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# A NOTE ON THE PERTURBED TRAPEZOID INEQUALITY

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ABSTRACT. In this paper, we utilize a variant of the Grüss inequality to obtain some new perturbed trapezoid inequalities. We improve the error bound of the trapezoid rule in numerical integration in some recent known results. Also we give a new Iyengar's type inequality involving a second order bounded derivative for the perturbed trapezoid inequality.

*Key words and phrases:* Grüss inequality, Perturbed trapezoid inequality, Sharp bounds.

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In the literature [2], [4] – [8], [11], [12] on numerical integration, the following estimation is well known as the trapezoid inequality:

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \le \frac{1}{12}M_2(b-a)^3, \tag{1.1}$$

where the mapping $f : [a,b] \to \mathbb{R}$ is supposed to be twice differentiable on the interval $(a,b)$, with the second derivative bounded on $(a,b)$ by $M_2 = \sup_{x\in(a,b)} |f''(x)| < +\infty$. In [5], the authors derived the error bounds for the trapezoid inequality (1.1) by different norm of mapping $f$. In [2, 7, 11], the authors obtained the trapezoid inequality by the difference of sup and inf bound of the first derivative, that is,

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \le \frac{1}{8}(\Gamma_1-\gamma_1)(b-a)^2,$$

where $\Gamma_1 = \sup_{x\in(a,b)} f'(x) < +\infty$ and $\gamma_1 = \inf_{x\in(a,b)} f'(x) > -\infty$.

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For the perturbed trapezoid inequality, S. Dragomir et al. [5] obtained the following inequality by an application of the Grüss inequality:

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a))\right| \leq \frac{1}{32}(\Gamma_2-\gamma_2)(b-a)^3, \tag{1.2}$$

where $f$ is supposed to be twice differentiable on the interval $(a,b)$, with the second derivative bounded on $(a,b)$ by $\Gamma_2 = \sup_{x\in(a,b)} f''(x) < +\infty$ and $\gamma_2 = \inf_{x\in(a,b)} f''(x) > -\infty$. The constant $\frac{1}{32}$ is smaller than $\frac{1}{6\sqrt{5}}$ given in [11] and $\frac{1}{18\sqrt{3}}$ given in [2].

In this note we first improve the constant $\frac{1}{32}$ in the inequality (1.2) to the best possible one of $\frac{1}{36\sqrt{3}}$. Then we give two new perturbed trapezoid inequalities for high-order differentiable mappings. We need the following variant of the Grüss inequality:

**Theorem 1.1.** *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\phi \leq g(x) \leq \Phi$ for some constants $\phi$, $\Phi$ for all $x \in [a,b]$, then*

$$\left|\frac{1}{b-a}\int_a^b h(x)g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b h(x)\,dx\int_a^b g(x)\,dx\right| \leq \frac{1}{2}\left(\int_a^b \left|h(x) - \frac{1}{b-a}\int_a^b h(y)dy\right|dx\right)(\Phi-\phi). \tag{1.3}$$

*Proof.* We write the left hand of inequality (1.3) as

$$\int_a^b h(x)g(x)\,dx - \frac{1}{b-a}\int_a^b h(x)\,dx\int_a^b g(x)\,dx = \int_a^b (h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy)g(x)\,dx.$$

Denote

$$I^+ = \int_a^b \max(h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy, 0)\,dx$$

and

$$I^- = \int_a^b \min(h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy, 0)\,dx.$$

Obviously $I^+ + I^- = 0$. For $\phi \leq g(x) \leq \Phi$, then

$$\int_a^b (h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy)g(x)\,dx \leq I^+\Phi + I^-\phi$$

and

$$-\int_a^b (h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy)g(x)\,dx \leq -I^+\phi - I^-\Phi$$

and hence the obtained result (1.3) follows. □

**Theorem 1.2.** *Let $f : [a,b] \to \mathbb{R}$ be a twice differentibale mapping on $(a,b)$ with $\Gamma_2 = \sup_{x\in(a,b)} f''(x) < +\infty$ and $\gamma_2 = \inf_{x\in(a,b)} f''(x) > -\infty$, then we have the estimation*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a))\right| \leq \frac{1}{36\sqrt{3}}(\Gamma_2-\gamma_2)(b-a)^3, \tag{1.4}$$

*where the constant $\frac{1}{36\sqrt{3}}$ is the best one in the sense that it cannot be replaced by a smaller one.*

