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ON SOME RETARDED INTEGRAL INEQUALITIES AND APPLICATIONS

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ABSTRACT. The aim of this paper is to establish explicit bounds on certain retarded integral inequalities which can be used as convenient tools in some applications. The two independent variable generalizations of the main results and some applications are also given.

Key words and phrases: Retarded Integral Inequalities, Explicit Bounds, Two Independent Variable Generalizations, Qualitative Study, Nondecreasing, Change of Variable, Integrodifferential Equation, Uniqueness of Solutions.

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1. INTRODUCTION

Integral inequalities which provide explicit bounds on unknown functions have played a fundamental role in the development of the theory of differential and integral equations. Over the years, various investigators have discovered many useful integral inequalities in order to achieve a diversity of desired goals, see [1] – [6] and the references given therein. In a recent paper [5] Lipovan has given a useful nonlinear generalisation of the celebrated Gronwall inequality and presented some of its applications. However, the integral inequalities available in the literature do not apply directly in certain general situations and it is desirable to find integral inequalities useful in some new applications. The main purpose of the present paper is to establish explicit bounds on more general retarded integral inequalities which can be used as tools in the qualitative study of certain retarded integrodifferential equations. Some immediate applications of one of the result to convey the importance of our results to the literature are also given.

2. STATEMENT OF RESULTS

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0,\infty)$, $I = [t_0, T)$, $J_1 = [x_0, X)$, $J_2 = [y_0, Y)$ are the given subsets of $\mathbb{R}$, $\Delta = J_1 \times J_2$ and $'$ denotes the derivative. The partial derivatives of a function $z(x, y)$, $x, y \in \mathbb{R}$ with respect to x and y are denoted by $D_1 z(x, y)$ and $D_2 z(x, y)$ respectively.

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Our main results are given in the following theorems.

Theorem 2.1. *Let* $u(t), a(t) \in C(I, \mathbb{R}_+), b(t,s) \in C(I^2, \mathbb{R}_+)$ *for* $t_0 \leq s \leq t \leq T$ *and* $\alpha(t) \in C^1(I, I)$ *be nondecreasing with* $\alpha(t) \leq t$ *on* I *and* $k \geq 0$ *be a constant.*

(a_1) *If*

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) u(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma) u(\sigma) d\sigma \right] ds, \tag{2.1}$$

for $t \in I$, *then*

$$u(t) \leq k \exp(A(t)), \tag{2.2}$$

for $t \in I$, *where*

$$A(t) = \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + \int_{\alpha(t_0)}^{s} b(s,\sigma) d\sigma \right] ds, \tag{2.3}$$

for $t \in I$.

(a_2) *Let* $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ *be a nondecreasing function with* $g(u) > 0$ *for* $u > 0$. *If*

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) g(u(s)) + \int_{\alpha(t_0)}^{s} b(s,\sigma) g(u(\sigma)) d\sigma \right] ds, \tag{2.4}$$

for $t \in I$, *then for* $t_0 \leq t \leq t_1$,

$$u(t) \leq G^{-1}[G(k) + A(t)], \tag{2.5}$$

where $A(t)$ *is defined by (2.3),* G^{-1} *is the inverse function of*

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r > 0, \ r_0 > 0, \tag{2.6}$$

and $t_1 \in I$ *is chosen so that*

$$G(k) + A(t) \in Dom\left(G^{-1}\right),$$

for all t *lying in the interval* $[t_0, t_1]$.

Theorem 2.2. *Let* $u(x,y), a(x,y) \in C(\Delta, \mathbb{R}_+), b(x,y,s,t) \in C(\Delta^2, \mathbb{R}_+)$, *for* $x_0 \leq s \leq x \leq X$, $y_0 \leq t \leq y \leq Y$, $\alpha(x) \in C^1(J_1, J_1)$, $\beta(y) \in C^1(J_2, J_2)$ *be nondecreasing with* $\alpha(x) \leq x$ *on* J_1, $\beta(y) \leq y$ *on* J_2 *and* $k \geq 0$ *be a constant.*

(b_1) *If*

$$u(x,y) \leq k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \Big[a(s,t) u(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta) u(\sigma,\eta) d\eta d\sigma \Big] dt ds, \tag{2.7}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leq k \exp(A(x,y)), \tag{2.8}$$

for $(x,y) \in \Delta$, *where*

$$A(x,y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta) d\eta d\sigma \right] dt ds, \tag{2.9}$$

for $(x,y) \in \Delta$.

