

Journal of Inequalities in Pure and Applied Mathematics

SOME INTEGRAL INEQUALITIES INVOLVING TAYLOR'S REMAINDER

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©2000 Victoria University
ISSN (electronic): 1443-5756
068-01



volume 3, issue 2, article 26,
2002.

*Received 17 September, 2001;;
accepted 23 January, 2001.*

Communicated by: G. Anastassiou

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

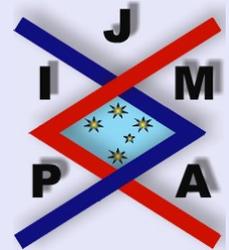
In this paper, using Steffensen's inequality we prove several inequalities involving Taylor's remainder. Among the simplest particular cases we obtain Iyengar's inequality and one of Hermite-Hadamard's inequalities for convex functions.

2000 Mathematics Subject Classification: 26D15.

Key words: Taylor's remainder, Steffensen's inequality, Iyengar's inequality, Hermite-Hadamard's inequality.

Contents

1	Introduction and Statement of Main Results	3
2	Proofs of Theorems 1.1 and 1.2	6
3	Applications of Theorem 1.1	12
4	Applications of Theorem 1.2	17
	References	



Some Integral Inequalities Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 2 of 20

1. Introduction and Statement of Main Results

In this paper, using Steffensen's inequality we prove several inequalities (Theorems 1.1 and 1.2) involving Taylor's remainder. In Sections 3 and 4 we give several applications of Theorems 1.1 and 1.2. Among the simplest particular cases we obtain Iyengar's inequality and one of Hermite-Hadamard's inequalities for convex functions. We prove Theorems 1.1 and 1.2 in Section 2.

In what follows n denotes a non-negative integer, $I \subseteq \mathbb{R}$ is a generic interval, and I° is the interior of I . We will denote by $R_{n,f}(c, x)$ the n th Taylor's remainder of function $f(x)$ with center c , i.e.

$$R_{n,f}(c, x) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k.$$

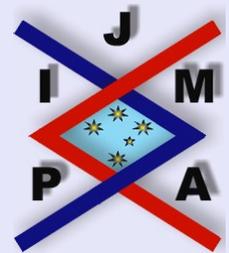
The following two theorems are the main results of the present paper.

Theorem 1.1. *Let $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be two mappings, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a, b])$, $g \in C([a, b])$. Assume that $m \leq f^{(n+1)}(x) \leq M$, $m \neq M$, and $g(x) \geq 0$ for all $x \in [a, b]$. Set*

$$\lambda = \frac{1}{M - m} [f^{(n)}(b) - f^{(n)}(a) - m(b - a)].$$

Then

$$(i) \quad \frac{1}{(n+1)!} \int_{b-\lambda}^b (x - b + \lambda)^{n+1} g(x) dx \\ \leq \frac{1}{M - m} \int_a^b \left[R_{n,f}(a, x) - m \frac{(x - a)^{n+1}}{(n+1)!} \right] g(x) dx$$



Title Page

Contents



Go Back

Close

Quit

Page 3 of 20

$$\leq \frac{1}{(n+1)!} \int_a^b [(x-a)^{n+1} - (x-a-\lambda)^{n+1}] g(x) dx$$

$$+ \frac{(-1)^{n+1}}{(n+1)!} \int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x) dx;$$

and

$$(ii) \quad \frac{1}{(n+1)!} \int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x) dx$$

$$\leq \frac{(-1)^{n+1}}{M-m} \int_a^b \left[R_{n,f}(b,x) - m \frac{(x-b)^{n+1}}{(n+1)!} \right] g(x) dx$$