*Proof.* We choose in (1.3), $h(x) = -\frac{1}{2}(x-a)(b-x)$ and $g(x) = f''(x)$, we get

$$\begin{aligned}\frac{1}{2}\int_a^b \left| h(x) - \frac{1}{b-a}\int_a^b h(y)dy \right| dx &= \left| \int_{x_1}^{x_2} \left( h(x) + \frac{1}{12}(b-a)^2 \right) dx \right| \\ &= \frac{1}{36\sqrt{3}}(b-a)^3,\end{aligned}$$

where $x_1 = a + \frac{3-\sqrt{3}}{6}(b-a)$ and $x_2 = a + \frac{3+\sqrt{3}}{6}(b-a)$. Thus from (1.3), we derive

$$\begin{aligned}&\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a)) \right| \\ &\quad = \left| \int_a^b -\frac{1}{2}(x-a)(b-x)f''(x)\,dx - \frac{1}{b-a}\int_a^b -\frac{1}{2}(x-a)(b-x)\,dx \int_a^b f''(x)\,dx \right| \\ &\quad\le \frac{1}{36\sqrt{3}}(\Gamma_2-\gamma_2)(b-a)^3.\end{aligned}$$

To explain the best constant $\frac{1}{36\sqrt{3}}$ in the inequality (1.4), we can construct the function $f(x) = \int_a^x \left( \int_a^y j(z)\,dz \right) dy$ to attain the inequality in (1.4),

$$j(x) = \begin{cases} \gamma_2, & a \le x < x_1 = a + \dfrac{3-\sqrt{3}}{6}(b-a), \\ \Gamma_2, & x_1 \le x < x_2 = a + \dfrac{3+\sqrt{3}}{6}(b-a), \\ \gamma_2, & x_2 \le x \le b. \end{cases}$$

The proof is complete. □

**Theorem 1.3.** *Let $f : [a,b] \to \mathbb{R}$ be a third-order differentibale mapping on $(a,b)$ with $\Gamma_3 = \sup_{x\in(a,b)} f'''(x) < +\infty$ and $\gamma_3 = \inf_{x\in(a,b)} f'''(x) > -\infty$, then we have the estimation*

$$\begin{aligned}\left| \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a)) \right| \\ \le \frac{1}{384}(\Gamma_3-\gamma_3)(b-a)^4,\end{aligned} \tag{1.5}$$

*where the constant $\frac{1}{384}$ is the best one in the sense that it cannot be replaced by a smaller one.*

*Proof.* We choose in (1.3), $h(x) = \frac{1}{12}(x-a)(2x-a-b)(b-x)$, $g(x) = f'''(x)$, to get

$$\frac{1}{2}\int_a^b |h(x) - \frac{1}{b-a}\int_a^b h(y)\,dy|\,dx = \int_a^{\frac{a+b}{2}} h(x)\,dx = \frac{1}{384}(b-a)^4,$$

Thus from (1.3) in Theorem 1.1, we can derive the inequality (1.5) immediately. Finally, we construct the function $f(x) = \int_a^x \left( \int_a^y \left( \int^z j(s)\,ds \right) dz \right) dy$, where $j(x) = \Gamma_3$ for $a \le x < \frac{a+b}{2}$ and $j(x) = \gamma_3$ for $\frac{a+b}{2} \le x \le b$, then the equality holds in (1.5). □

**Theorem 1.4.** *Let* $f : [a,b] \to \mathbb{R}$ *be a fourth-order differentibale mapping on* $(a,b)$ *with* $M_4 = \sup_{x\in(a,b)} |f^{(4)}(x)| < +\infty$*, then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a))\right| \le \frac{1}{720}M_4(b-a)^5, \tag{1.6}$$

*where* $\frac{1}{720}$ *is the best possible constant.*

*Proof.* We may write the remainder of the perturbed trapezoid inequality in the kernel form

$$\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a)) = \int_a^b f^{(4)}(x)k_4(x)\,dx, \tag{1.7}$$

where $k_4(x) = \frac{1}{24}(x-a)^2(b-x)^2$. Then we get

$$\int_a^b |k_4(x)|\,dx = \frac{1}{24}\int_0^1 x^2(1-x)^2\,dx = \frac{1}{720}. \tag{1.8}$$

