Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 3, Issue 2, Article 23, 2002

# INEQUALITIES FOR LATTICE CONSTRAINED PLANAR CONVEX SETS

POH WAH HILLOCK AND PAUL R. SCOTT

4/38 BEAUFORT STREET,
ALDERLEY, QUEENSLAND 4051, AUSTRALIA.

DEPARTMENT OF PURE MATHEMATICS,
UNIVERSITY OF ADELAIDE,
S.A. 5005 AUSTRALIA.
pscott@maths.adelaide.edu.au
*URL*: http://www.maths.adelaide.edu.au/pure/pscott/

*Received 20 September, 2000; accepted 28 November, 2001.*
*Communicated by C.E.M. Pearce*

ABSTRACT. Every convex set in the plane gives rise to geometric functionals such as the area, perimeter, diameter, width, inradius and circumradius. In this paper, we prove new inequalities involving these geometric functionals for planar convex sets containing zero or one interior lattice point. We also conjecture two results concerning sets containing one interior lattice point. Finally, we summarize known inequalities for sets containing zero or one interior lattice point.

*Key words and phrases:* Planar Convex Set, Lattice, Lattice Point Enumerator, Lattice-Point-Free, Sublattice, Area, Perimeter, Diameter, Width, Inradius, Circumradius.

2000 *Mathematics Subject Classification.* 52A10, 52A40, 52C05, 11H06.

## 1. INTRODUCTION

Let $\mathcal{K}^2$ denote the set of all planar, compact, convex sets. Let $K$ be a set in $\mathcal{K}^2$ with area $A = A(K)$, perimeter $p = p(K)$, diameter $d = d(K)$, width $w = w(K)$, inradius $r = r(K)$ and circumradius $R = R(K)$. Let $K^o$ denote the interior of $K$. Let $\Gamma$ denote the integer lattice. The lattice point enumerator $G(K^o, \Gamma)$ is defined to be the number of points of $\Gamma$ contained in $K^o$. In the case where $G(K^o, \Gamma) = 0$, we say that $K$ is lattice-point-free.

In this article, we prove new inequalities involving the geometric functionals $A, p, d, w, r$ and $R$ for sets $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 0$ and $G(K^o, \Gamma) = 1$. These may be found in Sections 2 and 3 respectively. In Section 4, we conjecture two results concerning sets $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 1$. Finally, in Sections 5 and 6, we summarize known inequalities in one and two functionals for sets $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 0$ and $G(K^o, \Gamma) = 1$ respectively (see [26] for a summary of inequalities involving two and three functionals for sets $K \in \mathcal{K}^2$ without lattice constraints). Although there are extensive bibliographies for lattice constrained convex sets [8, 10, 11, 12, 24], this article attempts to organise the numerous results for sets $K \in \mathcal{K}^2$ with

ISSN (electronic): 1443-5756

036-00

$G(K^o, \Gamma) = 0$ and $G(K^o, \Gamma) = 1$. Although these results are rather special, they are a natural starting point for problems in the area and have in fact served as a springboard for many new and interesting problems.

In the statements of the theorems and the conjecture, each inequality is followed by a set for which the inequality is sharp.

## 2. Some Elementary Results for Lattice-Point-Free Sets

**Theorem 2.1.** *Let* $K \in \mathcal{K}^2$ *with* $G(K^o, \Gamma) = 0$. *Let* $\lambda = 2\sqrt{2}\sin\phi/2$, $\phi$ *being the unique solution of the equation* $\sin\theta = \pi/2 - \theta$, *(*$\phi \approx 0.832 \approx 47.4^o$*). Then*

$$r \leq \frac{\sqrt{2}}{2}, \quad \mathcal{C}_0 \text{ (Figure 5.1a)}, \tag{2.1}$$

$$\frac{A}{R} \leq 2\lambda \approx 2.288, \quad \mathcal{H}_0 \text{ (Figure 5.1c)}, \tag{2.2}$$

$$\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}\left(1 + \frac{\sqrt{3}}{2}\right)^{-1} \approx 0.309, \quad \mathcal{E}_0 \text{ (Figure 5.1b)}, \tag{2.3}$$

$$(2r-1)p \leq \frac{4}{r}(\sqrt{2}-1), \quad \mathcal{S}_0 \text{ (Figure 5.1e)}. \tag{2.4}$$

