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SOME INTEGRAL INEQUALITIES INVOLVING TAYLOR'S REMAINDER

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Abstract

In this paper, using Steffensen's inequality we prove several inequalities involving Taylor's remainder. Among the simplest particular cases we obtain Iyengar's inequality and one of Hermite-Hadamard's inequalities for convex functions.

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Key words: Taylor's remainder, Steffensen's inequality, Iyengar's inequality, Hermite-Hadamard's inequality.

Contents

1. Introduction and Statement of Main Results

In this paper, using Steffensen's inequality we prove several inequalities (Theorems 1.1 and 1.2) involving Taylor's remainder. In Sections 3 and 4 we give several applications of Theorems 1.1 and 1.2. Among the simplest particular cases we obtain Iyengar's inequality and one of Hermite-Hadamard's inequalities for convex functions. We prove Theorems 1.1 and 1.2 in Section 2.

In what follows n denotes a non-negative integer, $I \subseteq \mathbb{R}$ is a generic interval, and I° is the interior of I. We will denote by $R_{n,f}(c,x)$ the nth Taylor's remainder of function $f(x)$ with center c, i.e.

$$R_{n,f}(c,x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

The following two theorems are the main results of the present paper.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two mappings, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a,b])$, $g \in C([a,b])$. Assume that $m \le f^{(n+1)}(x) \le M$, $m \ne M$, and $g(x) \ge 0$ for all $x \in [a,b]$. Set*

$$\lambda = \frac{1}{M-m}\left[f^{(n)}(b) - f^{(n)}(a) - m(b-a)\right].$$

Then

(i) $$\frac{1}{(n+1)!}\int_{b-\lambda}^{b} (x-b+\lambda)^{n+1} g(x)dx \le \frac{1}{M-m}\int_a^b \left[R_{n,f}(a,x) - m\frac{(x-a)^{n+1}}{(n+1)!}\right] g(x)dx$$

$$\leq \frac{1}{(n+1)!}\int_a^b \left[(x-a)^{n+1}-(x-a-\lambda)^{n+1}\right]g(x)dx$$
$$+\frac{(-1)^{n+1}}{(n+1)!}\int_a^{a+\lambda}(a+\lambda-x)^{n+1}g(x)dx;$$

and

(ii)
$$\frac{1}{(n+1)!}\int_a^{a+\lambda}(a+\lambda-x)^{n+1}g(x)dx$$
$$\leq \frac{(-1)^{n+1}}{M-m}\int_a^b\left[R_{n,f}(b,x)-m\frac{(x-b)^{n+1}}{(n+1)!}\right]g(x)dx$$
$$\leq \frac{1}{(n+1)!}\int_a^b\left[(b-x)^{n+1}-(b-\lambda-x)^{n+1}\right]g(x)dx$$
$$+\frac{(-1)^{n+1}}{(n+1)!}\int_{b-\lambda}^b(x-b+\lambda)^{n+1}g(x)dx.$$

Theorem 1.2. *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two mappings, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a,b])$, $g \in C([a,b])$. Assume that $f^{n+1}(x)$ is increasing on $[a,b]$ and $m \leq g(x) \leq M$, $m \neq M$, for all $x \in [a,b]$. Set*

$$\lambda_1 = \frac{1}{(M-m)(b-a)^{n+1}}\int_a^b(x-a)^{n+1}g(x)dx-\frac{m}{M-m}\cdot\frac{b-a}{n+2},$$
$$\lambda_2 = \frac{1}{(M-m)(b-a)^{n+1}}\int_a^b(b-x)^{n+1}g(x)dx-\frac{m}{M-m}\cdot\frac{b-a}{n+2}.$$

Then

(i)
$$f^{(n)}(a-\lambda_1) - f^{(n)}(a) \le \frac{(n+1)!}{(M-m)(b-a)^{n+1}} \int_a^b R_{n,f}(a,x)(g(x)-m)dx \le f^{(n)}(b) - f^{(n)}(b-\lambda_1);$$

and

(ii)
$$f^{(n)}(a+\lambda_2) - f^{(n)}(a) \le (-1)^{n+1} \frac{(n+1)!}{(M-m)(b-a)^{n+1}} \int_a^b R_{n,f}(b,x)\,(g(x)-m)\,dx \le f^{(n)}(b) - f^{(n)}(b-\lambda_2).$$