Then (1.7) – (1.8) imply (1.6). The equality holds for $f(x) = x^4$, $a \le x \le b$ in inequality (1.6). □

**Remark 1.5.** We also can prove Theorem 1.2 and 1.3 in the kernel form

$$\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{12}(b-a)^2(f'(b)-f'(a)) = \int_a^b f^{(n)}(x)k_n(x)\,dx, \tag{1.9}$$

where $k_2(x) = -\frac{1}{2}(x-a)(b-x) + \frac{1}{12}$ and $k_3(x) = \frac{1}{12}(x-a)(2x-a-b)(b-x)$. By the formula (1.7) and (1.9), and derive the perturbed trapezoid inequality for different norms as shown in [5].

Now we present the composite perturbed trapezoid quadrature for an equidistant partitioning of interval $[a,b]$ into $n$ subintervals. Applying Theorems 1.2 – 1.4, we obtain

$$\int_a^b f(x)\,dx = T_n(f) + R_n(f),$$

where

$$T_n(f) = \frac{b-a}{2n}\sum_{i=0}^{n-1}\left[f\left(a + i\frac{b-a}{n}\right) + f\left(a + (i+1)\frac{b-a}{n}\right)\right] - \frac{(b-a)^2}{12n^2}(f'(b)-f'(a)),$$

and the remainder $R_n(f)$ satisfies the error estimate

$$|R_n(f)| \le \begin{cases} \dfrac{(b-a)^3}{36\sqrt{3}n^2}(\Gamma_2-\gamma_2), & \text{if } \gamma_2 \le f''(x) \le \Gamma_2,\ \forall x\in(a,b), \\[2ex] \dfrac{(b-a)^4}{384n^3}(\Gamma_3-\gamma_3), & \text{if } \gamma_3 \le f'''(x) \le \Gamma_3,\ \forall x\in(a,b) \\[2ex] \dfrac{(b-a)^5}{720n^4}M_4, & \text{if } |f^{(4)}(x)| \le M_4,\ \forall x\in(a,b). \end{cases} \tag{1.10}$$

Then we can use (1.10) to get different error estimates of the composite perturbed trapezoid quadrature.

As in [5], we may also apply the Theorems 1.2, 1.3 and 1.4 to special means. In this case we may improve some of the bounds related to inequalities about special means as given in [5, p. 492-494].

Furthermore, we discuss the Iyengar's type inequality for the perturbed trapezoidal quadrature rule for functions whose first and second order derivatives are bounded. In [1, 3, 9, 10] they proved the following interesting inequality involving bounded derivatives.

If $f$ is a differentiable function on $(a,b)$ and $|f'(x)| \le M_1$, then

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \le \frac{M_1(b-a)^2}{4} - \frac{(f(b)-f(a))^2}{4M_1}.$$

If $|f''(x)| \le M_2,\ x\in[a,b]$ for positive constant $M_2\in\mathbb{R}$, then

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \le \frac{M_2}{24}\left((b-a)^3 - \left(\frac{|\Delta|}{M_2}\right)^3\right),$$

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \le \frac{M_2}{24}(b-a)^3 - \frac{\Delta_1^2(b-a)}{16M_2},$$

where

$$\Delta = f'(a) - 2f'\left(\frac{a+b}{2}\right) + f'(b), \qquad \Delta_1 = f'(a) - 2\frac{(f(b)-f(a))}{b-a} + f'(b). \tag{1.11}$$

We will prove the following inequality.

**Theorem 1.6.** *Let $f: I\to\mathbb{R}$, where $I\subseteq\mathbb{R}$ is an interval. Suppose that $f$ is twice differentiable in the interior $\overset{\circ}{I}$ of $I$, and let $a,b\in\overset{\circ}{I}$ with $a<b$. If $|f''(x)|\le M_2,\ x\in[a,b]$ for positive constant $M_2\in\mathbb{R}$. Then*

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \le \frac{1}{24}M_2(b-a)^3 - \sqrt{\frac{|\Delta_1|^3(b-a)^3}{72M_2}}, \tag{1.12}$$

*where $\Delta_1$ is defined as (1.11).*

*Proof.* Denote

$$J_f = \int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a)).$$

It is easy to see that

$$J_f = \int_a^b \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2 f''(x)\,dx,$$

and

$$\Delta_1 = f'(a) - \frac{2(f(b)-f(a))}{b-a} + f'(b) = \frac{1}{b-a}\int_a^b 2\left(x-\frac{a+b}{2}\right) f''(x)\,dx. \tag{1.13}$$

