situation, we transform (1.1) into the following auxiliary system: find $U = (u^1, \ldots, u^M) \in (H_0^1(\Omega))^M$ such that:

$$\begin{cases} b^i(u^i, v - u^i) \geqq (f^i + \lambda u^i, v - u^i)\ \forall v \in H_0^1(\Omega) \\ u^i \leq k + u^{i+1},\ v \leq k + u^{i+1} \\ u^{M+1} = u^1, \end{cases} \tag{1.2}$$

where

$$b^i(u,\,v) = a^i(u,v) + \lambda(v,v) \tag{1.3}$$

and $\lambda > 0$ is large enough such that:

$$b^i(v,v) \geq \gamma \left\| v \right\|_{H^1(\Omega)}^2\ \ \gamma > 0;\ \forall v \in H_0^1(\Omega). \tag{1.4}$$

Under this condition, using a monotone approach inspired from [5], we shall prove that both the continuous and discrete problems admit a unique solution.

On the numerical analysis side, using piecewise linear finite elements, we shall establish a quasi-optimal $L^\infty-$convergence order. To that end, we propose a new approach which consists of characterizing both the continuous and the finite element solution as fixed points of contractions in L^∞.

This new approach appears to be quite simple. It also offers the advantage of providing an iterative scheme useful for the numerical computation of the solution.

The paper is organized as follows. In Section 2, we discuss existence and uniqueness of a solution to problem (1.1). Section 3 deals with its discretization by the standard finite element method where, under a discrete maximum principle assumption, analogous discrete qualitative results are given as well. Finally, in Section 4 we respectively associate with both the continuous and discrete systems appropriate contractions and give an $L^\infty-$error estimate.

2. The Continuous Problem

Let us begin with some necessary notations, assumptions and qualitative properties of elliptic variational inequalities.

2.1. **Notations, Assumptions and Preliminaries.** We are given functions

$$a_{jk}^i(x),\ b_k^i(x),\ a_0^i(x) \in C^2(\overline{\Omega}),\ x \in \overline{\Omega},\ 1 \leq k, j \leq N;\ 1 \leq i \leq M \tag{2.1}$$

such that:

$$\sum_{1 \leq\ j,\ k \leq N} a_{jk}^i(x) \xi_j \xi_k \geqq \alpha \left| \xi \right|^2;\ (x \in \overline{\Omega},\ \xi \in R^N,\ \ \alpha > 0) \tag{2.2}$$

$$a_{jk}^i = a_{kj}^i;\ a_0^i(x) \geqq \beta > 0;\ x \in \overline{\Omega}. \tag{2.3}$$

We define the bilinear forms: for any $u, v \in H^1(\Omega)$,

$$a^i(u,v) = \int_\Omega \left(\sum_{1\leq\ j,k\leq N} a^i_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_k} + \sum_{k=1}^{N} b^i_k(x)\frac{\partial u}{\partial x_k}v + a^i_0(x).uv \right). \tag{2.4}$$

We are also given regular functions f^i such that

$$f^i \in C^2(\overline{\Omega}) \text{ and } f^{\,i} \geq 0;\ \forall\, i = 1,\ldots,M. \tag{2.5}$$

and the following norm: $\forall\, W = (w^1,\ldots,w^M) \in \prod_{i=1}^M L^\infty(\Omega)$

$$\|W\|_\infty = \max_{1\leq i\leq M} \left\|w^i\right\|_{L^\infty(\Omega)}, \tag{2.6}$$

where $\|\cdot\|_{L^\infty(\Omega)}$ denotes the well-known $L^\infty-$norm.

2.2. **Elliptic Variational Inequalities.** Let f in $L^\infty(\Omega)$ and ψ in $W^{2,\infty}(\Omega)$ such that $\psi \geq 0$ on $\partial\Omega$. Let also $b(\cdot,\cdot)$ be a continuous and coercive bilinear form of the same form as those defined in (1.2) and consider $u = \sigma(f,\psi)$ a solution to the following elliptic variational inequality VI: find $u \in H^1_0(\Omega)$ such that

$$\begin{cases} b(u, v-u) \geqq (f, v-u)\ \forall v \in H^1_0(\Omega) \\ u \leq \psi;\ v \leq \psi. \end{cases} \tag{2.7}$$

Theorem 2.1. *(cf.* [3],[4]*) Under the above assumptions, there exists a unique solution to the variational inequality (VI) (2.7). Moreover,* $u \in\ W^{2,p}(\Omega),\ 1 \leq p < \infty$.

2.2.1. ***A Monotonicity property for VI (2.7)***. Let (f,ψ), $\left(\tilde{f},\tilde{\psi}\right)$ be a pair of data and $u = \sigma(f,\psi)$, $\widetilde{u} = \sigma\left(\tilde{f},\tilde{\psi}\right)$ the respective solutions to (2.7).

Theorem 2.2. *(cf.* [4]*) If* $f \geq \tilde{f}$ *and* $\psi \geq \tilde{\psi}$ *then* $\sigma(f,\psi) \geq \sigma(\tilde{f},\tilde{\psi})$.

2.3. **Existence and uniqueness.** As mentioned earlier, we solve the noncoercive system of QVI's by considering the following auxiliary system: find a vector $U = \left(u^1,\ldots,u^M\right) \in \left(H^1_0(\Omega)\right)^M$ such that

$$\begin{cases} b^i\left(u^i, v-u^i\right) \geqq \left(f^i + \lambda u^i, v-u^i\right)\ \forall v \in H^1_0(\Omega) \\ u^i \leq k + u^{i+1},\ \ v \leq k + u^{i+1} \\ u^{M+1} = u^1. \end{cases} \tag{2.8}$$

It can readily be noticed in the above system, that besides the obstacles "$k + u^{i+1}$", the right hand sides depend upon the solution as well. Therefore, the increasing property of the solution of VI with respect to the obstacle and the right hand side, reduces the problem (2.8) to finding a fixed point of an increasing mapping as in [5].