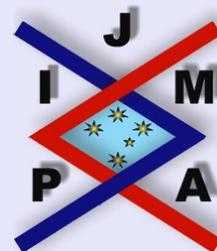
$$\leq \frac{1}{(n+1)!} \int_a^b [(b-x)^{n+1} - (b-\lambda-x)^{n+1}] g(x) dx$$

$$+ \frac{(-1)^{n+1}}{(n+1)!} \int_{b-\lambda}^b (x-b+\lambda)^{n+1} g(x) dx.$$

Theorem 1.2. Let $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be two mappings, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a, b])$, $g \in C([a, b])$. Assume that $f^{(n+1)}(x)$ is increasing on $[a, b]$ and $m \leq g(x) \leq M$, $m \neq M$, for all $x \in [a, b]$. Set

$$\lambda_1 = \frac{1}{(M-m)(b-a)^{n+1}} \int_a^b (x-a)^{n+1} g(x) dx - \frac{m}{M-m} \cdot \frac{b-a}{n+2},$$

$$\lambda_2 = \frac{1}{(M-m)(b-a)^{n+1}} \int_a^b (b-x)^{n+1} g(x) dx - \frac{m}{M-m} \cdot \frac{b-a}{n+2}.$$



**Some Integral Inequalities
Involving Taylor's Remainder. I**

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 4 of 20

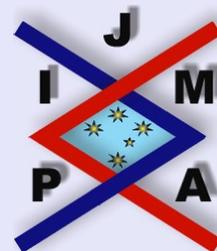
Then

$$\begin{aligned} \text{(i)} \quad & f^{(n)}(a - \lambda_1) - f^{(n)}(a) \\ & \leq \frac{(n+1)!}{(M-m)(b-a)^{n+1}} \int_a^b R_{n,f}(a, x)(g(x) - m) dx \\ & \leq f^{(n)}(b) - f^{(n)}(b - \lambda_1); \end{aligned}$$

and

$$\begin{aligned} \text{(ii)} \quad & f^{(n)}(a + \lambda_2) - f^{(n)}(a) \\ & \leq (-1)^{n+1} \frac{(n+1)!}{(M-m)(b-a)^{n+1}} \int_a^b R_{n,f}(b, x)(g(x) - m) dx \\ & \leq f^{(n)}(b) - f^{(n)}(b - \lambda_2). \end{aligned}$$

Remark 1.1. It is easy to verify that the inequalities in Theorems 1.1 and 1.2 become equalities if $f(x)$ is a polynomial of degree $\leq n + 1$.



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 5 of 20

2. Proofs of Theorems 1.1 and 1.2

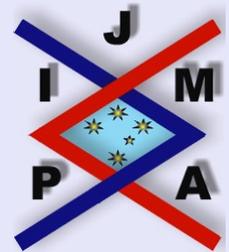
The following is well-known Steffensen's inequality:

Theorem 2.1. [4]. Suppose the f and g are integrable functions defined on (a, b) , f is decreasing and for each $x \in (a, b)$, $0 \leq g(x) \leq 1$. Set $\lambda = \int_a^b g(x)dx$. Then

$$\int_{b-\lambda}^b f(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^{a+\lambda} f(x)dx.$$

Proposition 2.2. Let $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be two maps, $a, b \in I^\circ$ with $a < b$ and let $f \in C^{n+1}([a, b])$, $g \in C[a, b]$. Assume that $0 \leq f^{(n+1)}(x) \leq 1$ for all $x \in [a, b]$ and $\int_x^b (t-x)^n g(t)dt$ is a decreasing function of x on $[a, b]$. Set $\lambda = f^{(n)}(b) - f^{(n)}(a)$. Then

$$\begin{aligned} (2.1) \quad & \frac{1}{(n+1)!} \int_{b-\lambda}^b (x-b+\lambda)^{n+1} g(x) dx \\ & \leq \int_a^b R_{n,f}(a, x) g(x) dx \\ & \leq \frac{1}{(n+1)!} \int_a^b [(x-a)^{n+1} - (x-a-\lambda)^{n+1}] g(x) dx \\ & \quad + \frac{(-1)^{n+1}}{(n+1)!} \int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x) dx. \end{aligned}$$



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 6 of 20

Proof. Set

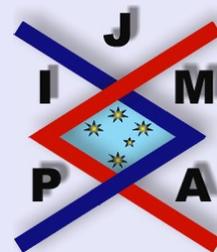
$$\begin{aligned}
 F_n(x) &= \frac{1}{n!} \int_x^b (t-x)^n g(t) dt, \\
 G_n(x) &= f^{n+1}(x), \\
 \lambda &= \int_a^b G_n(x) dx = f^{(n)}(b) - f^{(n)}(a).
 \end{aligned}$$