For any $|\varepsilon| \leq \frac{1}{8}$, we get for $|f''(x)| \leq M_2,\ x \in [a,\ b]$,

$$\begin{aligned} J_f + \varepsilon(b-a)^2\Delta_1 &= \int_a^b \left( \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + 2\varepsilon(b-a)\left(x - \frac{a+b}{2}\right)\right) f''(x)\,dx \\ &\leq F(\varepsilon)M_2(b-a)^3, \end{aligned}$$

where

$$\begin{aligned} F(\varepsilon) &= \frac{1}{(b-a)^3}\int_a^b \left| \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2 + 2\varepsilon(b-a)\left(x - \frac{a+b}{2}\right)\right| dx \\ &= \int_0^1 \left| \frac{1}{2}\left(x - \frac{1}{2}\right)^2 + 2\varepsilon\left(x - \frac{1}{2}\right)\right| dx. \end{aligned}$$

For the case $0 \leq \varepsilon \leq \frac{1}{8}$, we have

$$\begin{aligned} F(\varepsilon) &= \int_0^1 \left| \frac{1}{2}\left(x - \frac{1}{2}\right)^2 + 2\varepsilon\left(x - \frac{1}{2}\right)\right| dx \\ &= \left\{ \int_0^{\frac{1}{2}-4\varepsilon} \left( \frac{1}{2}\left(x - \frac{1}{2}\right)^2 + 2\varepsilon\left(x - \frac{1}{2}\right)\right) dx \right. \\ &\qquad - \int_{\frac{1}{2}-4\varepsilon}^{\frac{1}{2}} \left( \frac{1}{2}\left(x - \frac{1}{2}\right)^2 + 2\varepsilon\left(x - \frac{1}{2}\right)\right) dx \\ &\qquad\qquad \left. + \int_{\frac{1}{2}}^1 \left( \frac{1}{2}\left(x - \frac{1}{2}\right)^2 + 2\varepsilon\left(x - \frac{1}{2}\right)\right) dx \right\} \\ &= \frac{1}{24} + \frac{32}{3}\varepsilon^3. \end{aligned}$$

For the case $-\frac{1}{8} \leq \varepsilon \leq 0$, we have similarly

$$F(\varepsilon) = \frac{1}{24} - \frac{32}{3}\varepsilon^3.$$

We can prove $|\Delta_1| \leq \frac{1}{2}(b-a)M_2$ easily from (1.13). Thus we choose the parameter

$$\varepsilon_* = sign(\Delta_1)\sqrt{\frac{|\Delta_1|}{32(b-a)M_2}}, \qquad |\varepsilon_*| \leq \frac{1}{8}.$$

By the above inequalities, we obtain

$$J_f \leq F(\varepsilon_*)M_2 - \varepsilon_*(b-a)^2\Delta_1 \leq \frac{1}{24}(b-a)^3M_2 - \sqrt{\frac{(b-a)^3|\Delta_1|^3}{72M_2}}.$$

Replacing $f$ with $-f$, we have

$$J_{-f} = -J_f \leq \frac{1}{24}(b-a)^3M_2 - \sqrt{\frac{(b-a)^3|\Delta_1|^3}{72M_2}}.$$

Thus we obtain bounds for $|J_f|$ and prove the inequality (1.12). □

**Remark 1.7.** As $|\Delta_1| \leq \frac{1}{2}M_2(b-a)$, we have

$$\sqrt{\frac{|\Delta_1|}{M_2}} \geq \sqrt{\frac{2}{b-a}}\,\frac{|\Delta_1|}{M_2},$$

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b)) + \frac{1}{8}(b-a)^2(f'(b)-f'(a))\right| \leq \frac{M_2}{24}(b-a)^3 - \frac{\Delta_1^2(b-a)}{6M_2}.$$

For the case $f'(a) = f'(b) = 0$, we have

$$\left|\int_a^b f(x)\,dx - \frac{1}{2}(b-a)(f(a)+f(b))\right| \leq \frac{M_2}{24}(b-a)^3 - \frac{2}{3}\,\frac{|f(b)-f(a)|^2}{M_2(b-a)}. \tag{1.14}$$

The inequality (1.14) is sharper than that stated in [9, p. 69].

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