(b_2) *Let g be as in Theorem 2.1, part (a_2). If*

$$u(x,y) \le k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[a(s,t) g(u(s,t)) + \int_{\alpha(x_0)}^{s} \int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta) g(u(\sigma,\eta)) d\eta d\sigma \right] dtds, \tag{2.10}$$

for $(x,y) \in \Delta$, then for $x_0 \le x \le x_1$, $y_0 \le y \le y_1$,

$$u(x,y) \le G^{-1}[G(k) + A(x,y)], \tag{2.11}$$

where $A(x,y)$ is defined by (2.9), G, G^{-1} are as defined in Theorem 2.1, part (a_2) and $x_1 \in J_1$, $y_1 \in J_2$ are chosen so that

$$G(k) + A(x,y) \in Dom\left(G^{-1}\right),$$

for all x and y lying in $[x_0, x_1]$ and $[y_0, y_1]$ respectively.

3. PROOFS OF THEOREMS 2.1 AND 2.2

From the hypotheses, we observe that $\alpha'(t) \ge 0$ for $t \in I$, $\alpha'(x) \ge 0$ for $x \in J_1$, $\beta'(y) \ge 0$ for $y \in J_2$.

(a_1) Let $k > 0$ and define a function $z(t)$ by the right hand side of (2.1). Then $z(t) > 0$, $z(t_0) = k$, $u(t) \le z(t)$ and

$$\begin{aligned} z'(t) &= \left[a(\alpha(t)) u(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} [b(\alpha(t), \sigma) u(\sigma) d\sigma] \right] \alpha'(t) \\ &\le \left[a(\alpha(t)) z(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} [b(\alpha(t), \sigma) z(\sigma) d\sigma] \right] \alpha'(t). \end{aligned} \tag{3.1}$$

From (3.1) it is easy to observe that

$$\frac{z'(t)}{z(t)} \le \left[a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma \right] \alpha'(t). \tag{3.2}$$

Integrating (3.2) from t_0 to t, $t \in I$ and by making the change of variables yields

$$z(t) \le k \exp(A(t)), \tag{3.3}$$

for $t \in I$. Using (3.3) in $u(t) \le z(t)$ we get the inequality in (2.2). If $k \ge 0$, we carry out the above procedure with $k + \varepsilon$ instead of k, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (2.2).

(a_2) Let $k > 0$ and define a function $z(t)$ by the right hand side of (2.4). Then $z(t) > 0$, $z(t_0) = k$, $u(t) \le z(t)$ and as in the proof of (a_1) we get

$$\frac{z'(t)}{g(z(t))} \le \left[a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma \right] \alpha'(t). \tag{3.4}$$

From (2.6) and (3.4) we have

$$\frac{d}{dt} G(z(t)) = \frac{z'(t)}{g(z(t))} \le \left[a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma \right] \alpha'(t). \tag{3.5}$$

Integrating (3.5) from t_0 to t, $t \in I$ and by making the change of variables we have

$$G(z(t)) \le G(k) + A(t). \tag{3.6}$$

Since $G^{-1}(z)$ is increasing, from (3.6) we have

$$z(t) \leq G^{-1}[G(k) + A(t)]. \tag{3.7}$$

Using (3.7) in $u(t) \leq z(t)$ we get (2.5). The case $k \geq 0$ can be completed as mentioned in the proof of (a_1). The subinterval $t_0 \leq t \leq t_1$ for t is obvious.

(b_1) Let $k > 0$ and define a function $z(x,y)$ by the right hand side of (2.7). Then $z(x,y) > 0$, $z(x_0,y) = z(x,y_0) = k$, $u(x,y) \leq z(x,y)$ and

$$\begin{aligned} D_1 z(x,y) &= \left[\int_{\beta(y_0)}^{\beta(y)} \left[a(\alpha(x),t)\, u(\alpha(x),t) \right.\right. \\ &\qquad \left.\left. + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} b(\alpha(x),t,\sigma,\eta)\, u(\sigma,\eta)\, d\eta d\sigma\right] dt\right] \alpha'(x) \\ &\leq \left[\int_{\beta(y_0)}^{\beta(y)} \left[a(\alpha(x),t)\, z(\alpha(x),t) \right.\right. \\ &\qquad \left.\left. + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} b(\alpha(x),t,\sigma,\eta)\, z(\sigma,\eta)\, d\eta d\sigma\right] dt\right] \alpha'(x). \end{aligned} \tag{3.8}$$