Remark 1.1. *It is easy to verify that the inequalities in Theorems 1.1 and 1.2 become equalities if $f(x)$ is a polynomial of degree $\le n+1$.*

2. Proofs of Theorems 1.1 and 1.2

The following is well-known Steffensen's inequality:

Theorem 2.1. *[4]. Suppose the f and g are integrable functions defined on (a,b), f is decreasing and for each $x \in (a,b)$, $0 \le g(x) \le 1$. Set $\lambda = \int_a^b g(x)dx$. Then*

$$\int_{b-\lambda}^{b} f(x)dx \le \int_a^b f(x)g(x)dx \le \int_a^{a+\lambda} f(x)dx.$$

Proposition 2.2. *Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be two maps, $a, b \in I^\circ$ with $a < b$ and let $f \in C^{n+1}([a,b])$, $g \in C[a,b]$. Assume that $0 \le f^{(n+1)}(x) \le 1$ for all $x \in [a,b]$ and $\int_x^b (t-x)^n g(t)dt$ is a decreasing function of x on $[a,b]$. Set $\lambda = f^{(n)}(b) - f^{(n)}(a)$. Then*

$$\begin{aligned}
\text{(2.1)}\quad & \frac{1}{(n+1)!}\int_{b-\lambda}^{b} (x-b+\lambda)^{n+1} g(x)dx \\
&\le \int_a^b R_{n,f}(a,x)g(x)dx \\
&\le \frac{1}{(n+1)!}\int_a^b \left[(x-a)^{n+1} - (x-a-\lambda)^{n+1}\right] g(x)dx \\
&\qquad + \frac{(-1)^{n+1}}{(n+1)!}\int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x)dx.
\end{aligned}$$

Proof. Set

$$\begin{aligned} F_n(x) &= \frac{1}{n!}\int_x^b (t-x)^n g(t)dt, \\ G_n(x) &= f^{n+1}(x), \\ \lambda &= \int_a^b G_n(x)dx = f^{(n)}(b) - f^{(n)}(a). \end{aligned}$$

Then $F_n(x)$, $G_n(x)$, and λ satisfy the conditions of Theorem 2.1. Therefore

$$\int_{b-\lambda}^b F_n(x)dx \le \int_a^b F_n(x)G_n(x)dx \le \int_a^{a+\lambda} F_n(x)dx. \tag{2.2}$$

It is easy to see that $F_n'(x) = -F_{n-1}(x)$. Hence

$$\begin{aligned} \int_a^b F_n(x)G_n(x)dx &= \int_a^b F_n(x)df^{(n)}(x) \\ &= f^{(n)}(x)F_n(x)\Big|_a^b + \int_a^b f^{(n)}(x)F_{n-1}(x)dx \\ &= -\frac{f^{(n)}(a)}{n!}\int_a^b (x-a)^n g(x)dx + \int_a^b F_{n-1}(x)G_{n-1}(x)dx \\ &= -\frac{f^{(n)}(a)}{n!}\int_a^b (x-a)^n g(x)dx \\ &\quad - \frac{f^{(n-1)}(a)}{(n-1)!}\int_a^b (x-a)^{n-1} g(x)dx + \int_a^b F_{n-2}(x)G_{n-2}(x)dx \end{aligned}$$

$$= \dots$$

$$= -\frac{f^{(n)}(a)}{n!}\int_a^b (x-a)^n g(x)dx - \frac{f^{(n-1)}(a)}{(n-1)!}\int_a^b (x-a)^{n-1} g(x)dx$$
$$- \cdots - f(a)\int_a^b g(x)dx + \int_a^b f(x)g(x)dx.$$