Let $\mathbb{L}^\infty(\Omega) = \prod_{i=1}^M L^\infty_+(\Omega)$, where $L^\infty_+(\Omega)$ is the positive cone of $L^\infty(\Omega)$. We introduce the following mapping

$$\begin{aligned} T : \mathbb{L}^\infty(\Omega) &\longrightarrow \mathbb{L}^\infty(\Omega) \\ W &\longrightarrow TW = \left(\zeta^1,\ldots,\zeta^M\right) \end{aligned} \tag{2.9}$$

where $\forall\ i = 1, \ldots, M,\ \zeta^i = \sigma\left(f^i + \lambda w^i;\ k + w^{i+1}\right)$ is solution to the following VI:

$$\begin{cases} b^i\left(\zeta^i, v - \zeta^i\right) \geqq \left(f^i + \lambda w^i, v - \zeta^i\right)\ \forall v \in H_0^1(\Omega) \\ \zeta^i \leq k + w^{i+1},\ \ v \leq k + w^{i+1}. \end{cases} \tag{2.10}$$

Problem (2.10) being a coercive variational inequality, thanks to [3], [4], has a unique solution.

2.3.1. ***Properties of The Mapping T***. Let us first introduce the vector $\hat{U}^0 = \left(\hat{u}^{1,0}, \ldots, \hat{u}^{M,0}\right)$, where $\forall i = 1, \ldots, M,\ \hat{u}^{i,0}$ is solution to the equation

$$a^i(\hat{u}^{i,0}, v) = (f^i, v)\ \forall\ v \in H_0^1(\Omega). \tag{2.11}$$

Since $f^i \geq 0$, there exists a unique positive solution to problem (2.11). Moreover, $\hat{u}^{i,0} \in W^{2,,p}(\Omega),\ p < \infty$ (cf. e.g., [5]).

Proposition 2.3. *Under the preceding notations and assumptions, the mapping T is increasing, concave and satisfies: $TW \leq \hat{U}^0$, $\forall\ W \in \mathbb{L}^\infty(\Omega)$ such that $W \leq \hat{U}^0$.*

Proof. **1.** T is increasing.

Let $V = (v^1, \ldots, v^M)$, $W = (w^1, , \ldots, w^M)$ in $\mathbb{L}^\infty(\Omega)$ such that $v^i \leq w^i$, $\forall i = 1, , \ldots, M$. Then, by Theorem 2.2, it follows that $\sigma(f^i + \lambda w^i; k + w^{i+1}) \geq \sigma(f^i + \lambda v^i; k + v^{i+1})$. Thus, $TV \leq TW$.

2. $TW \leq \hat{U}^0\ \forall W \leq \hat{U}^0$.

Let us first recall that $u^+ = \sup(u, 0)$ and $u^- = \sup(-u, 0)$. The fact that both of the solutions ζ^i of (2.10) and $\hat{u}^{i,0}$ of (2.11) belong to $H_0^1(\Omega)$, we clearly have:

$$\zeta^i - \left(\zeta^i - \hat{u}^{i,0}\right)^+ \in H_0^1(\Omega).$$

Moreover, as $(\zeta^i - \hat{u}^{i,0})^+ \geq 0$, it follows that

$$\zeta^i - \left(\zeta^i - \hat{u}^{i,0}\right)^+ \leq \zeta^i \leq k + w^{i+1}.$$

Therefore, we can take $v = \zeta^i - (\zeta^i - \hat{u}^{i,0})^+$ as a trial function in (2.10). This gives:

$$b^i\left(\zeta^i, -\left(\zeta^i - \hat{u}^{i,0}\right)^+\right) \geqq \left(f^i + \lambda w^i, -\left(\zeta^i - \hat{u}^{i,0}\right)^+\right).$$

On the other hand, taking $v = \left(\zeta^i - \hat{u}^{i,0}\right)^+$ in equation (2.11) we get

$$b^i\left(\hat{u}^{i,0}, \left(\zeta^i - \hat{u}^{i,0}\right)^+\right) = \left(f^i + \lambda\hat{u}^{i,0}, \left(\zeta^i - \hat{u}^{i,0}\right)^+\right)$$

and, since $W \leq \hat{U}^0$, by addition, we obtain

$$-b^i\left(\left(\zeta^i - \hat{u}^{i,0}\right)^+, \left(\zeta^i - \hat{u}^{i,0}\right)^+\right) \geqq 0$$

which, by (1.4), yields

$$\left(\zeta^i - \hat{u}^{i,0}\right)^+ = 0.$$

Thus

$$\zeta^i \leq \hat{u}^{i,0}\ \forall\ i = 1, 2, \ldots, M$$

i.e.,

$$TW \leq \hat{U}^0.$$

3. T **is concave**.

Let us agree on the following notations:

$$w_\theta^i = \theta w^i + (1-\theta)\tilde{w}^i;\ w_{\theta,k}^i = \theta(k+w^i) + (1-\theta)(k+\tilde{w}^i); \theta \in [0,1].$$