*Proof.* To prove (2.1), we use the following lemma from [3]:

**Lemma 2.2.** *Suppose that* $K \in \mathcal{K}^2$ *and* $G(K^o, \Gamma) = 0$. *Then there is a set* $K_* \in \mathcal{K}^2$ *with* $G({K_*}^o, \Gamma) = 0$ *satisfying the following conditions:*

*(a)* $r(K) \leq r(K_*)$,

*(b)* $K_*$ *is symmetric about the lines* $x = \frac{1}{2}, y = \frac{1}{2}$.

From the lemma, it suffices to prove (2.1) for sets $K$ which are symmetric about the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$. To fully utilise the symmetry of $K$ about the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$, we move the origin to the point $(\frac{1}{2}, \frac{1}{2})$. If $r \leq \frac{1}{2}$, then (2.1) is trivially true. Hence we may assume that $r > \frac{1}{2}$. Since $K^o$ does not contain the points $P_1(\frac{1}{2}, \frac{1}{2}), P_2(-\frac{1}{2}, \frac{1}{2}), P_3(-\frac{1}{2}, -\frac{1}{2})$ and $P_4(\frac{1}{2}, -\frac{1}{2})$, it follows by the convexity of $K$ that for each $i = 1, \ldots, 4$, $K$ is bounded by a line $l_i$ through the point $P_i$ with $l_1$ and $l_3$ having negative slope and $l_2$ and $l_4$ having positive slope. Furthermore, since $K$ is symmetric about the coordinate axes, $K$ is contained in a rhombus $Q$ determined by the lines $l_i, i = 1, \ldots, 4$. Since $K \subseteq Q$, $r(K) \leq r(Q)$. Clearly $r(Q) \leq \sqrt{2}/2$. Hence $r(K) \leq \sqrt{2}/2$ and (2.1) is proved. An example of a set for which the inequality is sharp is the circle $\mathcal{C}_0$ (Figure 5.1a).

(2.2) follows easily from a result by Scott [18], that if $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 0$, then

$$\frac{A}{d} \leq \lambda \approx 1.144, \tag{2.5}$$

where $\lambda$ is as defined in Theorem 2.1. The result is best possible with equality when and only when $K \cong \mathcal{H}_0$ (Figure 5.1c). Using $d \leq 2R$ and (2.5), it follows immediately that

$$\frac{A}{R} \leq 2\lambda \approx 2.288,$$

with equality when and only when $K \cong \mathcal{H}_0$ (Figure 5.1c).

The proof of (2.3) follows easily by combining two known results. The first is that of all sets in $\mathcal{K}^2$ with a given width, the equilateral triangle has the least area [27, p. 68]. Hence

$A \geq (1/\sqrt{3})w^2$. We also recall from [17] that if $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 0$, then

$$w \leq 1 + \frac{\sqrt{3}}{2},$$

with equality when and only when $K \cong \mathcal{E}_0$ (Figure 5.1b). Hence

$$\frac{A}{w^3} = \left(\frac{A}{w^2}\right)\frac{1}{w} \geq \frac{1}{\sqrt{3}}\left(1 + \frac{\sqrt{3}}{2}\right)^{-1} \approx 0.309.$$

Equality holds when and only when $K \cong \mathcal{E}_0$ (Figure 5.1b).