From (3.8) it is easy to observe that

$$\frac{D_1 z(x,y)}{z(x,y)} \leq \left[\int_{\beta(y_0)}^{\beta(y)} \left[a(\alpha(x),t) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{t} b(\alpha(x),t,\sigma,\eta)\, d\eta d\sigma\right] dt\right] \alpha'(x). \tag{3.9}$$

Keeping y fixed in (3.9), setting $x = \xi$ and integrating it with respect to ξ from x_0 to x and making the change of variables we get

$$z(x,y) \leq k \exp(A(x,y)). \tag{3.10}$$

Using (3.10) in $u(x,y) \leq z(x,y)$, we get the required inequality in (2.8). The case $k \geq 0$ follows as mentioned in the proof of (a_1).

(b_2) The proof can be completed by following the proof of (a_2) and closely looking at the proof of (b_1). Here we omit the details.

4. Some Applications

In this section, we present some immediate applications of the inequality (a_1) in Theorem 2.1 to study certain properties of solutions of the integrodifferential equation

$$x'(t) = F\left(t, x(t-h(t)), \int_{t_0}^{t} f(t,\sigma,x(\sigma - h(\sigma)))\, d\sigma\right), \tag{P}$$

with the given initial condition

$$x(t_0) = x_0, \tag{P_0}$$

where $f \in C(I^2 \times \mathbb{R}, \mathbb{R})$, $F \in C(I \times \mathbb{R}^2, \mathbb{R})$, x_0 is a real constant and $h \in C^1(I,I)$ be nondecreasing with $t - h(t) \geq 0$, $h'(t) < 1$, $h(t_0) = 0$.

The following theorem deals with the estimate on the solution of (P) – (P_0).

Theorem 4.1. *Suppose that*

$$|f(t,s,x)| \leq b(t,s)\,|x|, \tag{4.1}$$

$$|F(t,x,w)| \leq a(t)\,|x| + |w|, \tag{4.2}$$

where $a(t)$, $b(t,s)$ are as defined in Theorem 2.1 and let

$$M = \max_{t\in I} \frac{1}{1-h'(t)}. \tag{4.3}$$

If $x(t)$ is any solution of (P) – (P_0), then

$$\begin{aligned} |x(t)| &\\ &\le |x_0| \exp\left(\int_{t_0}^{t-h(t)} \left[Ma(s+h(\eta)) + \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma+h(\tau))\, d\sigma\right] ds\right), \end{aligned} \tag{4.4}$$

for t, η, τ in I.

Proof. The solution $x(t)$ of (P) – (P_0) can be written as

$$x(t) = x_0 + \int_{t_0}^{t} F\left(s, x(s-h(s)), \int_{t_0}^{s} f(s,\sigma, x(\sigma - h(\sigma)))\, d\sigma\right) ds. \tag{4.5}$$

Using (4.1) – (4.3) in (4.5) and making the change of variables we have

$$|x(t)| \le |x_0| + \int_{t_0}^{t-h(t)} \left[Ma(s+h(\eta))\,|x(s)| + \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma+h(\tau))\,|x(\sigma)|\, d\sigma\right] ds,$$

for t, η, τ in I. Now a suitable application of the inequality in (a_1) given in Theorem 2.1 yields the required estimate in (4.4). □

Next, we shall prove the uniqueness of the solutions of (P) – (P_0).

Theorem 4.2. *Suppose that the functions f, F in (P) satisfy the conditions*

$$|f(t,s,x) - f(t,s,y)| \le b(t,s)\,|x-y|, \tag{4.6}$$

$$|F(t,x,\bar{x}) - F(t,y,\bar{y})| \le a(t)\,|x-y| + |\bar{x}-\bar{y}|, \tag{4.7}$$

where $a(t)$, $b(t,s)$ are as defined in Theorem 2.1 and let M be as in (4.3). Then the problem (P) – (P_0) has at most one solution on I.