Then we have:

$$\begin{aligned} &T(\theta W + (1-\theta)\tilde{W}) \\ &= \left[\sigma\left(f^1 + \lambda w_\theta^1; k + w_\theta^2\right), \dots, \sigma\left(f^i + \lambda w_\theta^i; k + w_\theta^{i+1}\right), \dots, \sigma\left(f^M + \lambda w_\theta^M; k + w_\theta^1\right)\right] \\ &= \left[\sigma\left(f^1 + \lambda w_\theta^1; w_{\theta,k}^2\right), \dots, \sigma\left(f^1 + \lambda w_\theta^i; w_{\theta,k}^{i+1}\right), \dots, \sigma\left(f^M + \lambda w_\theta^M; w_{\theta,k}^1\right)\right]. \end{aligned}$$

Now, denoting by:

$$\zeta^i = \sigma\left(f^i + \lambda w^i; k + w^{i+1}\right),$$

$$\tilde{\zeta}^i = \sigma\left(f^i + \lambda \tilde{w}^i; k + \tilde{w}^{i+1}\right),$$

$$\zeta_\theta^i = \theta\zeta^i + (1-\theta)\tilde{\zeta}^i,$$

$$U_\theta^i = \sigma\left(f^i + \lambda w_\theta^i; w_\theta^{i+1}\right).$$

It is clear that ζ_θ^i is admissible for the problem which has U_θ^i as a solution. So

$$U_\theta^i + \left(U_\theta^i - \zeta_\theta^i\right)^-$$

is admissible for this problem. Therefore,

$$b\left(U_\theta^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right) \geq \left(f + \lambda w_\theta^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right). \tag{2.12}$$

Also, we can take $\zeta^i - (U_\theta^i - \zeta_\theta^i)^-$ as a test function in the problem where ζ^i is the solution and $\tilde{\zeta}^i - (U_\theta^i - \zeta_\theta^i)^-$ can be taken as a test function in the problem whose solution is $\tilde{\zeta}^i$. From this we deduce that

$$-b\left(\zeta^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right) \geq -\left(f + \lambda w^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right) \tag{2.13}$$

and

$$-b\left(\tilde{\zeta}^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right) \geq -\left(f + \lambda \tilde{w}^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right). \tag{2.14}$$

Now multiplying (2.13) by θ, and (2.14) by $1-\theta$, addition yields

$$-b\left(\zeta_\theta^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right) \geq -\left(f + \lambda w_\theta^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right)$$

which added to (2.12) gives

$$b\left(U_\theta^i - \zeta_\theta^i, \left(U_\theta^i - \zeta_\theta^i\right)^-\right) \geq 0.$$

Thus

$$\left(U_\theta^i - \zeta_\theta^i\right)^- = 0$$

which completes the proof i.e.,

$$T\left(\theta W + (1-\theta)\tilde{W}\right) \geq \theta TW + (1-\theta)T\tilde{W}.$$

□

2.3.2. ***A Continuous Iterative Scheme of Bensoussan-Lions Type.*** Starting from $\hat{U}^0$ solution of (2.11) and $\check{U}^0 = 0$, we define the iterations:

$$\hat{U}^{n+1} = T\hat{U}^n;\ n = 0, 1, \ldots \tag{2.15}$$

and

$$\check{U}^{n+1} = T\check{U}^n;\ n = 0, 1, \ldots, \tag{2.16}$$

respectively.

The analysis of the convergence of these iterations requires to prove the following intermediate results.

Lemma 2.4. *Assume* $f^i \geq f^0 > 0$; $1 \leq i \leq M$, *where* f^0 *is a positive constant, and let*

$$0 < \mu < \inf \left\{ \frac{k}{\left\| \hat{U}^0 \right\|_\infty}; \frac{f^0}{\lambda \left\| \hat{U}^0 \right\|_\infty + f^0} \right\}.$$

Then we have

$$T(0) \geqq \mu \hat{U}^0. \tag{2.17}$$

Proof. Indeed, from (2.16), $T(0) = \check{U}^1 = (\check{u}^{1,1}, \ldots, \check{u}^{1,M})$, where $\check{u}^{i,1}$ is the solution of the following variational inequality:

$$\begin{cases} b^i\left(\check{u}^{i,1}, v - \check{u}^{i,1}\right) \geqq \left(f^i + \lambda \check{u}^{i,0}, v - \check{u}^{i,1}\right)\ \forall v \in H_0^1(\Omega) \\ \\ \check{u}^{i,1} \leq k;\ v \leq k. \end{cases} \tag{2.18}$$

Then by the choice of μ it is clear that

$$v = \left(\check{u}^{i,1} - \mu \hat{u}^{i,0}\right)^- + \check{u}^{i,1}$$

can be taken as a trial function in the VI (2.18) inequality. So taking

$$v = -\left(\check{u}^{i,1} - \mu \hat{u}^{i,0}\right)^-$$

as a trial function in (2.11) and using the fact that $f^i \geq f^0$ and $\check{u}^{i,0} = 0$, we get by addition:

$$\begin{aligned} b^i\left(\check{u}^{i,1} - \mu \hat{u}^{i,0}, \left(\check{u}^{i,1} - \mu \hat{u}^{i,0}\right)^-\right) &\geqq \left(\left(f^i - \mu f^i - \mu \lambda \hat{u}^{i,0}\right), \left(\check{u}^{i,1} - \mu \hat{u}^{i,0}\right)^-\right) \\ &\geqq \left(\left(f^0(1-\mu) - \mu \lambda \hat{u}^{i,0}\right), \left(\check{u}^{i,1} - \mu \hat{u}^{i,0}\right)^-\right). \end{aligned}$$

But, again, by the choice of μ

$$f^0(1-\mu) - \mu\lambda \hat{u}^{i,0} \geq f^0(1-\mu) - \mu\lambda \left\| \hat{U}^0 \right\|_\infty \geq 0.$$