Thus

$$\int_a^b F_n(x)G_n(x)dx = \int_a^b R_{n,f}(a,x)g(x)dx. \tag{2.3}$$

In addition

$$\int_a^{a+\lambda} F_n(x)dx = \frac{1}{n!}\int_a^{a+\lambda}\left(\int_x^b (t-x)^n g(t)dt\right)dx.$$

Changing the order of integration, we obtain

$$\int_a^{a+\lambda} F_n(x)dx$$
$$= \frac{1}{n!}\int_a^{a+\lambda}\left(\int_a^t (t-x)^n g(t)dx\right)dt + \frac{1}{n!}\int_{a+\lambda}^b\left(\int_a^{a+\lambda} (t-x)^n g(t)dx\right)dt$$
$$= -\frac{1}{n!}\int_a^{a+\lambda} g(t)\frac{(t-x)^{n+1}}{n+1}\Bigg|_{x=a}^{x=t} dt - \frac{1}{n!}\int_{a+\lambda}^b g(t)\frac{(t-x)^{n+1}}{n+1}\Bigg|_{x=a}^{x=a+\lambda} dt$$

$$= \frac{1}{(n+1)!} \int_a^{a+\lambda} (t-a)^{n+1} g(t)dt - \frac{1}{(n+1)!} \int_{a+\lambda}^{b} \left[(t-a-\lambda)^{n+1} - (t-a)^{n+1} \right] g(t)dt$$

$$= \frac{1}{(n+1)!} \int_a^{b} (t-a)^{n+1} g(t)dt - \frac{1}{(n+1)!} \int_a^{b} (t-a-\lambda)^{n+1} g(t)dt + \frac{1}{(n+1)!} \int_a^{a+\lambda} (t-a-\lambda)^{n+1} g(t)dt.$$

Thus,

$$\int_a^{a+\lambda} F_n(x)dx = \frac{1}{(n+1)!} \int_a^{b} \left[(x-a)^{n+1} - (x-a-\lambda)^{n+1} \right] g(x)dx + \frac{(-1)^{n+1}}{(n+1)!} \int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x)dx. \tag{2.4}$$

Similarly we obtain

$$\int_{b-\lambda}^{b} F_n(x)dx = \frac{1}{(n+1)!} \int_{b-\lambda}^{b} (x-b+\lambda)^{n+1} g(x)dx \tag{2.5}$$

Substituting (2.3), (2.4), and (2.5) into (2.2), we obtain (2.1). □

Proposition 2.3. *Let* $f : I \to \mathbb{R}$ *and* $g : I \to \mathbb{R}$ *be two maps,* $a, b \in I^{\circ}$ *with* $a < b$ *and let* $f \in C^{n+1}([a,b])$, $g \in C([a,b])$. *Assume that* $m \leq f^{(n+1)}(x) \leq$

M for all $x \in [a,b]$ and $\int_x^b (t-x)^n g(t)dt$ is a decreasing function of x on $[a,b]$. Set $\lambda = \frac{1}{M-m}\left[f^{(n)}(b) - f^{(n)}(a) - m(b-a)\right]$. Then

$$\begin{aligned}(2.6)\quad \frac{1}{(n+1)!}\int_{b-\lambda}^{b}(x-b+\lambda)^{n+1}g(x)dx &\le \frac{1}{M-m}\int_a^b \left[R_{n,f}(a,x) - m\frac{(x-a)^{n+1}}{(n+1)!}\right]g(x)dx \\ &\le \frac{1}{(n+1)!}\int_a^b \left[(x-a)^{n+1} - (x-a-\lambda^{n+1})\right]g(x)dx \\ &\quad + \frac{(-1)^{n+1}}{(n+1)!}\int_a^{a+\lambda}(a+\lambda-x)^{n+1}g(x)dx.\end{aligned}$$