Then $F_n(x)$, $G_n(x)$, and λ satisfy the conditions of Theorem 2.1. Therefore

$$(2.2) \quad \int_{b-\lambda}^b F_n(x) dx \leq \int_a^b F_n(x) G_n(x) dx \leq \int_a^{a+\lambda} F_n(x) dx.$$

It is easy to see that $F'_n(x) = -F_{n-1}(x)$. Hence

$$\begin{aligned}
 \int_a^b F_n(x) G_n(x) dx &= \int_a^b F_n(x) df^{(n)}(x) \\
 &= f^{(n)}(x) F_n(x) \Big|_a^b + \int_a^b f^{(n)}(x) F_{n-1}(x) dx \\
 &= -\frac{f^{(n)}(a)}{n!} \int_a^b (x-a)^n g(x) dx + \int_a^b F_{n-1}(x) G_{n-1}(x) dx \\
 &= -\frac{f^{(n)}(a)}{n!} \int_a^b (x-a)^n g(x) dx \\
 &\quad - \frac{f^{(n-1)}(a)}{(n-1)!} \int_a^b (x-a)^{n-1} g(x) dx + \int_a^b F_{n-2}(x) G_{n-2}(x) dx
 \end{aligned}$$



**Some Integral Inequalities
Involving Taylor's Remainder. I**

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 7 of 20

= ...

$$= -\frac{f^{(n)}(a)}{n!} \int_a^b (x-a)^n g(x) dx - \frac{f^{(n-1)}(a)}{(n-1)!} \int_a^b (x-a)^{n-1} g(x) dx \\ - \dots - f(a) \int_a^b g(x) dx + \int_a^b f(x) g(x) dx.$$

Thus

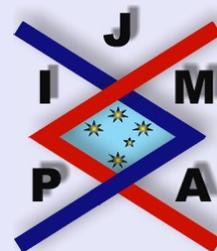
$$(2.3) \quad \int_a^b F_n(x) G_n(x) dx = \int_a^b R_{n,f}(a, x) g(x) dx.$$

In addition

$$\int_a^{a+\lambda} F_n(x) dx = \frac{1}{n!} \int_a^{a+\lambda} \left(\int_x^b (t-x)^n g(t) dt \right) dx.$$

Changing the order of integration, we obtain

$$\int_a^{a+\lambda} F_n(x) dx \\ = \frac{1}{n!} \int_a^{a+\lambda} \left(\int_a^t (t-x)^n g(t) dx \right) dt + \frac{1}{n!} \int_{a+\lambda}^b \left(\int_a^{a+\lambda} (t-x)^n g(t) dx \right) dt \\ = -\frac{1}{n!} \int_a^{a+\lambda} g(t) \frac{(t-x)^{n+1}}{n+1} \Big|_{x=a}^{x=t} dt - \frac{1}{n!} \int_{a+\lambda}^b g(t) \frac{(t-x)^{n+1}}{n+1} \Big|_{x=a}^{x=a+\lambda} dt$$



**Some Integral Inequalities
Involving Taylor's Remainder. I**

Hillel Gauchman

Title Page

Contents

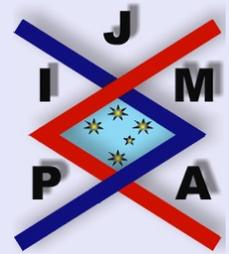


Go Back

Close

Quit

Page 8 of 20



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 9 of 20

$$\begin{aligned} &= \frac{1}{(n+1)!} \int_a^{a+\lambda} (t-a)^{n+1} g(t) dt \\ &\quad - \frac{1}{(n+1)!} \int_{a+\lambda}^b [(t-a-\lambda)^{n+1} - (t-a)^{n+1}] g(t) dt \\ &= \frac{1}{(n+1)!} \int_a^b (t-a)^{n+1} g(t) dt - \frac{1}{(n+1)!} \int_a^b (t-a-\lambda)^{n+1} g(t) dt \\ &\quad + \frac{1}{(n+1)!} \int_a^{a+\lambda} (t-a-\lambda)^{n+1} g(t) dt. \end{aligned}$$