Thus, by (1.4)

$$\left(\check{u}^{i,1} - \mu \hat{u}^{i,0}\right)^- = 0$$

i.e.,

$$\check{u}^{i,1} \geqq \mu \hat{u}^{i,0} \quad \forall\, i = 1, 2, \ldots, M.$$

□

Proposition 2.5. *Let* $\mathbb{C} = \{W \in \mathbb{L}^\infty(\Omega)$ *such that* $0 \leq W \leq \hat{U}^0\}$. *Let also* $\gamma \in]0\ ; 1]$, $W, \tilde{W} \in \mathbb{C}$ *such that:*

$$W - \tilde{W} \leq \gamma W. \tag{2.19}$$

Then, the following holds

$$TW - T\tilde{W} \leq \gamma(1-\mu) TW. \tag{2.20}$$

Proof. By (2.19), we have $(1-\gamma)\,W \leq \tilde{W}$. Then, using the fact that T is increasing and concave (see Proposition 2.3.), it follows that

$$\begin{aligned}(1-\gamma)TW + \gamma T(0) &\leq T[(1-\gamma)W + \gamma.0] \\ &\leq T\tilde{W}\end{aligned}$$

Finally, using Lemma 2.4. we get (2.20). □

Theorem 2.6. *Under conditions of Propositions 2.3 – 2.5, the sequences* $\left(\hat{U}^n\right)$ *and* $\left(\check{U}^n\right)$ *are monotone and well defined in* $\mathbb{C}$ *. Moreover, they converge respectively from above and below to the unique solution* U *of system of QVI's (1.1).*

Proof. The proof will be carried out in five steps.

Step 1. The sequence $(\hat{U}^n)$ stays in $\mathbb{C}$ and is decreasing.

From (2.15) it is easy to see that $\forall i$, $\hat{u}^{i,n}$ is solution to the following VI:

$$(2.21)\qquad \begin{cases} b^i(\hat{u}^{i,n}, v - \hat{u}^{i,n}) \geqq (f^i + \lambda \hat{u}^{i,n-1}, v - \hat{u}^{i,n})\ \forall v \in H_0^1(\Omega) \\ \\ \hat{u}^{i,n} \leq k + \hat{u}^{i+1,n-1};\ v \leq k + \hat{u}^{i+1,n-1} \\ \\ \hat{u}^{M+1,n} = \hat{u}^{1,n}. \end{cases}$$

Since $f^i \geq 0$ and $\hat{u}^{i,0} \geq 0,$ a simple induction combined with standard comparison results in variational inequalities lead to $\hat{u}^{i,n} \geq 0$ i.e.,

$$(2.22)\qquad \hat{U}^n \geq 0 \quad \forall n \geq 0.$$

Furthermore, by Proposition 2.3. and (2.15), we have:

$$\hat{U}^1 = T\hat{U}^0 \leq \hat{U}^0.$$

Thus, inductively

$$(2.23)\qquad 0 \leq \hat{U}^{n+1} = T\hat{U}^n \leq \hat{U}^n \leq \cdots \leq \hat{U}^0.$$

Step 2. $(\hat{U}^n)$ converges to the solution of the system (1.1).

From (2.22) and (2.23) it is clear that $\forall i = 1, 2, \ldots, M$

$$(2.24)\qquad \lim_{n\to\infty} \hat{u}^{i,n}(x) = \bar{u}^i(x),\ x \in \Omega \text{ and } (\bar{u}^1, \ldots, \bar{u}^M) \in \mathbb{C}.$$

Moreover, from (2.22) we have $k + \hat{u}^{i+1,n-1} \geq 0.$ Then we can take $v = 0$ as a trial function in (2.21), which yields

$$\gamma \left\|\hat{u}^{i,n}\right\|^2_{H^1(\Omega)} \leq b^i(\hat{u}^{i,n}, \hat{u}^{i,n}) \leq \left\|f^i + \lambda \hat{u}^{i,n-1}\right\|_{L^2(\Omega)} \left\|\hat{u}^{i,n}\right\|_{H^1(\Omega)}$$

or more simply

$$\left\|\hat{u}^{i,n}\right\|_{H^1(\Omega)} \leq C,$$

where C is a constant independent of $n.$ Hence $\hat{u}^{i,n}$ stays bounded in $H^1(\Omega)$ and consequently we can complete (2.24) by

$$(2.25)\qquad \lim_{n\to\infty} \hat{u}^{i,n} = \bar{u}^i \text{ weakly in } H^1(\Omega).$$

Step 3. $\overline{U} = (\bar{u}^1, \ldots, \bar{u}^M)$ coincides with the solution of system (1.1).

Indeed, since

$$\hat{u}^{i,n}(x) \leq k + \hat{u}^{i+1,n-1}(x)$$

then (2.24) implies

$$\bar{u}^i(x) \leq k + \bar{u}^{i+1}(x).$$

Now let $v \leq k + \bar{u}^{i+1}$ then $v \leq k + \hat{u}^{i+1,n-1}$, $\forall n \geq 0$. We can therefore take v as a trial function for the VI (2.21). Consequently, combining (2.24) and (2.25) with the weak lower semi continuity of $b^i(v,v)$ and passing to the limit in problem (2.21), we clearly get

$$b^i(\bar{u}^i, v - \bar{u}^i) \geqq (f^i + \lambda \bar{u}^i, v - \bar{u}^i)\ \forall v \in H_0^1(\Omega),\ \ v \leq k + \bar{u}^{i+1}.$$