To prove (2.4), we use a result from [3]: If $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 0$, then

$$(2r - 1)A \leq 2(\sqrt{2} - 1), \tag{2.6}$$

with equality when and only when $K \cong \mathcal{S}_0$ (Figure 5.1e). We also note from the same paper, that if $K$ is a convex polygon, $K$ may be partitioned into triangles by joining each vertex of $K$ to an in-centre of $K$. Summing the areas of these triangles gives

$$A \geq \frac{1}{2}pr,$$

with equality when and only when every edge of $K$ touches the unique incircle. Since any set in $\mathcal{K}^2$ is either a convex polygon, or may be approximated by a convex polygon, this inequality is valid for all sets in $\mathcal{K}^2$. By combining this inequality with (2.6), we have (2.4), with equality when and only when $K \cong \mathcal{S}_0$ (Figure 5.1e). □

## 3. SOME ELEMENTARY RESULTS FOR SETS CONTAINING ONE INTERIOR LATTICE POINT

**Theorem 3.1.** *Let $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 1$, . Let $\lambda$ be as defined in Theorem 2.1. Then*

$$r \leq 1, \quad \mathcal{C}_1 \text{ (Figure 6.1a)}, \tag{3.1}$$

$$\frac{A}{R} \leq 2\sqrt{2}\lambda \approx 3.232, \quad \mathcal{H}_1 \text{ (Figure 6.1d)}, \tag{3.2}$$

$$A(w - \sqrt{2}) \leq \frac{1}{2}w^2, \quad \mathcal{T}_1 \text{ (Figure 6.1e)}, \tag{3.3}$$

$$(2r - \sqrt{2})p \leq \frac{8}{r}(2 - \sqrt{2}), \quad \mathcal{S}_1 \text{ (Figure 6.1g)}. \tag{3.4}$$

We note that (3.1), (3.2) and (3.4) are the results for sets $K \in \mathcal{K}^2$ having $G(K^o, \Gamma) = 1$ corresponding to (2.1), (2.2) and (2.4) respectively. Furthermore, we recall from [22] that if $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 0$, then

$$A(w - 1) \leq \frac{1}{2}w^2, \tag{3.5}$$

with equality when and only when $K \cong \mathcal{T}_0$ (Figure 5.1f). We observe that (3.3) is the result corresponding to (3.5) for sets $K \in \mathcal{K}^2$ having $G(K^o, \Gamma) = 1$.

In fact, (3.3) has been proved in [14], where the method of proof is an adaptation of the method in [22]. In this paper we present a short and different proof for (3.3). We will see that all the inequalities of Theorem 3.1 follow immediately from their corresponding inequalities for lattice-point-free sets by using a simple sublattice argument.

*Proof.* Let

$$\Gamma' = \{(x, y) : x + y \equiv 1 \mod 2)\}.$$

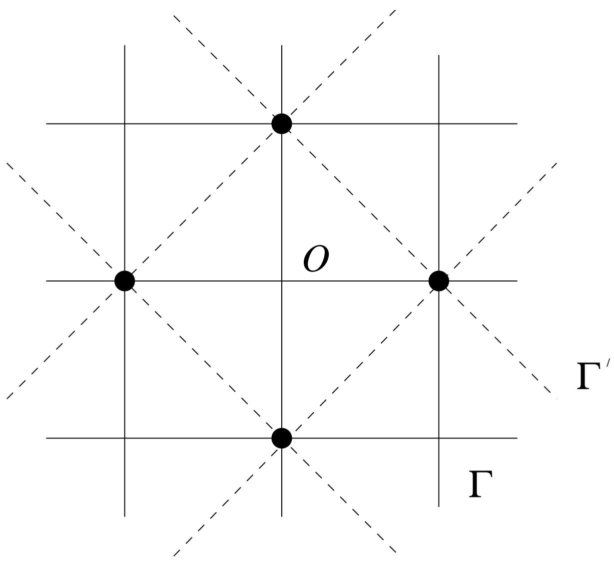



*Figure 3.1: The lattice* $\Gamma'$.