Thus,

$$(2.4) \quad \int_a^{a+\lambda} F_n(x) dx = \frac{1}{(n+1)!} \int_a^b [(x-a)^{n+1} - (x-a-\lambda)^{n+1}] g(x) dx \\ + \frac{(-1)^{n+1}}{(n+1)!} \int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x) dx.$$

Similarly we obtain

$$(2.5) \quad \int_{b-\lambda}^b F_n(x) dx = \frac{1}{(n+1)!} \int_{b-\lambda}^b (x-b+\lambda)^{n+1} g(x) dx$$

Substituting (2.3), (2.4), and (2.5) into (2.2), we obtain (2.1). \square

Proposition 2.3. *Let $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ be two maps, $a, b \in I^\circ$ with $a < b$ and let $f \in C^{n+1}([a, b])$, $g \in C([a, b])$. Assume that $m \leq f^{(n+1)}(x) \leq$*

M for all $x \in [a, b]$ and $\int_x^b (t-x)^n g(t) dt$ is a decreasing function of x on $[a, b]$.
 Set $\lambda = \frac{1}{M-m} [f^{(n)}(b) - f^{(n)}(a) - m(b-a)]$. Then

$$\begin{aligned}
 (2.6) \quad & \frac{1}{(n+1)!} \int_{b-\lambda}^b (x-b+\lambda)^{n+1} g(x) dx \\
 & \leq \frac{1}{M-m} \int_a^b \left[R_{n,f}(a,x) - m \frac{(x-a)^{n+1}}{(n+1)!} \right] g(x) dx \\
 & \leq \frac{1}{(n+1)!} \int_a^b [(x-a)^{n+1} - (x-a-\lambda^{n+1})] g(x) dx \\
 & \quad + \frac{(-1)^{n+1}}{(n+1)!} \int_a^{a+\lambda} (a+\lambda-x)^{n+1} g(x) dx.
 \end{aligned}$$

Proof. Set

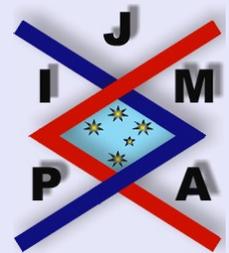
$$\tilde{f}(x) = \frac{1}{M-m} \left[f(x) - m \frac{(x-a)^{n+1}}{(n+1)!} \right].$$

Then $0 \leq \tilde{f}^{(n+1)}(x) \leq 1$ and

$$\lambda = \frac{1}{M-m} [f^{(n)}(b) - f^{(n)}(a) - m(b-a)] = \tilde{f}^{(n)}(b) - \tilde{f}^{(n)}(a).$$

Hence $\tilde{f}(x)$, $g(x)$, and λ satisfy the conditions of Proposition 2.2. Substituting $\tilde{f}(x)$ instead of $f(x)$ into (2.1), we obtain (2.6). \square

Proof of Theorem 1.1(i). If $g(x) \geq 0$ for all $x \in [a, b]$, then $\int_x^b (t-x)^n g(t) dt$ is a decreasing function of x on $[a, b]$. Hence Proposition 2.3 implies Theorem 1.1(i). \square



**Some Integral Inequalities
 Involving Taylor's Remainder. I**

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 10 of 20

Proof of Theorems 1.1(ii), 1.2(i), and 1.2(ii). Proofs of Theorems 1.1(ii), 1.2(i), and 1.2(ii) are similar to the above proof of Theorem 1.1(i). For the proof of Theorem 1.1(ii) we take

$$F_n(x) = -\frac{1}{n!} \int_a^x (x-t)^n g(t) dt, \quad G_n(x) = f^{n+1}(x).$$

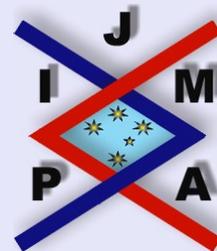
For the proof of Theorem 1.2(i) we take

$$F_n(x) = -f^{(n+1)}(x), \quad G_n(x) = \frac{1}{n!} \int_x^b (t-x)^n g(t) dt.$$

For the proof of Theorem 1.2(ii) we take

$$F_n(x) = -f^{(n+1)}(x), \quad G_n(x) = \frac{1}{n!} \int_a^x (x-t)^n g(t) dt.$$

□



**Some Integral Inequalities
Involving Taylor's Remainder. I**

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 11 of 20

3. Applications of Theorem 1.1

Theorem 3.1. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a, b])$. Assume that $m \leq f^{(n+1)}(x) \leq M$, $m \neq M$, for all $x \in [a, b]$.