Proof. Set

$$\tilde{f}(x) = \frac{1}{M-m}\left[f(x) - m\frac{(x-a)^{n+1}}{(n+1)!}\right].$$

Then $0 \le \tilde{f}^{(n+1)}(x) \le 1$ and

$$\lambda = \frac{1}{M-m}\left[f^{(n)}(b) - f^{(n)}(a) - m(b-a)\right] = \tilde{f}^{(n)}(b) - \tilde{f}^{(n)}(a).$$

Hence $\tilde{f}(x)$, $g(x)$, and λ satisfy the conditions of Proposition 2.2. Substituting $\tilde{f}(x)$ instead of $f(x)$ into (2.1), we obtain (2.6). □

Proof of Theorem 1.1(i). If $g(x) \ge 0$ for all $x \in [a,b]$, then $\int_x^b (t-x)^n g(t)dt$ is a decreasing function of x on $[a,b]$. Hence Proposition 2.3 implies Theorem 1.1(i). □

Proof of Theorems 1.1(ii), 1.2(i), and 1.2(ii). Proofs of Theorems 1.1(ii), 1.2(i), and 1.2(ii) are similar to the above proof of Theorem 1.1(i). For the proof of Theorem 1.1(ii) we take

$$F_n(x) = -\frac{1}{n!}\int_a^x (x-t)^n g(t)dt, \quad G_n(x) = f^{n+1}(x).$$

For the proof of Theorem 1.2(i) we take

$$F_n(x) = -f^{(n+1)}(x), \quad G_n(x) = \frac{1}{n!}\int_x^b (t-x)^n g(t)dt.$$

For the proof of Theorem 1.2(ii) we take

$$F_n(x) = -f^{(n+1)}(x), \quad G_n(x) = \frac{1}{n!}\int_a^x (x-t)^n g(t)dt.$$

□

3. Applications of Theorem 1.1

Theorem 3.1. *Let* $f : I \to \mathbb{R}$ *be a mapping,* $a, b \in I^{\circ}$ *with* $a < b$*, and let* $f \in C^{n+1}([a,b])$*. Assume that* $m \leq f^{(n+1)}(x) \leq M$*,* $m \neq M$*, for all* $x \in [a,b]$*. Set*

$$\lambda = \frac{1}{M-m}\left[f^{(n)}(b) - f^{(n)}(a) - m(b-a)\right].$$

Then

$$\text{(i)}\quad \frac{1}{(n+2)!}\left[m(b-a)^{n+2} + (M-m)\lambda^{n+2}\right] \leq \int_a^b R_{n,f}(a,x)dx \leq \frac{1}{(n+2)!}\left[M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}\right];$$

and

$$\text{(ii)}\quad \frac{1}{(n+2)!}\left[m(b-a)^{n+2} + (M-m)\lambda^{n+2}\right] \leq (-1)^{n+1}\int_a^b R_{n,f}(b,x)dx \leq \frac{1}{(n+2)!}\left[M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}\right].$$

Proof. Take $g(x) \equiv 1$ on $[a,b]$ in Theorem 1.1. □

Two inequalities of the form $A \leq X \leq B$ and $A \leq Y \leq B$ imply two new inequalities $A \leq \frac{1}{2}(X+Y) \leq B$ and $|X-Y| \leq B-A$. Applying this construction to inequalities (i) and (ii) of Theorem 3.1, we obtain the following two more symmetric with respect to a and b inequalities:

Theorem 3.2. *Let* $f : I \to \mathbb{R}$ *be a mapping,* $a, b \in I^{\circ}$ *with* $a < b$*, and let* $f \in C^{n+1}([a,b])$*. Assume that* $m \leq f^{n+1}(x) \leq M$*,* $m \neq M$*, for all* $x \in [a,b]$*. Set*

$$\lambda = \frac{1}{M-m}\left[f^{(n)}(b) - f^{(n)}(a) - m(b-a)\right].$$

Then

(i)
$$\begin{aligned} &\frac{1}{(n+2)!}[m(b-a)^{n+2} + (M-m)\lambda^{n+2}] \\ &\leq \int_a^b \frac{1}{2}\left[R_{n,f}(a,x) + (-1)^{n+1}R_{n,f}(b,x)\right] dx \\ &\leq \frac{1}{(n+2)!}\left[M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}\right]; \end{aligned}$$

and

(ii)
$$\begin{aligned} &\left|\int_a^b \left[R_{n,f}(a,x) + (-1)^{n}R_{n,f}(b,x)\right] dx\right| \\ &\qquad \leq \frac{M-m}{(n+2)!}\left[(b-a)^{n+2} - \lambda^{n+2} - (b-a-\lambda)^{n+2}\right]. \end{aligned}$$