Suppose that $K \in \mathcal{K}^2$, with $G(K^o, \Gamma) = 1$. Then clearly $G(K^o, \Gamma') = 0$ (Figure 3.1). We also observe that $\Gamma'$ is essentially an anticlockwise rotation of $\Gamma$ about $O$ through an angle $\pi/4$ and scaled by a factor of $\sqrt{2}$. Now let $A'$, $p'$, $d'$ $w'$, $r'$, and $R'$ be the area, perimeter, diameter, width, inradius and circumradius respectively of $K$ measured in the scale of $\Gamma'$. Then since $G(K^o, \Gamma') = 0$, the inequalities (2.1), (2.2), (3.5), and (2.4) apply, from which we have

$$\begin{aligned} r' &\le \frac{\sqrt{2}}{2}, \quad \mathcal{C}_0{}' \\ \frac{A'}{R'} &\le 2\lambda, \quad \mathcal{H}_0{}' \\ A'(w'-1) &\le \frac{1}{2}(w')^2, \quad \mathcal{T}_0{}' \\ (2r'-1)p' &\le \frac{4}{r'}(\sqrt{2}-1), \quad \mathcal{S}_0{}', \end{aligned}$$

where $\mathcal{C}_0{}', \mathcal{H}_0{}', \mathcal{T}_0{}'$, and $\mathcal{S}_0{}'$ are the sets $\mathcal{C}_0$, $\mathcal{H}_0$, $\mathcal{T}_0$ and $\mathcal{S}_0$ respectively rotated anticlockwise about $O$ through $\pi/4$ and scaled by a factor of $\sqrt{2}$. Hence $\mathcal{C}_0{}' = \mathcal{C}_1$ (Figure 6.1a), $\mathcal{H}_0{}' = \mathcal{H}_1$ (Figure 6.1d), $\mathcal{T}_0{}' = \mathcal{T}_1$ (Figure 6.1e), and $\mathcal{S}_0{}' = \mathcal{S}_1$ (Figure 6.1g). Furthermore, since $\Gamma'$ is a rotation of $\Gamma$ scaled by a factor of $\sqrt{2}$, we have

$$A' = \left(\frac{1}{\sqrt{2}}\right)^2 A,\ p' = \frac{1}{\sqrt{2}}p,\ w' = \frac{1}{\sqrt{2}}w,\ r' = \frac{1}{\sqrt{2}}r,\ R' = \frac{1}{\sqrt{2}}R.$$

Substituting these into the above inequalities, we obtain (3.1), (3.2), (3.3), and (3.4) respectively. □

## 4. Conjectures for Sets Containing One Interior Lattice Point

**Conjecture 4.1.** *Let $K \in \mathcal{K}^2$ with $G(K^o, \Gamma') = 1$. Let $O$ be the circumcentre of $K$ in* (4.2). *Then*

$$\frac{A}{w^3} \ge \frac{1}{\sqrt{3}}.\frac{4}{\sqrt{2}(5+\sqrt{3})} \approx 0.243, \quad \mathcal{E}_1 \textit{ (Figure 6.1b)}, \tag{4.1}$$

$$A \le \alpha \approx 4.05, \quad \mathcal{Q}_1 \textit{ (Figure 6.1f)}. \tag{4.2}$$

The problem which occurs in (4.1) is that for a set $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 1$, $w \leq 1+\sqrt{2} \approx 2.414$, with equality when and only when $K \cong \mathcal{I}_1$ (Figure 6.1e) [23]. Since this set of largest width is not an equilateral triangle, the method used to prove (2.3) cannot be applied.

A simple calculation shows that the width of $\mathcal{E}_1$ (Figure 6.1b) is $\frac{1}{4}\sqrt{2}(5+\sqrt{3}) \approx 2.38$. Hence if $0 < w \leq \frac{1}{4}\sqrt{2}(5+\sqrt{3})$, an equilateral triangle containing one interior lattice point may be constructed. Since $A \geq (1/\sqrt{3})w^2$ with equality when and only when $K$ is an equilateral triangle, for this range of $w$ we have

$$\frac{A}{w^3} = \left(\frac{A}{w^2}\right)\frac{1}{w} \geq \frac{1}{\sqrt{3}}.\frac{4}{\sqrt{2}(5+\sqrt{3})} \approx 0.243,$$

with equality when and only when $K \cong \mathcal{E}_1$ (Figure 6.1b).