Set

$$\lambda = \frac{1}{M - m} [f^{(n)}(b) - f^{(n)}(a) - m(b - a)].$$

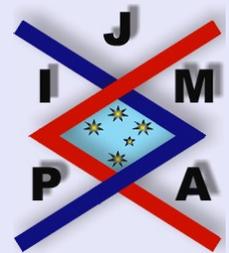
Then

$$\begin{aligned} \text{(i)} \quad & \frac{1}{(n+2)!} [m(b-a)^{n+2} + (M-m)\lambda^{n+2}] \\ & \leq \int_a^b R_{n,f}(a, x) dx \\ & \leq \frac{1}{(n+2)!} [M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}]; \end{aligned}$$

and

$$\begin{aligned} \text{(ii)} \quad & \frac{1}{(n+2)!} [m(b-a)^{n+2} + (M-m)\lambda^{n+2}] \\ & \leq (-1)^{n+1} \int_a^b R_{n,f}(b, x) dx \\ & \leq \frac{1}{(n+2)!} [M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}]. \end{aligned}$$

Proof. Take $g(x) \equiv 1$ on $[a, b]$ in Theorem 1.1. □



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 12 of 20

Two inequalities of the form $A \leq X \leq B$ and $A \leq Y \leq B$ imply two new inequalities $A \leq \frac{1}{2}(X + Y) \leq B$ and $|X - Y| \leq B - A$. Applying this construction to inequalities (i) and (ii) of Theorem 3.1, we obtain the following two more symmetric with respect to a and b inequalities:

Theorem 3.2. *Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{m+1}([a, b])$. Assume that $m \leq f^{n+1}(x) \leq M$, $m \neq M$, for all $x \in [a, b]$. Set*

$$\lambda = \frac{1}{M - m} [f^{(n)}(b) - f^{(n)}(a) - m(b - a)].$$

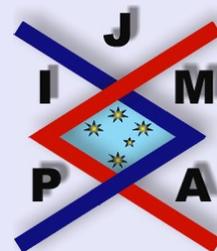
Then

$$\begin{aligned} \text{(i)} \quad & \frac{1}{(n+2)!} [m(b-a)^{n+2} + (M-m)\lambda^{n+2}] \\ & \leq \int_a^b \frac{1}{2} [R_{n,f}(a, x) + (-1)^{n+1} R_{n,f}(b, x)] dx \\ & \leq \frac{1}{(n+2)!} [M(b-a)^{n+2} - (M-m)(b-a-\lambda)^{n+2}]; \end{aligned}$$

and

$$\begin{aligned} \text{(ii)} \quad & \left| \int_a^b [R_{n,f}(a, x) + (-1)^n R_{n,f}(b, x)] dx \right| \\ & \leq \frac{M-m}{(n+2)!} [(b-a)^{n+2} - \lambda^{n+2} - (b-a-\lambda)^{n+2}]. \end{aligned}$$

We now consider the simplest cases of inequalities (i) and (ii) of Theorem 3.2, namely the cases when $n = 0$ or 1 .



Some Integral Inequalities Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 13 of 20

Case 3.1. $n = 0$

Inequality (i) of Theorem 3.2 for $n = 0$ gives us the following result.

Theorem 3.3. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$ and let $f \in C^1([a, b])$. Assume that $m \leq f'(x) \leq M$, $m \neq M$, for all $x \in [a, b]$. Set

$$\lambda = \frac{1}{M - m} [f(b) - f(a) - m(b - a)].$$

Then

$$m + \frac{(M - m)\lambda^2}{(b - a)^2} \leq \frac{f(b) - f(a)}{b - a} \leq M - \frac{(M - m)(b - a - \lambda)^2}{(b - a)^2}.$$

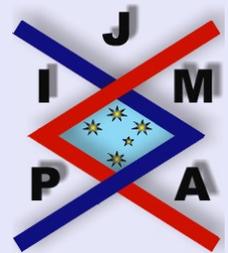
Remark 3.1. Theorem 3.3 is an improvement of a trivial inequality $m \leq \frac{f(b) - f(a)}{b - a} \leq M$.