Step 4. **Uniqueness**. Let U, $\tilde{U}$ be two solutions of the system (1.1). These are fixed points of T. Therefore, since $U - \tilde{U} \leq U,$ by taking $W = U$ and $\tilde{W} = \tilde{U}$ in (2.19) with $\gamma = 1 - \mu$ we have

$$U - \tilde{U} \leq (1-\mu)U.$$

Doing this again with $\gamma = 1 - \mu$, we obtain

$$U - \tilde{U} \leq (1-\mu)^2 U$$

and inductively

$$U - \tilde{U} \leq (1-\mu)^n U \leq (1-\mu)^n \left\| \hat{U}^0 \right\|_\infty .$$

Thus, making n tend to ∞ yields $U \leq \tilde{U}$. Finally, interchanging the roles of U and $\tilde{U}$, we obtain $U = \tilde{U}$

Step 5. The monotone property of the sequence $(\check{U}^n)$ can be shown in a similar way to that of sequence $(\hat{U}^n)$. Let us prove its convergence to the solution of system (1.1). Indeed, apply (2.19),(2.20) with

$$W = \hat{U}^0;\ \ \tilde{W} = \check{U}^0;\ \ \gamma = 1$$

then

$$T\hat{U}^0 - T\check{U}^0 \leq (1-\mu) T\hat{U}^0,$$

so

$$0 \leq \hat{U}^1 - \check{U}^1 \leq (1-\mu)\hat{U}^1.$$

Applying (2.20) again, yields

$$0 \leq \hat{U}^2 - \check{U}^2 \leq (1-\mu)^2 \hat{U}^2$$

and quite generally

$$0 \leq \hat{U}^n - \check{U}^n \leq (1-\mu)^n \hat{U}^n \leq (1-\mu)^n \hat{U}^0 \leq (1-\mu)^n \left\| \hat{U}^0 \right\|_\infty .$$

Thus

$$\hat{U}^n - \check{U}^n \to 0\ \ a.e$$

from which it follows that

$$\check{U}^n \to \underline{U} = U$$

is the unique solution of system of QVI's (1.1).

□

2.3.3. ***Regularity of the solution of system (1.1)***. The following is a theorem on the regularity of (1.1).

Theorem 2.7. *(cf. e.g,*[1]*). Let assumptions (2.1)-(2.5) hold. Then each component of the solution of system (1.1) belongs to* $C(\bar{\Omega}) \cap W^{2,p}(\Omega)$; $\forall 2 \leq p < \infty$.

3. THE DISCRETE PROBLEM

Let Ω be decomposed into triangles and let τ_h denote the set of all those elements; $h > 0$ is the mesh size. We assume that the family τ_h is regular and quasi-uniform.

Let V_h denote the standard piecewise linear finite element space,

$$V_h = \{v \in C(\Omega) \cap H_0^1(\Omega) \text{ such that } v/_K \ \in P_1\ ,\ \forall K \in \tau_h\ \}. \tag{3.1}$$

Let $\mathbb{B}^i$ be the matrices with generic coefficients

$$\left(\mathbb{B}^i\right)_{ls} = b^i(\varphi_l, \varphi_s)\ 1 \le i \le M\ ;\ 1 \le l, s \le m(h), \tag{3.2}$$

where, $\{\varphi_l\},\ l = 1, 2, \dots m(h)$ is the basis of V_h.

Let r_h be the usual restriction operator defined by

$$\forall v \in C(\Omega) \cap H_0^1(\Omega)\ ,\ r_h v = \sum_{l=1}^{m(h)} v_l \varphi_l. \tag{3.3}$$

In the sequel of the paper we shall make use of the **discrete maximum principle (d.m.p)** assumption. In other words, we shall assume that $\mathbb{B}^i$, $1 \le i \le M$ are M-matrices (see [7]).

3.1. **Discrete Variational Inequality.** Let $u_h \in V_h$ be the solution of the following discrete variational inequality

$$\begin{cases} b(u_h\,, v - u_h) \geqq (f, v - u_h)\ \forall v \in V_h \\ \\ u_h \le r_h \psi;\ v \le r_h \psi. \end{cases} \tag{3.4}$$

3.1.1. ***A Discrete Monotonicity Property for VI (3.4)***. Let $(f,\ \psi)$, $(\tilde{f},\ \tilde{\psi})$ be a pair of data and $u = \sigma_h(f, \psi)$, $\widetilde{u} = \sigma_h(\tilde{f}, \tilde{\psi})$ the respective solutions of (3.4). Then we have the discrete analogue of Theorem 2.2.

Theorem 3.1. *Under the* **d.m.p**, *If $f \ge \tilde{f}$ and $\psi \ge \tilde{\psi}$ then $\sigma_h(f, \psi) \ge \sigma_h(\tilde{f}, \tilde{\psi})$.*

3.2. **The Noncoercive Discrete System of QVI's.** Let $\mathbb{V}_h = (V_h)^M$. We define the noncoercive discrete system of QVI's as follows: find $U_h = (u_h^1, \dots, u_h^M) \in \mathbb{V}_h$ solution of

$$\begin{cases} a^i(u_h^i, v - u^i) \geqq (f^i, v - u_h^i)\ \forall v \in V_h \\ \\ u_h^i \le k + u_h^{i+1},\ \ v \le k + u_h^{i+1} \\ \\ u_h^{M+1} = u_h^1. \end{cases} \tag{3.5}$$

And, similarly to the continuous problem, we solve (3.5) via the following implicit coercive system: find $U_h = (u_h^1\ , \dots, u_h^M) \in \mathbb{V}_h$ solution to

$$\begin{cases} b^i(u_h^i, v - u^i) \geqq (f^i + \lambda u_h^i, v - u_h^i)\ \forall v \in V_h; \\ \\ u_h^i \le k + u_h^{i+1},\ v \le k + u_h^{i+1}; \\ \\ u_h^{M+1} = u_h^1. \end{cases} \tag{3.6}$$

Let us also note that all the properties established in the continuous case remain conserved in the discrete case, provided the **d.m.p** is satisfied. The proofs of these will not be given as they are respectively identical to their continuous analogue ones.