This leaves unresolved those cases for which $\frac{1}{4}\sqrt{2}(5+\sqrt{3}) < w \leq 1+\sqrt{2}$. We believe that the set for which $A/w^3$ is minimal is congruent to the equilateral triangle $\mathcal{E}_1$ (Figure 6.1b).

In [21], Scott conjectures a result concerning the maximal area of a set $K \in \mathcal{K}^2$ with $G(K^o, \Gamma) = 1$ and having circumcentre $O$. Using a computer run, we discover that the conjecture is false. We revise the conjecture as stated in (4.2), with equality when and only when $K \cong \mathcal{Q}_1$ (Figure 6.1f).

## 5. INEQUALITIES INVOLVING ONE AND TWO FUNCTIONALS FOR LATTICE-POINT-FREE SETS

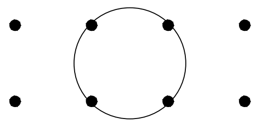
Tables 5.1 and 6.1 list the known inequalities (including conjectures) involving one and two functionals for lattice-point-free sets and sets containing one interior lattice point respectively. The extremal sets referred to in the tables may be found in Figures 5.1 and 6.1 respectively. Where a star ($\star$) appears in the inequality column, no inequality is known for the corresponding functionals.

| Parameters | Inequality | Extremal Set | Reference |
|---|---|---|---|
| $A$ | unbounded | | |
| $p$ | unbounded | | |
| $d$ | unbounded | | |
| $w$ | $w \leq \frac{1}{2}(2+\sqrt{3}) \approx 1.866$ | $\mathcal{E}_0$ | [17] |
| $R$ | unbounded | | |
| $r$ | $r \leq \sqrt{2}/2$ | $\mathcal{C}_0$ | (2.1) |
| $A, p$ | $A < \frac{1}{2}p$ | $\mathcal{P}_0$ | [6] |
| $A, d$ | $A/d \leq \lambda, \lambda \approx 1.144$ | $\mathcal{H}_0$ | [18] |
| $A, w$ | 1. $(w-1)A \leq \frac{1}{2}w^2$<br>2. $\frac{A}{w^3} \geq \frac{1}{\sqrt{3}}(1+\frac{\sqrt{3}}{2})^{-1} \approx 0.309$ | $\mathcal{T}_0$<br>$\mathcal{E}_0$ | [20]<br>(2.3) |
| $A, R$ | $A/R \leq 2\lambda$, $\lambda \approx 1.144$ | $\mathcal{H}_0$ | (2.2) |
| $A, r$ | 1. $(2r-1)A \leq 2(\sqrt{2}-1) \approx 0.828$<br>2. $(2r-1)\lvert A-1\rvert < \frac{1}{2}$ | $\mathcal{S}_0$<br>$\mathcal{P}_0$ | [3]<br>[3] |
| $p, d$ | $\star$ | | |
| $p, w$ | $(w-1)p \leq 3w$ | $\mathcal{E}_0$ | [20] |
| $p, R$ | $\star$ | | |
| $p, r$ | 1. $(2r-1)\lvert p-4\rvert < 2$<br>2. $(2r-1)p \leq \frac{4}{r}(\sqrt{2}-1)$ | $\mathcal{P}_0$<br>$\mathcal{S}_0$ | [3]<br>(2.4) |

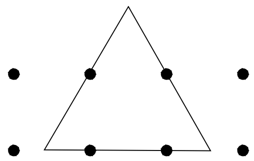
*Continued . . .*

| **Parameters** | **Inequality** | **Extremal Set** | **Reference** |
|---|---|---|---|
| $d, w$ | $(w-1)(d-1) \leq 1$ | $\mathcal{T}_0$ | [19] |
| $d, R$ | $2R - d \leq \frac{1}{3}$ | $\mathcal{E}_0$ | [4] |
| $d, r$ | $(2r-1)(d-1) < 1$ | $\mathcal{P}_0$ | [3] |
| $w, R$ | 1. $(w-1)R \leq \frac{1}{\sqrt{3}}w$ | $\mathcal{E}_0$ | [20] |
| | 2. $(w-1)(2R-1) \leq \frac{\sqrt{3}}{6} + 1 \approx 1.289$ | $\mathcal{E}_0$ | [25] |
| $w, r$ | $w - 2r \leq \frac{1}{3} + \frac{1}{6}\sqrt{3} \approx 0.622$ | $\mathcal{E}_0$ | [4] |
| $R, r$ | $(2r-1)(2R-1) \leq 1$ | $\mathcal{P}_0$ | [25] |