For $n = 0$, inequality (ii) of Theorem 3.2 gives the following result:

Theorem 3.4. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let $f \in C^1([a, b])$. Assume that $m \leq f'(x) \leq M$, $m \neq M$ for all $x \in [a, b]$. Then

$$\left| \int_a^b f(x) dx - \frac{f(a) + f(b)}{2}(b - a) \right| \leq \frac{[f(b) - f(a) - m(b - a)][M(b - a) - f(b) + f(a)]}{2(M - m)}.$$

Theorem 3.4 is a modification of Iyengar's inequality due to Agarwal and Dragomir [1]. If $|f'(x)| \leq M$, then taking $m = -M$ in Theorem 3.4, we



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 14 of 20

obtain

$$\left| \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b-a) \right| \leq \frac{M(b-a)^2}{4} - \frac{1}{4M} [f(b) - f(a)]^2.$$

This is the original Iyengar's inequality [2]. Thus, inequality (ii) of Theorem 3.2 can be considered as a generalization of Iyengar's inequality.

Case 3.2. $n = 1$

In the case $n = 1$, inequality (i) of Theorem 3.2 gives us the following result:

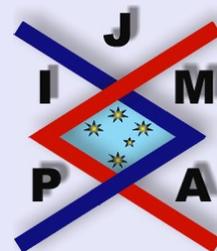
Theorem 3.5. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$ and let $f \in C^2([a, b])$. Assume that $m \leq f''(x) \leq M$, $m \neq M$, for all $x \in [a, b]$. Set

$$\lambda = \frac{1}{M - m} [f'(b) - f'(a) - m(b - a)].$$

Then

$$\begin{aligned} & \frac{1}{6} [m(b - a)^3 + (M - m)\lambda^3] \\ & \leq \int_a^b f(x)dx - \frac{f(a) + f(b)}{2}(b - a) + \frac{f'(b) - f'(a)}{4}(b - a)^2 \\ & \leq \frac{1}{6} [M(b - a)^3 - (M - m)(b - a - \lambda)^3]. \end{aligned}$$

In the case $n = 1$, inequality (ii) of Theorem 3.2 implies that if $f \in C^2([a, b])$



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

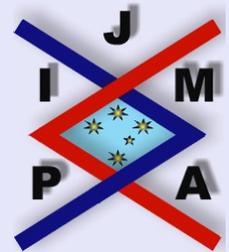
Quit

Page 15 of 20

and $m \leq f''(x) \leq M$, then

$$\left| \frac{f(b) - f(a)}{b - a} - \frac{f'(a) + f'(b)}{2} \right| \leq \frac{[f'(b) - f'(a) - m(b - a)][M(b - a) - f'(b) + f'(a)]}{2(b - a)(M - m)}.$$

This result follows readily from Iyengar's inequality if we take $f'(x)$ instead of $f(x)$ in Theorem 3.4.



**Some Integral Inequalities
Involving Taylor's Remainder. I**

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 16 of 20

4. Applications of Theorem 1.2

Take $g(x) = M$ on $[a, b]$ in Theorem 1.2. Then $\lambda_1 = \lambda_2 = \frac{b-a}{n+2}$ and Theorem 1.2 implies

Theorem 4.1. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let $f \in C^{n+1}([a, b])$. Assume that $f^{(n+1)}(x)$ is increasing on $[a, b]$. Then

$$(i) \quad \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)} \left(a + \frac{b-a}{n+2} \right) - f^{(n)}(a) \right] \\ \leq \int_a^b R_{n,f}(a, x) dx \leq \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)}(b) - f^{(n)} \left(b - \frac{b-a}{n+2} \right) \right];$$

and

$$(ii) \quad \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)} \left(a + \frac{b-a}{n+2} \right) - f^{(n)}(a) \right] \\ \leq (-1)^{n+1} \int_a^b R_{n,f}(b, x) dx \\ \leq \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)}(b) - f^{(n)} \left(b - \frac{b-a}{n+2} \right) \right].$$

The next theorem follows from Theorem 4.1 in exactly the same way as Theorem 3.2 follows from Theorem 3.1.