3.3. **Existence and Uniqueness.** Let us first define $\hat{U}_h^0$ to be the piecewise linear approximation of $\hat{U}^0$ defined in (2.11):

$$a^i(\hat{u}_h^{i,0}, v) = (f^i, v) \ \forall v \in V_h; \ \ 1 \leq i \leq M. \tag{3.7}$$

We consider the following mapping

$$T_h : \mathbb{L}^\infty(\Omega) \longrightarrow \mathbb{V}_h, \tag{3.8}$$

$$W \longrightarrow TW = (\zeta_h^1, \ldots, \zeta_h^M),$$

where, $\forall \ i = 1, \ldots, M$, $\zeta_h^i = \sigma_h(f^i + \lambda w^i, k + w^{i+1})$ is the solution of the following discrete VI:

$$\begin{cases} b^i(\zeta_h^i, v - \zeta_h^i) \geqq (f^i + \lambda w^i, v - \zeta_h^i) \ \forall v \in V_h, \\ \\ \zeta_h^i \leq r_h(k + w^{i+1}), \ v \leq r_h(k + w^{i+1}). \end{cases} \tag{3.9}$$

Proposition 3.2. *Under the* ***d.m.p****, T_h is increasing, concave and satisfies $T_h W \leq \hat{U}_h^0 \ \ \forall W \in \mathbb{L}^\infty(\Omega)$, $W \leq \hat{U}_h^0$.*

3.4. **A Discrete Iterative Scheme of Bensoussan-Lions Type.** We associate with the mapping T_h the following discrete iterative scheme: starting from $\hat{U}_h^0$ defined in (3.7) and $\check{U}_h^0 \ = 0$, we define:

$$\hat{U}_h^{n+1} = T_h \hat{U}_h^n \tag{3.10}$$

and

$$\check{U}_h^{n+1} = T_h \check{U}_h^n \tag{3.11}$$

respectively.

Similarly to Theorem 2.6, the convergence of the above algorithm rests on the discrete analogues of Lemma 2.4. and Proposition 2.5, respectively.

Lemma 3.3. *Assume $f^i \geq f^0 > 0; \ 1 \leq i \leq M$, where f^0 is a positive constant, and let*

$$0 < \mu < \inf \left\{ \frac{k}{\left\| \hat{U}_h^0 \right\|_\infty}; \frac{f^0}{\lambda \left\| \hat{U}_h^0 \right\|_\infty + f^0} \right\}.$$

Then we have

$$T_h(0) \geqq \mu \hat{U}_h^0 \tag{3.12}$$

Proposition 3.4. *Let $\mathbb{C}_h = \{W \in \mathbb{L}^\infty(\Omega)$ such that $0 \leq W \leq \hat{U}_h^0\}$. Let also $\gamma \in]0, 1]$, $W, \ \tilde{W} \in \mathbb{C}_h$ such that:*

$$W - \tilde{W} \leq \gamma W. \tag{3.13}$$

Then the following holds

$$T_h W - T_h \tilde{W} \leq \gamma(1 - \mu) T_h W. \tag{3.14}$$

Theorem 3.5. *Under conditions of Proposition 3.2 – 3.4, the sequences $(\hat{U}_h^n)$ and $(\check{U}_h^n)$ are monotone and well defined in $\mathbb{C}_h$. Moreover, they converge respectively from above and below to the unique solution U_h of system of QVI's (3.5).*

4. THE FINITE ELEMENT ERROR ANALYSIS

In what follows, we prove the convergence of the approximation and establish a uniform error estimate. Our approach consists of characterizing both the solution of systems (1.1) and (3.5) as the unique fixed points of appropriate contractions in $\mathbb{L}^\infty(\Omega)$. To that end we need first to introduce a coercive system of quasi-variational inequalities and prove that its solution is monotone with respect to the right hand side.

Let $F = (F^1, \dots, F^M) \in \mathbb{L}^\infty(\Omega)$. We denote by $Z = (z^1, \dots, z^M)$ the solution of the coercive system of QVI's:

$$\text{(4.1)} \qquad \begin{cases} b^i(z^i, v - z^i) \geqq (F^i,\ v - z^i)\ \forall v \in H_0^1(\Omega) \\ z^i \leq k + z^{i+1} \\ z^{M+1} = z^1. \end{cases}$$

Denoting by $z^i = \sigma(F^i, k + z^{i+1})$, we introduce the sequences $\overline{Z}^n = (\bar{z}^{1,n}, \dots, \bar{z}^{M,n})$ and $\underline{Z}^n = (\underline{z}^{1,n}, \dots, \underline{z}^{M,n})$ defined by

$$\bar{z}^{i,n+1} = \sigma(F^i, k + \bar{z}^{i+1,n}),$$

and

$$\underline{z}^{i,n+1} = \sigma(F^i, k + \underline{z}^{i+1,n}),$$

where $\bar{z}^{i,0}$ is the unique solution of $b(\bar{z}^{i,0}, v) = (F^i,\ v)\ \forall v \in H_0^1(\Omega)$ and $\underline{z}^{i,0} = 0$.