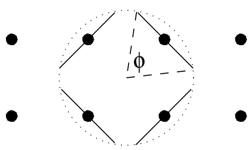
*Table 5.1: Inequalities for the case $G(K^o, \Gamma) = 0$.*



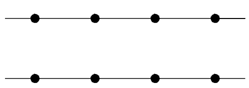
(a) The circle $\mathcal{C}_0$



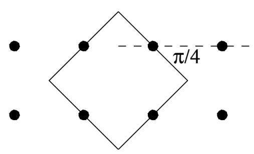
(b) The equilateral triangle $\mathcal{E}_0$



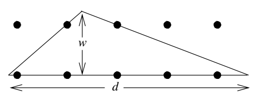

(c)
The truncated diagonal square $\mathcal{H}_0, \phi \approx 47.7^o$



(d) The parallel strip $\mathcal{P}_0$




(e) The diagonal square $\mathcal{S}_0$



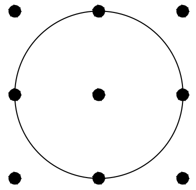

(f) The triangle $\mathcal{T}_0$

*Figure 5.1: Extremal sets for the case $G(K^o, \Gamma) = 0$*

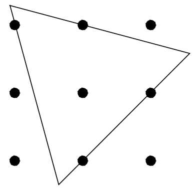
## 6. Inequalities Involving One and Two Functionals for Sets Containing One Interior Lattice Point

| **Parameters** | **Inequality** | **Extremal Set** | **Reference** |
|---|---|---|---|
| $A$ | 1. $A \le 4$ if $O$ is centre of $K$<br>2. $A \le 4.5$ if $O$ is the C.G.<br>3. *Conjecture*:<br>If $O$ is the circumcentre then $A \approx 4.05$ | e.g. $\mathcal{S}_1$<br>Ehrhart's $\triangle$<br><br>$\mathcal{Q}_1$ | [16]<br>[9]<br><br>(4.2) |
| $p$ | unbounded | | |
| $d$ | unbounded | | |
| $w$ | 1. $w \le 1 + \sqrt{2} \approx 2.414$<br>2. If $O$ is the C.G. then $w \le 3\sqrt{2}/2$<br>for the family of triangles | $\mathcal{I}_1$<br><br>Ehrhart's $\triangle$ | [23]<br><br>[13] |
| $R$ | $R \le \alpha \approx 1.685$ or $R$ unbounded | $\mathcal{T}$ | [2] |
| $r$ | $r \le 1$ | $\mathcal{C}_1$ | (3.1) |
| $A, p$ | $A/p \le 2(2 + \sqrt{\pi})^{-1} \approx 0.53$<br>($O$ is centre of $K$) | $\mathcal{U}_1$ | [1, 7] |
| $A, d$ | $A/d \le \sqrt{2}\lambda, \lambda \approx 1.144$ | $\mathcal{H}_1$ | [15] |
| $A, w$ | 1. $A(w - \sqrt{2}) \le \frac{1}{\sqrt{2}} w^2$<br>2. *Conjecture:*<br>$\frac{A}{w^3} \ge \frac{1}{\sqrt{3}} . \frac{4}{\sqrt{2}(5+\sqrt{3})} \approx 0.243$ | $\mathcal{T}_1$<br><br>$\mathcal{E}_1$ | (3.3), [14]<br><br>(4.1) |
| $A, R$ | $A/R \le 2\sqrt{2}\lambda$ | $\mathcal{H}_1$ | (3.2) |
| $A, r$ | $A(2r - \sqrt{2}) \le 4(2 - \sqrt{2}) \approx 2.343$ | $\mathcal{S}_1$ | [3] |
| $p, d$ | $\star$ | | |
| $p, w$ | $\star$ | | |
| $p, R$ | $\star$ | | |
| $p, r$ | $p(2r - \sqrt{2}) \le \frac{8}{r}(2 - \sqrt{2})$ | $\mathcal{S}_1$ | (3.4) |
| $d, w$ | $(w - \sqrt{2})(d - \sqrt{2}) \le 2$ | $\mathcal{T}_1$ | [23] |
| $d, R$ | *Conjecture:*<br>$2R - d \le \frac{\sqrt{2}}{6}.(7 - 3\sqrt{3}) \approx 0.425$ | <br>$\mathcal{E}_1$ | <br>[5] |
| $d, r$ | $\star$ | | |
| $w, R$ | $\star$ | | |
| $w, r$ | *Conjecture:*<br>$w - 2r \le \frac{\sqrt{2}}{12}(5 + \sqrt{3}) \approx 0.793$ | <br>$\mathcal{E}_1$ | <br>[5] |
| $R, r$ | $\star$ | | |