Theorem 4.2. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 17 of 20

$f \in C^{n+1}([a, b])$. Assume that $f^{n+1}(x)$ is increasing on $[a, b]$. Then

$$(i) \quad \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)} \left(a + \frac{b-a}{n+2} \right) - f^{(n)}(a) \right] \\ \leq \frac{1}{2} \int_a^b [R_{n,f}(a, x) + (-1)^{n+1} R_{n,f}(b, x)] dx \\ \leq \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)}(b) - f^{(n)} \left(b - \frac{b-a}{n+2} \right) \right];$$

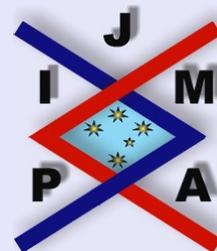
$$(ii) \quad \left| \int_a^b [R_{n,f}(a, x) + (-1)^n R_{n,f}(b, x)] dx \right| \\ \leq \frac{(b-a)^{n+1}}{(n+1)!} \left[f^{(n)}(b) - f^{(n)} \left(b - \frac{b-a}{n+2} \right) - f^{(n)} \left(a + \frac{b-a}{n+2} \right) + f^{(n)}(a) \right].$$

We now consider inequalities (i) and (ii) of Theorem 4.2 in the simplest cases when $n = 0$ or 1.

Case 4.1. $n = 0$.

Inequality (i) of Theorem 4.2 gives a trivial fact: If $f'(x)$ increases then $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$. Inequality (ii) of Theorem 4.2 gives the following result: If $f'(x)$ is increasing, then

$$(4.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq f(a) + f(b) - f\left(\frac{a+b}{2}\right).$$



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 18 of 20

The left inequality (2.1) is a half of the famous Hermite-Hadamard's inequality [3]: If $f(x)$ is convex, then

$$(4.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Note that the right inequality (4.1) is weaker than the right inequality (4.2). Thus, inequality (ii) of Theorem 4.2 can be considered as a generalization of the Hermite-Hadamard's inequality $f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx$, where $f(x)$ is convex.

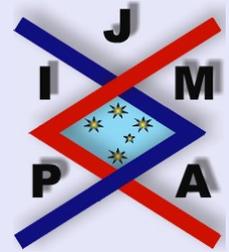
Case 4.2. $n = 1$.

In this case Theorem 4.2 implies the following two results:

Theorem 4.3. Let $f : I \rightarrow \mathbb{R}$ be a mapping, $a, b \in I^\circ$ with $a < b$, and let $f \in C^2([a, b])$. Assume that $f''(x)$ is increasing on $[a, b]$. Then

$$(i) \quad \begin{aligned} & \frac{(b-a)^2}{2} \left[f' \left(a + \frac{b-a}{3} \right) + f'(a) \right] \\ & \leq \int_a^b f(x)dx - \frac{f(a)+f(b)}{2}(b-a) + \frac{f'(b)-f'(a)}{4}(b-a)^2 \\ & \leq \frac{(b-a)^2}{2} \left[f'(b) - f' \left(\frac{b-a}{3} \right) \right]; \end{aligned}$$

$$(ii) \quad \begin{aligned} & \left| \frac{f(b)-f(a)}{b-a} - \frac{f'(a)+f'(b)}{2} \right| \\ & \leq \frac{1}{2} \left[f'(a) - f' \left(a + \frac{b-a}{3} \right) - f' \left(b - \frac{b-a}{3} \right) + f'(b) \right]. \end{aligned}$$



Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

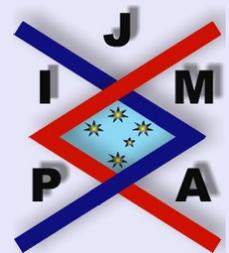
Close

Quit

Page 19 of 20

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Some Integral Inequalities
Involving Taylor's Remainder. I

Hillel Gauchman

Title Page

Contents



Go Back

Close

Quit

Page 20 of 20