Theorem 4.1. *(cf.* [6]*) The sequence* $(\overline{Z}^n)$ *and* $(\underline{Z}^n)$ *converge respectively from above and below to the unique solution of system (4.1). Moreover* $z^i \in W^{2,p}(\Omega)$ $1 \leq i < M$*;* $1 \leq p < \infty$*.*

Proposition 4.2. *Let* $\left(F^1, \dots, F^M\right)$*;* $\left(\tilde{F}^1, \dots, \tilde{F}^M\right)$ *be two families of right hands side and* $Z = \left(z^1, \dots, z^M\right)$*;* $\tilde{Z} = \left(\tilde{z}^1, \dots, \tilde{z}^M\right)$ *be the respective solutions of system (4.1). Then the following holds. If* $F \geq \tilde{F}$*, then* $Z \geq \tilde{Z}$*.*

Proof. Let $\bar{Z}^0 = \left(\bar{z}^{1,0}, \dots, \bar{z}^{M,0}\right)$ and $\overline{\tilde{Z}}^0 = \left(\overline{\tilde{z}}^{1,0}, \dots, \overline{\tilde{z}}^{,M,0}\right)$ such that $\bar{z}^{i,0}$ and $\overline{\tilde{z}}^{i,0}$ are solutions to equations $b(\bar{z}^{i,0}, v) = (F^i, v)$ and $b\left(\overline{\tilde{z}}^{i,0}, v\right) = \left(\tilde{F}^i, v\right)$, respectively. Then the respective associated decreasing sequences

$$\overline{Z}^n = (\bar{z}^{1,n}, \dots, \bar{z}^{M,n}) \text{ and } \overline{\tilde{Z}}^n = \left(\overline{\tilde{z}}^{1,n}, \dots, \overline{\tilde{z}}^{M,n}\right)$$

satisfy the following assertion.

$$\text{If } F^i \geq \tilde{F}^i \text{ then } \bar{z}^{i,n} \geq \overline{\tilde{z}}^{i,n}\ \ \forall i = 1, \dots, M.$$

Indeed, since

$$\bar{z}^{i,n+1} = \sigma\left(F^i, k + \bar{z}^{i+1,n}\right),$$

$$\overline{\tilde{z}}^{i,n+1} = \sigma\left(\tilde{F}^i, k + \overline{\tilde{z}}^{i+1,n}\right),$$

$F^i \geq \tilde{F}^i$ implies $\bar{z}^{i,0} \geq \overline{\tilde{z}}^{i,0}$, $\forall i = 1, 2, \dots, M$. So, $k + \bar{z}^{i+1,0} \geq k + \overline{\tilde{z}}^{i+1,0}$ and thus, from standard comparison results in coercive variational inequalities, it follows that

$$\bar{z}^{i,1} \geq \overline{\tilde{z}}^{i,1}.$$

Now assume that $\bar{z}^{i,,n-1} \geq \bar{\tilde{z}}^{i,n-1}$. Then, as $F^i \geq \tilde{F}^i$, applying the same comparison argument as before, we get:

$$\bar{z}^{i,,n} \geq \bar{\tilde{z}}^{i,n}.$$

Finally, by Theorem 4.1, passing to the the limit as $n \to \infty$, we get $Z \geq \tilde{Z}$. □

Remark 4.3. Proposition 4.2 remains true in the discrete case provided the **d.m.p** is satisfied.

4.1. **A Contraction Associated with System of QVI's (1.1).** Consider the following mapping

$$\begin{aligned} \mathbb{S} &: \mathbb{L}^\infty(\Omega) \to \mathbb{L}^\infty(\Omega) \\ W &\to \mathbb{S}W = Z = \left(z^1, \ldots, z^M\right), \end{aligned} \tag{4.2}$$

where Z is solution to the coercive system of QVI's below

$$\begin{cases} b^i(z^i, v - z^i) \geqq (f^i + \lambda w^i, v - z^i) \ \forall v \in H_0^1(\Omega) \\ z^i \leq k + z^{i+1}, \ v \leq k + z^{i+1}; \ i = 1, .., M \\ z^{M+1} = z^1. \end{cases} \tag{4.3}$$

By Theorem 4.1, problem (4.3) has one and only one solution.

Proposition 4.4. *The mapping* $\mathbb{S}$ *is a contraction in* $\mathbb{L}^\infty(\Omega)$. *i.e.,*

$$\left\|\mathbb{S}W - \mathbb{S}\tilde{W}\right\|_\infty \leq \frac{\lambda}{\lambda + \beta} \left\|W - \tilde{W}\right\|_\infty.$$

Therefore, there exists a unique fixed point which coincides with the solution U *of the system of QVI's (1.1).*