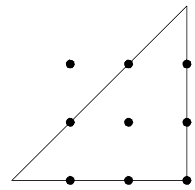
*Table 6.1: Inequalities for the case $G(K^o, \Gamma) = 1$*



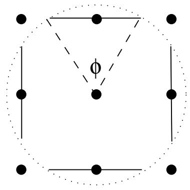
(a) The circle $\mathcal{C}_1$



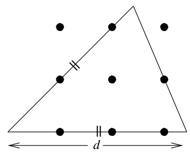
(b) The equilateral triangle $\mathcal{E}_1$



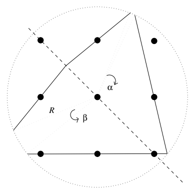
(c) Ehrhart's $\triangle$



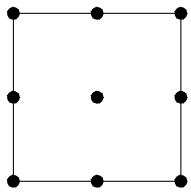

(d) The truncated square $\mathcal{H}_1$, $\phi \approx 47.7^o$



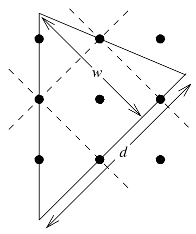

(e) The isosceles triangle $\mathcal{I}_1$



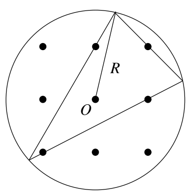

(f) The truncated quadrilateral $\mathcal{Q}_1$, $R \approx 1.593$, $\alpha \approx 5.47^o$, $\beta \approx 20.23^o$



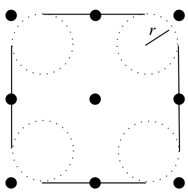
(g) The square $\mathcal{S}_1$




(h) The triangle $\mathcal{T}_1$




(i) The triangle $\mathcal{T}$, $R \approx 1.685$




(j) The rounded square $\mathcal{U}_1$, $r \approx 0.530$

*Figure 6.1: Extremal sets for the case* $G(K^o, \Gamma) = 1$

## REFERENCES

[1] J.R. ARKINSTALL AND P.R. SCOTT, An isoperimetric problem with lattice point constraints. *J. Austral. Math. Soc. Ser. A*, **27** (1979), 27–36.

[2] P.W. AWYONG AND P.R. SCOTT, On the maximal circumradius of a planar convex set containing one lattice point, *Bull. Austral. Math. Soc.*, **52** (1995), 137–151.

[3] P.W. AWYONG AND P.R. SCOTT, New inequalities for planar convex sets with lattice point constraints, *Bull. Austral. Math. Soc.*, **54** (1996), 391–396.

[4] P.W. AWYONG, An inequality relating the circumradius and diameter of two-dimensional lattice-point-free convex bodies, *Amer. Math. Monthly*, **106**(3) (1999), 252–255.