We now consider the simplest cases of inequalities (i) and (ii) of Theorem 3.2, namely the cases when $n = 0$ or 1.

Case 3.1. $\underline{n = 0}$

Inequality (i) of Theorem 3.2 for $n = 0$ gives us the following result.

Theorem 3.3. *Let $f : I \to \mathbb{R}$ be a mapping, $a, b \in I^{\circ}$ with $a < b$ and let $f \in C^1([a,b])$. Assume that $m \le f'(x) \le M$, $m \ne M$, for all $x \in [a,b]$. Set*

$$\lambda = \frac{1}{M-m}\left[f(b) - f(a) - m(b-a)\right].$$

Then

$$m + \frac{(M-m)\lambda^2}{(b-a)^2} \le \frac{f(b)-f(a)}{b-a} \le M - \frac{(M-m)(b-a-\lambda)^2}{(b-a)^2}.$$

Remark 3.1. *Theorem 3.3 is an improvement of a trivial inequality $m \le \frac{f(b)-f(a)}{b-a} \le M$.*

For $n = 0$, inequality (ii) of Theorem 3.2 gives the following result:

Theorem 3.4. *Let $f : I \to \mathbb{R}$ be a mapping, $a, b \in I^{\circ}$ with $a < b$, and let $f \in C^1([a,b])$. Assume that $m \le f'(x) \le M$, $m \ne M$ for all $x \in [a,b]$. Then*

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{\left[f(b)-f(a)-m(b-a)\right]\left[M(b-a)-f(b)+f(a)\right]}{2(M-m)}.$$

Theorem 3.4 is a modification of Iyengar's inequality due to Agarwal and Dragomir [1]. If $|f'(x)| \le M$, then taking $m = -M$ in Theorem 3.4, we *obtain*

$$\left|\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a)\right| \le \frac{M(b-a)^2}{4} - \frac{1}{4M}\left[f(b)-f(a)\right]^2.$$

This is the original Iyengar's inequality [2]. Thus, inequality (ii) of Theorem 3.2 can be considered as a generalization of Iyengar's inequality.

Case 3.2. $\underline{n=1}$
In the case $n=1$, inequality (i) of Theorem 3.2 gives us the following result:

Theorem 3.5. *Let $f : I \to \mathbb{R}$ be a mapping, $a,b \in I^\circ$ with $a<b$ and let $f \in C^2([a,b])$. Assume that $m \le f''(x) \le M$, $m \ne M$, for all $x \in [a,b]$. Set*

$$\lambda = \frac{1}{M-m}\left[f'(b) - f'(a) - m(b-a)\right].$$

Then

$$\frac{1}{6}\left[m(b-a)^3 + (M-m)\lambda^3\right] \le \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) + \frac{f'(b)-f'(a)}{4}(b-a)^2 \le \frac{1}{6}\left[M(b-a)^3 - (M-m)(b-a-\lambda)^3\right].$$

In the case $n=1$, inequality (ii) of Theorem 3.2 implies that if $f \in C^2([a,b])$

and $m \le f''(x) \le M$, *then*

$$\left| \frac{f(b)-f(a)}{b-a} - \frac{f'(a)+f'(b)}{2} \right| \le \frac{[f'(b)-f'(a)-m(b-a)]\,[M(b-a)-f'(b)+f'(a)]}{2(b-a)(M-m)}.$$

This result follows readily from Iyengar's inequality if we take $f'(x)$ *instead of* $f(x)$ *in Theorem 3.4.*