Proof. Let $W, \tilde{W} \in \mathbb{L}^\infty(\Omega)$. We consider $Z = \mathbb{S}W = (z^1, \ldots, z^M)$ and $\tilde{Z} = \mathbb{S}\tilde{W} = (\tilde{z}^1, \ldots, \tilde{z}^M)$ solutions to system of QVI's (4.3) with right hands side $F = (F^1, \ldots, F^M)$ and $\tilde{F} = (\tilde{F}^1, \ldots, \tilde{F}^M)$, where $F^i = f^i + \lambda w^i$ and $\tilde{F}^i = f^i + \lambda \tilde{w}^i$. Now setting

$$\Phi = \frac{1}{\lambda + \beta} \left\|F - \tilde{F}\right\|_\infty ; \ \Phi^i = \frac{1}{\lambda + \beta} \left\|F^i - \tilde{F}^i\right\|_\infty$$

it follows that

$$F^i \leq \tilde{F}^i + \left\|F^i - \tilde{F}^i\right\|_\infty$$

and

$$\tilde{F}^i + \frac{a_0(x) + \lambda}{\lambda + \beta} \left\|F - \tilde{F}^i\right\|_\infty \leq \tilde{F}^i + (a_0(x) + \lambda\Phi) \ (\text{because } a_0^i(x) \geqq \beta > 0)$$

so, by Proposition 4.2, we obtain:

$$z^i \leq \tilde{z}^i + \Phi^i.$$

Interchanging the roles of W and $\tilde{W}$, we similarly get

$$\tilde{z}^i \leq z^i + \Phi^i.$$

Thus

$$\left\|z^i - \tilde{z}^i\right\|_{L^\infty(\Omega)} \leq \Phi^i$$

which completes the proof. □

In a similar way to that of the continuous problem, we are also able to characterize the solution of the system of QVI's (3.5) as the unique fixed point of a contraction.

4.2. **A Contraction Associated with The Discrete System of QVI's (3.5).** We consider the following mapping:

$$\begin{aligned} &\mathbb{S}_h : \mathbb{L}^\infty(\Omega) \to \mathbb{V}_h \\ &W \to \mathbb{S}_h W = Z_h = \left(z_h^1, \ldots, z_h^M\right), \end{aligned} \tag{4.4}$$

where z_h^i is solution to the discrete coercive system of QVI's:

$$\begin{cases} b(z_h^i, v - z_h^i) \geqq (f + \lambda w^i, v - z_h^i) \ \forall v \in V_h \\ z_h^i \leq k + z_h^{i+1};\ v \leq k + z_h^{i+1} \\ z_h^{M+1} = z_h^1. \end{cases} \tag{4.5}$$

Thanks to [6], [8] system (4.5) has one and only one solution.

Next, making use of Proposition 4.2 and Remark 4.3 we have the contraction property of $\mathbb{S}_h$.

Proposition 4.5. *The mapping* $\mathbb{S}_h$ *is a contraction in* $\mathbb{L}^\infty(\Omega)$. *i.e.,*

$$\left\|\mathbb{S}_h W - \mathbb{S}_h \tilde{W}\right\|_\infty \leq \frac{\lambda}{\lambda + \beta} \left\|W - \tilde{W}\right\|_\infty.$$

Therefore, there exists a unique fixed point which coincides with the solution U_h *of the system of QVI (3.5)*

Now, guided by Propositions 4.4 and 4.5, we are in a position to establish a uniform error estimate for the noncoercive system of QVI's (1.1). To this end, we need first to introduce the following auxiliary discrete coercive system of QVI's.

4.3. **An Auxiliary Coercive System of QVI's.** We consider the following coercive system of QVI's: find $\bar{Z}_h = \left(\bar{z}_h^1, \ldots, \bar{z}_h^M\right)$ solution to

$$\begin{cases} b\left(\bar{z}_h^i, v - \bar{z}_h^i\right) \geqq \left(f + \lambda u^i, v - \bar{z}_h^i\right) \ \forall v \in V_h \\ \bar{z}_h^i \leq k + \bar{z}_h^{i+1};\ v \leq k + \bar{z}_h^{i+1};\ i = 1, \ldots, M \\ \bar{z}_h^{i,M+1} = \bar{z}_h^1. \end{cases} \tag{4.6}$$

Clearly, (4.6) is a coercive system whose right hand side depends on $U = \left(u^1, \ldots, u^M\right)$ the continuous solution of system (1.1). So, in view of (4.4), we readily have:

$$\bar{Z}_h = \mathbb{S}_h U. \tag{4.7}$$

Therefore, using the result of [6], we have the following error estimate.

Theorem 4.6. *(cf.* [6]*)*

$$\left\|\bar{Z}_h - U\right\|_\infty \leq Ch^2 \left|Logh\right|^3. \tag{4.8}$$

4.4. **$\mathbf{L}^\infty$- Error Estimate For the Noncoercive System of QVI's (1.1).** Let U and U_h be the solutions of system (1.1) and (3.5), respectively. Then we have:

Theorem 4.7.

$$\left\|U - U_h\right\|_\infty \leq Ch^2 \left|Logh\right|^3.$$

Proof. In view of (4.8) and Propositions 4.4 and 4.5, we clearly have

$$U = \mathbb{S}U;\ U_h = \mathbb{S}_h U_h;\ \bar{Z}_h = \mathbb{S}_h U.$$

Then , using estimation (4.8), we have

$$\begin{aligned}\|\mathbb{S}_h U - \mathbb{S}U\|_\infty &= \left\|\bar{Z}_h - U\right\|_\infty \\ &\leq Ch^2\,|Logh|^3\,.\end{aligned}$$

Therefore

$$\begin{aligned}\|U_h - U\|_\infty &\leq \|U_h - \mathbb{S}_h U\|_\infty + \|\mathbb{S}_h U - \mathbb{S}U\|_\infty \\ &\leq \|\mathbb{S}_h U_h - S_h U\|_\infty + \|\mathbb{S}_h U - \mathbb{S}U\|_\infty \\ &\leq \frac{\lambda}{\lambda+\beta}\,\|U - U_h\|_\infty + Ch^2\,|Logh|^3\,.\end{aligned}$$

Thus

$$\|U - U_h\|_\infty \leq \frac{Ch^2\,|Logh|^3}{\left(1 - \dfrac{\lambda}{\lambda+\beta}\right)}.$$

This completes the proof. □

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