[5] P.W. AWYONG AND P.R. SCOTT, Circumradius-diameter and width-inradius relations for lattice constrained convex sets, *Bull. Austral. Math. Soc.*, **59** (1999), 147–152.

[6] E.A. BENDER, Area-perimeter relations for two-dimensional lattices, *Amer. Math. Monthly*, **69** (1962), 742–744.

[7] H.T. CROFT, Cushions, cigars and diamonds: an area-perimeter problem for symmetric ovals, *Math. Proc. Cambridge Philos. Soc.*, **85** (1979), 1–16.

[8] H.T. CROFT, K.J. FALCONER AND R.K. GUY, *Unsolved problems in geometry*, Springer-Verlag, New York, 1991.

[9] E. EHRHART, Une généralisation du théorème de Minkowski, *C.R. Acad. Sci. Paris*, **240** (1955), 483–485.

[10] P. ERDÖS, P.M. GRUBER AND J. HAMMER, *Lattice points*, Longman, Essex, 1989

[11] P. GRITZMANN AND J.M. WILLS, Lattice points, In *Handbook of Convex Geometry, Vol. A,B*, eds. P.M. Gruber and J.M. Wills, North-Holland, Amsterdam, 1993, 765–797.

[12] J. HAMMER, *Unsolved problems concerning lattice points*, Pitman, London, 1977

[13] M.A. HERNÁNDEZ CIFRE, P.R. SCOTT AND S. SEGURA GOMIS, On the centre of gravity and width of lattice-constrained convex sets in the plane, *Beitrage zur Algebra und Geometrie (Contributions to Algebra and Geometry)*, **38**(2), 1997, 423–427.

[14] M.A. HERNÁNDEZ CIFRE AND S. SEGURA GOMIS, Some inequalities for planar convex sets containing one lattice point, *Bull. Austral. Math. Soc.*, **58** (1998), 159–166.

[15] M.A. HERNÁNDEZ CIFRE AND S. SEGURA GOMIS, Some area-diameter inequalities for two-dimensional lattices, *Geometriae Dedicata*, **72** (1998), 325–330.

[16] H. MINKOWSKI, *Geometrie der Zahlen*, Teubner, Leipzig, 1991.

[17] P.R. SCOTT, A lattice problem in the plane, *Mathematika*, **20** (1973), 247–252.

[18] P.R. SCOTT, Area-diameter relations for two-dimensional lattices, *Math. Mag.*, **47** (1974), 218–221.

[19] P.R. SCOTT, Two inequalities for convex sets in the plane, *Bull. Austral. Math. Soc.*, **19** (1978), 131–133.

[20] P.R. SCOTT, Further inequalities for convex sets with lattice point constraints in the plane, *Bull. Austral. Math. Soc.*, **21** (1980), 7–12.

[21] P.R. SCOTT, Two problems in the plane, *Amer. Math. Monthly*, **89** (1982), 460–461.

[22] P.R. SCOTT, Area, width and diameter of planar convex sets with lattice point constraints, *Indian J. Pure Appl. Math.*, **14** (1983), 444–448.

[23] P.R. SCOTT, On planar convex sets containing one lattice point, *Quart. J. Math. Oxford Ser.* (2), 36 (1985), 105–111.

[24] P.R. SCOTT, Modifying Minkowski's Theorem, *J. Number Theory*, **29** (1988), 13–20.

[25] P.R. SCOTT AND P.W. AWYONG, Inradius and circumradius for planar convex bodies containing

no lattice points, *Bull. Austral. Math. Soc.*, **59** (1999), 163–168.

[26] P.R. SCOTT AND P.W. AWYONG, Inequalities for convex sets. *J. Ineq. Pure and Appl. Mathematics* **1**(1) Art. 6, 2000. [ONLINE] Available online at `http://jipam.vu.edu.au/v1n1/016_99.html`

[27] I.M. YAGLOM AND V.G. BOLTYANSKII, *Convex Figures*, Translated by P.J. Kelly and L.F. Walton, Holt, Rinehart and Winston, New York, 1961