4. Applications of Theorem 1.2

Take $g(x) = M$ on $[a,b]$ in Theorem 1.2. Then $\lambda_1 = \lambda_2 = \frac{b-a}{n+2}$ and Theorem 1.2 implies

Theorem 4.1. *Let $f : I \to \mathbb{R}$ be a mapping, $a,b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a,b])$. Assume that $f^{n+1}(x)$ is increasing on $[a,b]$. Then*

$$\text{(i)}\quad \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}\left(a+\frac{b-a}{n+2}\right)-f^{(n)}(a)\right] \le \int_a^b R_{n,f}(a,x)dx \le \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}(b)-f^{(n)}\left(b-\frac{b-a}{n+2}\right)\right];$$

and

$$\text{(ii)}\quad \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}\left(a+\frac{b-a}{n+2}\right)-f^{(n)}(a)\right] \le (-1)^{n+1}\int_a^b R_{n,f}(b,x)dx \le \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}(b)-f^{(n)}\left(b-\frac{b-a}{n+2}\right)\right].$$

The next theorem follows from Theorem 4.1 in exactly the same way as Theorem 3.2 follows from Theorem 3.1.

Theorem 4.2. *Let $f : I \to \mathbb{R}$ be a mapping, $a,b \in I^\circ$ with $a < b$, and let*

$f \in C^{n+1}([a,b])$. Assume that $f^{n+1}(x)$ is increasing on $[a,b]$. Then

$$\text{(i)}\qquad \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}\left(a+\frac{b-a}{n+2}\right)-f^{(n)}(a)\right] \le \frac{1}{2}\int_a^b \left[R_{n,f}(a,x)+(-1)^{n+1}R_{n,f}(b,x)\right]dx \le \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}(b)-f^{(n)}\left(b-\frac{b-a}{n+2}\right)\right];$$

$$\text{(ii)}\qquad \left|\int_a^b \left[R_{n,f}(a,x)+(-1)^{n}R_{n,f}(b,x)\right]dx\right| \le \frac{(b-a)^{n+1}}{(n+1)!}\left[f^{(n)}(b)-f^{(n)}\left(b-\frac{b-a}{n+2}\right)-f^{(n)}\left(a+\frac{b-a}{n+2}\right)+f^{(n)}(a)\right].$$

We now consider inequalities (i) and (ii) of Theorem 4.2 in the simplest cases when $n=0$ or 1.

Case 4.1. $\underline{n=0}$.
Inequality (i) of Theorem 4.2 gives a trivial fact: If $f'(x)$ increases then $f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2}$. Inequality (ii) of Theorem 4.2 gives the following result: If $f'(x)$ is increasing, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le f(a)+f(b)-f\left(\frac{a+b}{2}\right). \tag{4.1}$$

The left inequality (2.1) is a half of the famous Hermite-Hadamard's inequality [3]: If $f(x)$ is convex, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{4.2}$$

Note that the right inequality (4.1) is weaker than the right inequality (4.2). Thus, inequality (ii) of Theorem 4.2 can be considered as a generalization of the Hermite-Hadamard's inequality $f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx$, where $f(x)$ is convex.

Case 4.2. $\underline{n=1}$.
In this case Theorem 4.2 implies the following two results:

Theorem 4.3. *Let $f : I \to \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let $f \in C^2([a,b])$. Assume that $f''(x)$ is increasing on $[a,b]$. Then*

(i)
$$\frac{(b-a)^2}{2}\left[f'\left(a+\frac{b-a}{3}\right)+f'(a)\right] \le \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) + \frac{f'(b)-f'(a)}{4}(b-a)^2 \le \frac{(b-a)^2}{2}\left[f'(b) - f'\left(\frac{b-a}{3}\right)\right];$$

(ii)
$$\left|\frac{f(b)-f(a)}{b-a} - \frac{f'(a)+f'(b)}{2}\right| \le \frac{1}{2}\left[f'(a) - f'\left(a+\frac{b-a}{3}\right) - f'\left(b-\frac{b-a}{3}\right) + f'(b)\right].$$

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