Proof. Let $x(t)$ and $\bar{x}(t)$ be two solutions of (P) – (P_0) on I, then we have

$$\begin{aligned} x(t) - \bar{x}(t) = \int_{t_0}^{t} \Bigg\{ &F\left(s, x(s-h(s)), \int_{t_0}^{s} f(s,\sigma,x(\sigma-h(\sigma)))\, d\sigma\right) \\ &- F\left(s, \bar{x}(s-h(s)), \int_{t_0}^{s} f(s,\sigma,\bar{x}(\sigma-h(\sigma)))\, d\sigma\right)\Bigg\} ds. \end{aligned} \tag{4.8}$$

Using (4.6), (4.7) in (4.8) and making the change of variables we have

$$\begin{aligned} |x(t) - \bar{x}(t)| \le \int_{t_0}^{t-h(t)} \Bigg[&Ma(s+h(\eta))\,|x(s) - \bar{x}(s)| \\ &+ \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma+h(\tau))\,|x(\sigma) - \bar{x}(\sigma)|\, d\sigma\Bigg] ds \end{aligned} \tag{4.9}$$

for t, η, τ in I. A suitable application of the inequality in (a_1) given in Theorem 2.1 yields $|x(t) - \bar{x}(t)| \le 0$. Therefore $x(t) = \bar{x}(t)$, i.e., there is at most one solution of (P) – (P_0). □

Our next result shows the dependency of solutions of (P) – (P_0) on initial values.

Theorem 4.3. *Let $x_1(t)$ and $x_2(t)$ be the solutions of (P) with the given initial conditions*

$$(P_1) \qquad x_1(t_0) = x_1,$$

and

$$(P_2) \qquad x_2(t_0) = x_2,$$

respectively, where x_1, x_2, are real constants. Suppose that the functions f and F in (P) satisfy the conditions (4.6) and (4.7) in Theorem 4.2 and let M be as in (4.3). Then

$$(4.10) \quad |x_1(t) - x_2(t)| \le |x_1 - x_2| \times \exp\left(\int_{t_0}^{t-h(t)} \left[Ma(s+h(\eta)) + \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma + h(\tau))\, d\sigma\right] ds\right),$$

for t, η, τ in I.

Proof. By using the facts that $x_1(t)$ and $x_2(t)$ are the solutions of (P) – (P_1) and (P) – (P_2) respectively, we have

$$(4.11) \quad x_1(t) - x_2(t) = x_1 - x_2 + \int_{t_0}^{t} \left\{ F\left(s, x_1(s-h(s)), \int_{t_0}^{s} f(s,\sigma,x_1(\sigma - h(\sigma)))\, d\sigma\right) - F\left(s, x_2(s-h(s)), \int_{t_0}^{s} f(s,\sigma,x_2(\sigma - h(\sigma)))\, d\sigma\right)\right\} ds.$$

Using (4.6), (4.7) in (4.11) and by making the change of variables, we have

$$(4.12) \quad |x_1(t) - x_2(t)| \le |x_1 - x_2| + \int_{t_0}^{t-h(t)} \left[Ma(s+h(\eta)) |x_1(s) - x_2(s)| + \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma + h(\tau)) |x_1(\sigma) - x_2(\sigma)|\, d\sigma \right] ds,$$

for t, η, τ in I. Now a suitable application of the inequality in (a_1) given in Theorem 2.1 to (4.12) yields the required estimate in (4.10). □

In concluding we note that the inequality in (b_1) given in Theorem 2.2 can be used to study the similar properties as in Theorems 4.1 – 4.3 for the hyperbolic partial integrodifferential equation

$$(4.13) \qquad D_1 D_2 z(x,y) = F(x, y, z(x - h_1(x), y - h_2(y)), Tz(x,y)),$$

with the given initial boundary conditions

$$(4.14) \qquad z(x, y_0) = a_1(x), \quad z(x_0, y) = a_2(y), \quad a_1(x_0) = a_2(y_0),$$

where

$$(4.15) \qquad Tz(x,y) = \int_{x_0}^{x} \int_{y_0}^{y} K(x, y, s, t, z(s - h_1(s), t - h_2(t)))\, dt ds,$$

under some suitable conditions on the functions involved in (4.13) – (4.15). Since the formulations of these results are very close to those given above, we omit it here. Various other applications of the inequalities given here is left to another work.

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